APPENDIX A.

VOL. I.

CENSUS OF THE COMMONWEALTH OF AUSTRALIA.

The Mathematical Theory of Population, of its Character and Fluctuations, and of the Factors which influence them,

BEING AN

Examination of the general scheme of Statistical Representation, with deductions of necessary formulae; the whole being applied to the data of the Australian Census of 1911, and to the elucidation of Australian Population Statistics generally.

BY

G. H. KNIBBS, C.M.G., F.S.S., F.R.A.S., etc.,

Member of the International Institute of Statistics,
Honorary Member of the Société de Statistique of Paris, and of the
American Statistical Association, etc., etc.

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FOREWORD.

The following monograph on the Mathematical Theory of Population, in form an appendix to the Report on the Australian Census of 1911, is intended to serve a double purpose. It aims on the one hand at supplying the elements of a mathematical technique, such as are needed for the analysis of the various aspects of vital phenomena that come under statistical review, and, on the other, at interpreting material made available by the first Census of Australia which has been carried out upon uniform lines and by a central authority. The earlier portion of the appendix has consequently been almost wholly devoted to the creation of the requisite technique. Later technical solutions are introduced only when required by way of application to any statistical analysis under immediate review.

In the realm of official statistics there is an enormous amount of accumulated material, which, decade after decade, remains unanalysed and uninterpreted. This is due to several things, viz., to the fact that routine tabulations largely occupy the energies of the staffs of statistical bureaux; to the fact that much of the mass of material itself is defective and its correction involves more time than is available; and perhaps still more to the fact that appropriate schemes of mathematical analysis have as yet either not been developed, or are regarded as inapplicable.

The present analyses and interpretations have yielded many results which, it is believed, will be seen to be of value. They have brought into clearer relief the necessity for recognising that the variation of any one statistical element affects all other statistical elements, so that the satisfactory reduction of "crude data" to a common system is by no means an easy undertaking, and the comparability of the statistic of two communities can never be rigorously exact in all particulars. It is fortunate, however, that practically exactitude means merely "a precision sufficient for any particular purpose in view."

In substance this monograph consists of two elements, viz., (i.) a technical one, and (ii.) an interpretative one. Formulæ essential for the purposes of interpretation have been deduced, and their use has been illustrated by application to the data of the Australian Censuses, or to intercensal statistical data which, subject otherwise to considerable uncertainty, could be adjusted only by means of information derived from the Census. Thus results of immediate value are obtained simultaneously with an exposition of the theory and technique of the subject.

The various formulæ developed have been carefully checked throughout, but it is too much to hope that among so many results error has been completely avoided. The author will, therefore, be grateful if any discoverer of errors or misprints will communicate with him.

As a rule corrections to data have been pushed as far as seemed to be desirable; theoretically it is often possible to push them even still farther. It is doubted, however, whether the precision of the data would justify this. An example will illustrate the point. In determining the ratios which reveal the age of maximum fecundity, if the number of women at risk be taken as the total of the same age-group, the denominator will be too large and the derived ratio too small. Hence allowances must be made for the diminution of risk for prior cases of child-birth. But there is no well-defined time-limit at which these allowances should stop. In general, however, their applicability becomes more questionable as they become smaller.

A synopsis shews the general treatment of the subject, and an index. at the end of this appendix, makes reference thereto easy. Where it has been deemed necessary to coin technical expressions their derivation has been indicated.

Finally it may be mentioned that many of the formulæ developed will be found serviceable in other investigations in which statistical methods are called into requisition.

G. H. KNIBBS.

Commonwealth Bureau of Census and Statistics.
Melbourne, March 1917.

CORRIGENDA.

Page 3.—Under figures in footnote: after "small figures" read "in brackets."

Page 4.—§ 4, line 8: for "an" read "on."

Page 7.—Line 3; for "acurately" read "accurately."
Line 3, footnote, for "Gesellschaftsehre" read "Gesellschaftslehre."

Page 8.—Sub-heading (iv.). For "interpolation" read "interpolations."
Line 7, last paragraph, insert "the" after "given."

Page 40.—Line 11 from bottom, after log x, insert "and κ being log k."
Line 9 from bottom, for "k," "$2k$," "$3k$," read "κ," "2κ," "3κ."

Page 55.—Line 1, for "of a curve" read "of the curve."

Page 68.—Formula (197d), for (" 1—" read " (i—."
Line 13, after "above" add "the numerical coefficients remaining, of course, the same."

Page 72.—In formula (211), the y should follow the sign of integration.

Page 81.—Line 4 from bottom, for ("n" read ("h."

Page 104.—Line 5, for "difference" read "the differences."
Line 27, for "the comparison of" read "comparisons among."

Page 144.—Lines 10 and 11, for "section" and "sections," read "Part" and "Parts," and for XII., read XI.

Page 163.—Line 4, for "M" read "M."
Line 7, for "$2Mr$ +" read "$2Mr$,."

Page 213.—Line 3 from bottom, for "occupying" read "occurring."

Page 233.—Line 4 of paragraph, for "in part of the" read "in part the."

Page 240.—Line 4, § 8, add after "maternity," "each birth being regarded a case of maternity."

Page 242.—Throughout table read "births" for "maternity."

Page 277.—Table LXXXVIII., in "Duration," for 251-160, read "251-260," and for "251-170" read "261-270."

Page 306.—Add to end of paragraph:—"Twins produced from one ovum have been called 'univitellins' and those from two ova 'bivitellins'."

Page 307.—Line 3 from bottom, for "uniovulate" read "uniovular."

Note. Formulæ 374, and 396 are omitted.

SYNOPSIS.

THE MATHEMATICAL THEORY OF POPULATION, OF ITS CHARACTERS AND FLUCTUATIONS, AND THE FACTORS WHICH INFLUENCE THEM.

APPENDIX A.

THE MATHEMATICAL THEORY OF POPULATION, OF ITS CHARACTER AND FLUCTUATIONS, AND OF THE FACTORS WHICH INFLUENCE THEM.

I.—INTRODUCTORY.

1. **General.**—The fundamental elements of social statistics are the fluctuations of the numbers and constitution of the population and of its various characteristics. These fluctuations are profoundly affected by many factors, only some of which are susceptible of physical expression. For example, the extraordinary development, characteristic in the last few decades, of every branch of science and technology, and the skill with which acquired knowledge has been applied to the exploitation of Nature's resources, have probably created the possibility of developing a considerably *larger population* than the world has yet carried, at least in historic times. On the other hand, the social standards have been so profoundly altered as to strongly counteract the effect indicated. Thus *the raising of the standard of living*, and an increased complexity in social organisation have held in check, more or less, that increase of population which might otherwise have been possible.

The opposition of tendency involved by the coëxistence of these two factors necessarily reinforces the interest, while it increases the difficulty of the problems which depend for solution on an evaluation of the degree of influence exerted by particular factors. The interest of any theory is evident when we ask: "What, on the whole, is indicated by past statistical history as to the future populations of the various races of the world?" This is a question, the correct answer to which is a necessary guide for national policy, and one which involves not only the accumulation of statistical facts that have now become available, but also a theory by means of which a forecast can be made as to what the immediate future has in store for each community.

An interesting illustration of this may be drawn from the history of the United States. In the year 1815, Elkanah Watson predicted with extraordinary accuracy the population of the United States up to the year 1860, by some method which, though not absolutely doing so, was sensibly equivalent to simply assuming a constant rate of increase. As a matter of fact, had Watson actually assumed that the rate of increase from 1790 to 1800 would remain constant till 1860, he would have predicted the population with still greater accuracy than he actually did. This will be made apparent hereinafter; see also Figs. 3 and 4.

The more complex conditions of the world to-day and the rapidity of the development of the arts and sciences, make the accuracy of prediction for so lengthy a period extremely doubtful; nevertheless an attempt to forecast the affairs of any country, to be well founded, must be based upon the results of a review, among other things, of all the facts of its population development, and upon a study of this development in all other parts of the world.

Of no less interest is the constitution of a population in respect of age, sex and race, and the influence of birth-rates and death-rates thereupon. The effect of age at marriage, the reproductivity as measured by frequency of childbirth, and the age at which it occurs, the probability of living at every age, and the variation of this probability with increasing scientific, hygienic and economic knowledge, are problems of the first order of importance.

The attempt is here made to give a rough outline of the theory of the subject, elucidating that theory where it seemed desirable by quantitative examples.

2. **Significance of analysis.**—The fluctuations in the number and constitution and other characters of populations present, in general, complex and dissimilar changes, and depend upon elements which will not readily lend themselves to prediction. They would thus appear at first sight not to be amenable to mathematical analysis. Nevertheless, when the fluctuations are analysed and expressed in mathematical form, *their trend often becomes much more definite*, and their true significance is more clearly revealed.[1]

[1] An example will illustrate what is meant. The populations in the United States in 1790 and 1820 were respectively 3.93 and 9.64 millions of people. If the number were supposed to increase at each instance at a uniform rate so as to give these numbers in the years mentioned, the deduced populations would be very nearly the actual ones, not only for the intermediate decades, but even up to the year 1860, as is evident from the following table, viz.:—

Year	1790	1800	1810	1820	1830	1840	1850	1860
Population supposed to increase at uniform rate (millions)	3.93	5.30	7.15	9.64	13.00	17.53	23.65	31.89
Actual population (millions) ..	3.93	5.31	7.24	9.64	12.87	17.07	23.19	31.44
Difference (millions)	.00	.01	.09	00	.13	.46	.46	.45

A remarkable prediction by Elkanah Watson is referred to later: see Figs. 3 and 4.

This fact, viz., that the supposition made is approximately true, throws light on the other facts. Thus, that to accord with this supposition the figures for 1800 and 1810 are very slightly too small, while those for 1830 to 1860 are somewhat in excess; and the excess is constant for 1840, 1850, and 1860; illustrate the value of the scheme of analysis by means of which the fundamental idea is ascertained. The deviations of the actual values from those computed on the assumption of uniform rate of increase may thus, indeed, become in turn the starting point of a further analysis undertaken with a view to the interpretation of the departure from the law of uniform increase, arbitrarily adopted as the norm of the phenomena.

For this reason it is proposed to develop the mathematical conceptions which may serve as the foundation of definite analyses of the fluctuation of any population; to express these conceptions by formulæ; to so develop and resolve the formulæ that they may be readily applied; and, where necessary, to illustrate their application.

3. **The nature of the problem.**—An ideal theory of population is one which would enable the statistician not only to determine definitely the influences thereupon of the various elements of human development, and of the phenomena of Nature, but also to examine all facts of interest to mankind, as they stand in relation to population. And however hopeless may be the expectations of establishing such a theory with meticulous precision and in all detail, it nevertheless remains true that fluctuations of population can often be adequately understood only when they are analysed by means of definite mathematical conceptions. Moreover, since all important facts concerning population are susceptible of numerical expression, analytical conceptions formulated for the purpose of giving exactitude to a knowledge of its variations, should be ultimately cast, if possible, in a mathematical mould.[1]

The total population-aggregates of some countries have been found to increase almost exactly at a uniform rate; in general, however, the rate fluctuates. "Can the characteristics of such fluctuations be subsumed under any conception?" is a question which naturally presents itself.

[1] To revert to a previous illustration, for example, if we ask: "What uniform rate of increase would cause a population of 3.93 millions to become 9.64 millions in 30 years?" the answer is that it would be necessary that each million persons should receive *at each instant* an addition at the rate of 29,910 persons *per annum*, that is to say, the rate of continuous increase would have to be 0.02991 per annum. More exactly, this would give the following figures, viz.:—

$$3{,}930{,}000\,;\ (+\ 1{,}370{,}173)\ =\ 5{,}300{,}173\,;\ (+\ 1{,}847{,}877)\ =\ 7{,}148{,}050\,;$$
$$(+\ 2{,}492{,}128)\ =\ 9{,}640{,}178.$$

The differences, shewn by the small figures, do not in themselves disclose the fact that the increase is at a uniform rate, but *on dividing each by the preceding population figures* it is seen to be equivalent to adding 348,644 persons per million per decennium. Hence, obviously, the *rate* of increase was constant. This rate will be found to be equal to an increase of 30,361.8 annually per million of the population at the beginning of each year.

The facts just indicated, viz., that starting with a population of 3,930,000, and uniform increases at the rate of 0.02991 per annum, gives a population of 5,300,173 in ten years, etc.; that an equivalent figure is given for the population if, at the end of each year, there is added to it an absolute increment of the amount of 0.0303618 of the population at its beginning; that the figures at the end of a decennium are given by adding an increment of 0.348644 of the population at the beginning of the decennium—can be elucidated only by formulating a definite conception of rate, and studying the consequences that flow therefrom. It is, for example, by no means immediately obvious that, used with the limitations above indicated, the three sets of figures will give identical results. The last will accurately give only decennial results; the middle value only annual; the rate of continuous increase is the only one which is appropriate to furnish correct results for any moment during the whole period under review: see Fig. 4.

Such answer as may be given must, if it is to be explicit, obviously be in the form of a mathematical theory of the subject. Such a theory will be found to involve two elements, viz. :—

(*a*) The appropriation of suitable conceptions of a mathematical character, and

(*b*) The development of a scheme of using them.

The propriety of the application of such conceptions is to be measured by the extent to which they are capable of illuminating the actual facts, and of reducing them to system.

What has been said regarding total population, applies equally to each constituent part, viz., to the totals for each sex, to the number of both sexes or of either sex at birth or at a particular age, to the ratio of the sexes, to the fluctuations in the rates of birth or death, and to all the circumstances of migration.

In other words, any fact, either of the condition or constitution of population at any moment, or of the relation of these at different moments can be readily subsumed under appropriate mathematical conceptions with sufficient precision for practical purposes.

Again, in dealing with the co-ordination of population with other related facts susceptible of statistical statement, the question often arises : "How can the nature of the relation be best defined or best disclosed ?" The selection of appropriate mathematical conceptions, and the means of bringing the facts under them, also constitute phases of the theory to be considered.

4. **Necessity for the mathematical expression of the conditions of the problem.**—Although, in the nature of the case, the population of any territory necessarily changes through births and deaths by whole units, and in instances of immigration and emigration sometimes by relatively large groups of units, no appreciable error will ordinarily be committed, at least where the aggregate population is large, if all its fluctuations be supposed to take place continuously and by infinitesimal increments. This supposition, which might appear an insufficient consideration to be physically invalid, very fairly represents, after all, the actual facts in their totality.[1]

[1] For, when all the circumstances are taken into account, it is obvious that the extent or degree to which the individuals of a community participate in its economic and general life, or in territorial occupation, passes through a wide range of values. These considerations have application even to the circumstances of birth and death, and even moreover to those of immigration and emigration. The ordinary involvement of a community by each individual through the circumstances preceding birth and following upon death, shew clearly that in many important respects the introduction and disappearance of a unit of the population is, *virtually*, not quite instantaneous.

It is obvious, too, that this consideration would apply even if registration, or rather the statistical recognition of that fact, were contemporaneous with birth and death, which, however, it is not, since ordinarily it follows these events by a period of varying length. In cases of birth it also stretches over a longer period. It

Thus the fluctuations of population therein may at least in ordinary cases, be represented with precision by an *imaginary* or *fictitious population*, the ideal fluctuations of which, varying with time, conform to all the laws of infinitesimal increment or decrement, in this way rendering those fluctuations amenable to a rigorous analysis by the methods of the infinitesimal calculus. Such an imaginary population, changing continually by infinitesimal amounts, not only accurately represents the totality of facts, but is amenable to mathematical treatment.

It is nevertheless important to bear in mind that actual population-changes may be oscillatory, as will later be shewn.

5. **Conception applies equally to a population "de facto" or a population "de jure."**—Population may be related to territory in two ways, viz., by actual presence, and by legal relationship therewith; that is to say, the relationship may be "*de facto*" or "*de jure*"; and official statements regarding population are of each kind. In some countries, as where the floating population is large, or where citizens are under special obligations (*e.g.*, military service, etc.), the main concern may be to ascertain the population which may be said to belong to, or to be domiciled in the place, the foreign migratory element, whatever its magnitude, being regarded as of relatively little moment. Again, where communal rights are explicit and of an important character, the general reasons for deciding to adopt the "*de jure*" relationship for the official enumeration of population may be very cogent.[1]

The association of a human being, however, with any particular territory, delimited by frontiers of any type whatever, is, after all, only one of degree, so that any criterion (*e.g.*, nationality, domicile, etc.), other than that of mere presence in the territory, however necessary for certain purposes, is more or less indeterminate for others, particularly in countries where the freedom of movement of the individual is practically unrestricted. The actual presence of an individual in any territory involves, in varying degree,[2] the whole scheme of general relationship which every unit has to the general community in which he finds himself, and which that community has to the territory it is occupying. He is

is considerably influenced by legal prescriptions in regard thereto, as well as by the traditions and circumstances of the community. Thus the registration of death must perforce quickly follow on its occurrence; not so the registration of birth. In a sparsely-populated district, the registration of birth may be very late as compared with registration in a densely-populated area.

We may remark in passing, that official estimates of population, at least when based upon accurate vital and migration records, as ordinarily kept and reported, are usually slightly in error as regards actual populations, viz., to an extent corresponding to the want of balance between inclusions at the beginning of a period of record, really belonging to a previous record, and exclusions at the end of the period owing to complete information not being to hand. In an increasing population the error tends on the whole to be one of defect.

[1] As, for example, in some of the Cantons in Switzerland.

[2] The economics and general relationship of individual with a community passes through a wide range of values, and in each individual the value varies with his age.

subject to the laws and to the same extent also the general civic and other responsibilities of the place, while the community, on the other hand, is concerned with his protection and well-being. Hence the "*de facto*" population may often be statistical desideratum. For other purposes obviously the "*de jūre*" population is a necessity.

For the general purposes of economics there are features characteristic of population which may be considered either in the "*de jure*" or the "*de facto*" relationship, which may call for specialisation in any mathematical treatment. For mere enumeration, however, the mathematical conception as above defined will apply with equal rigour to either.

6. **Nature of population fluctuations.**—The fluctuations of the entire population of the earth, if available for long periods, would probably disclose in their most general aspect the secular characteristics of its increase, which must have greatly varied. Merely local effects would to a large extent disappear in the total; opposite periodicities, dependent on seasons, would be balanced by the inclusion of results from both hemispheres; by taking quinquennial, decennial, or longer means or averages, the effect of minor fluctuations would be correspondingly eliminated; and the broad outlines of the facts of the growth of the world's population would be brought into relief. Were the curve of secular increase of population for the entire earth available, it would obviously constitute the most suitable *norm* for general comparative purposes. Statistic unfortunately, has, however, not yet attained to this. All we can assert with certainity is that the present rate of increase can have existed for a relatively short time only.

Limiting the consideration to particular countries, changes will be found exhibiting the following features, viz.:—

(i.) The rate of appearance of individuals by birth, and disappearance by death is not, in general, uniform throughout the year, but shews more or less definitely an annual period.

(ii.) The movement of floating population is also non-uniform, disclosing, in many instances, definite annual periodicity.

(iii.) Improvements of natural conditions are in general followed by changed rate of increment to the population, which may have a period of a considerable number of years, or may be brief.

(iv.) Variations of social and economic traditions profoundly affect the rate of increase of population.

For the larger purposes of statistic, elements of the type (i.) and (ii.) are ordinarily negligible; while those of the type (iii.) and (iv.) are of the first order of importance. For minor purposes the converse may be true. Hence, the scheme of any investigation must be adapted to the element under consideration.

In general, secular and long-period changes must be eliminated in order to accurately study minor and short-period changes; and conversely, minor periodic changes must be eliminated in order to acurately ascertain the characteristics of the secular changes.

7. **Changes in the constitution of populations.**—The ratio of the total numbers of each sex, the proportion of the sexes at each age, the relative birth, marriage, and death rates, the circumstances affecting fecundity, the consequences upon all of these of migration, of disease, of war, and of economic and social traditions and developments, as well as their fluctuations with the lapse of time, are necessarily matters of statistical concern. Such changes may be called "*constitutive changes*," or perhaps "*organic changes*," and their analysis and subsumption under mathematical expressions are often of importance and are essential in various statistical analyses.

8. **Organic adjustments of populations.**—In reviewing the constitution of population as a whole, it is obvious that organic adjustments occur.[1]

The nature and drift of such adjustment as has been indicated, or of the deviations of the actual constitution of a population at any moment from *some norm adopted for comparison*, and the changes in such deviations, can be effectively studied only by the establishment of a system of suitable mathematical relations. For such deviations to be made the subject of prediction, the law of their fluctuation with time, must, of course, be ascertained. The principles guiding the constitution of a norm will be illustrated hereinafter.

9. **Continuous and finite fluctuations.**—The scope of the mathematical theory of the fluctuation of population reveals its fundamental importance. Every form of fluctuation, whether of total population, or of its constitutive elements, of its characters, or of the influences to which these are subject, may ordinarily be regarded as changing continuously by infinitesimal increments or decrements within the period during which it is assumed to vary. In special cases the fluctuations may even be discontinuous.

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[1] In Europe, for example, of those born living, there are about 105 male births to every 100 female births: of those still-born the proportion is about 133 (see "Die Geborenen nach dem Geschlecht," in "Statistik und Gesellschaftsehre," by Prof. Dr. Georg von Mayr. Bd. II., § 56, p. 189), and the deviation from these figures for different countries is, in general, small. Nevertheless, in the total population of Europe there is a ratio of only about 97.6 males to 100 females. To war and unhealthy occupation, and accident, the death of a considerable number of males is directly attributed. Thus there aro no less than about 108 deaths of males to 100 deaths of females, for a number of countries. Nevertheless, because of the larger number of male births, the percentage does not materially change.

The aim of any definitive consideration of the subject is to express the fluctuations of population or of its constituent elements, and of its characters, in forms which will serve—

(i.) To render intelligible the characteristics of such fluctuations.

(ii.) To assist attempts at tracing the cause and effect of fluctuations.

(iii.) To determine means and averages, etc.

(iv.) To make all required interpolation of values.

(v.) To make prediction by extrapolation possible, or to make it possible by the result of a general analysis.

(vi.) To bring into clear relief the various characters of a population.

10. **Curves required to represent various fluctuations and the solution of the same.**—When a curve or "graph" representing a series of statistical results can be defined with sufficient accuracy by some form which is susceptible of geometrical or algebraical representation, such definition constitutes an advance as regards the understanding of the essential nature of the facts: a clearer conception of the statistical results is attained. For example, if the rate at which a population is growing be constant, then the curve passing through the terminals of the ordinates (whose length represents the successive values of the population) plotted against distances along an axis representing time, is a curve which is concave upward. This curve is of character such that, if, instead of plotting the ordinates on the natural scale, their logarithms be plotted, the terminals will be found to lie upon a straight line. Thus, if when the logarithms of the numbers of any population at different dates are plotted as ordinates, and the times as abscissæ, the points are found to lie on a straight line, we know that the rate of increase is constant.

To thoroughly represent and to analyse the nature of the changes in the size of any population or the changes in its constituent elements or characters, a considerable command of schemes of curve-representation is a desideratum. For the mathematical representation of fluctuation, therefore, it is, in general, necessary to know the geometrical form or graph of various algebraic or other mathematical expressions; in order that, given geometrical form or *graph* of a series of results, the mathematical expression appropriate to represent it will be recognised. For this reason a considerable number of type-curves and a knowledge of their graphs must be at the disposal of the statistical analyst, so that the appropriate expression may be selected. As soon as it is decided upon, the mode of solving for the constants of the representative expression becomes of importance. With this in view, it has been found desirable to give a considerable number of formulæ, and to indicate the methods by means of which the constants that make the expression definitive can be found.

This has been the more necessary, because, after all, the scheme of statistical representation, or the "*fitting of curves*," is an art of much difficulty, and one which is only in its infancy.

The fluctuations of the numbers representing population and its various characters make considerable demands in regard to knowledge of this kind, and consequently not only are formulæ given herein from time to time, but their "graphs" are also drawn. These exhibit the character of the curves represented. It will be seen that the interpretation of statistical results therefore make considerable demands of what is called *curve-tracing*.[1]

[1] The "Spezielle algebraische und transzendente ebene Kurven, Theorie und Geschichte," of Dr. Gino Loria, 2 vols., Teubner, Leipzig, 1910-1911; the "Sammlung von Formeln der reinen und angewandten Mathematik," by Dr. W. Laska, Fr. Vieweg und Sohn, Braunschweig, 1888-1894; and Frost's well-known "Curve Tracing," give much valuable information in regard to the possibility of representing certain important forms. These works, however, are neither adequate nor exhaustive. The work of Felix Auerbach on "Physik in graphischen Darstellungen," Teubner, Leipzig, 1912, has also a large number of forms of importance to statisticians.

II.—VARIOUS TYPES OF POPULATION FLUCTUATIONS.

1. **Mathematical conception of rate of increase.**—Whether diminishing or gaining, any actual population may be replaced by a "representative population," assumed to change at every moment by infinitesimal amounts at some *rate* (ρ say) per unit of time. That is to say, ρ will denote the fraction of a unit which, at the instant under consideration, measures the rate of change of the population for a unit of time. Hence, if P_t be the population at the time t, and $P_{t+\delta t}$ that at the time $t + \delta t$, then where δt is small we shall have

$$(1) \ldots\ldots\ldots P_{t+\delta t} = P_t (1 + \rho\delta t) = P_t\, e^{\rho\delta t}$$

as the fundamental expression for its fluctuation. In other words $P_t\,\rho\delta t$ is the absolute change in the time δt. If ρ be positive, the change is an increase ; if negative, it is a decrease.

The rate ρ may be either constant, in which case we shall denote it by r, or it may on the other hand vary in some determinate way with time, in which case we shall retain the Greek letter. If the rate be regarded as a function of time, then we should have

$$(1a) \ldots\ldots\ldots P_{t+\delta t} = P_t \{\rho\,(1 + \phi t)\,dt\}$$

We shall consider initially the case where it is constant.

2. **Determination of a population for any instant when the rate is constant.**[1] If increments of population be supposed to be added at N uniform intervals of time, extending over the period t, at the uniform rate r per unit of population per unit of time, then, putting P_0 for the initial population and P_t for that at the end of the time t, we shall have,

$$(2) \ldots\ldots P_t = P_0 \left(1 + \frac{r}{N}\right)^{Nt} ; = P_0\, e^{rt}$$

when N becomes finite.

As usual e denotes the base of Napierian logarithms, viz. :—2.7182818284590, etc.

It is sometimes convenient to put this expression in the form of a series : thus, by the exponential theorem, we have

$$(2a) \ldots\ldots P_t = P_0 \left(1 + rt + \frac{r^2t^2}{2!} + \frac{r^3t^3}{3!} + \text{etc.}\right)$$

Taking logarithms of both sides of (2), we notice that

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[1] When ρ is constant the investigation is analogous to that for determining the increase in a sum of money when interest is supposed to accrue at every instant of time. For a development of the theory of continuous interest and a kindred investigation of population, see a paper by J. M. Allen, Journ. Inst. Actuaries, Vol. XLI., p. 305.

$$(3)\ldots\ldots \log P_t = \log P_0 + (r \log e)\, t$$

hence, if r be constant, the *graph* obtained by passing a line through the points formed by plotting as ordinates the logarithms of the population for successive years, quinquenniums or decenniums, opposite the corresponding values of t as abscissæ, will be a *straight line*, the tangent of whose angle with the axis of abscissæ is $r \log e$. We shall call this graph the partial[1] *logarithmic homologue* of the graph of equation (2).

The value of $\log_{10} e$ is 0.4342944819032, etc., and of $\log_{10} (\log_{10} e)$ is 9.6377843113005, etc.[2] Both are required in practical calculation, to, however, only few places of decimals.

To find the *constant rate of increase*, we have

$$(4)\ldots\ldots r = (\log P_t - \log P_0) / (t \log e)$$

3. **Relation of instantaneous rate to the ratio of increase for various periods.**—We may call the constant r the *constant* rate of continuous increase, and similarly the variable ρ the instantaneous rate of continuous increase. It is often necessary, however, to substitute for r the equivalent rate for a year, or for five or ten years, that is to say, to measure the ratio at which the population at the beginning of the period must be increased in order to give it its proper value at the end thereof. Calling this r_t, we have

$$(5)\ldots\ldots r_t = (P_t - P_0) / P_0 = e^{rt} - 1\ ;\ \text{or } e^{rt} = 1 + r_t$$

4. **Determination of the mean population for any period: rate constant.**—Let P_0 denote the population at the beginning of any period and P_t the population after the time t: then, since $\int e^{rt}\,dt = e^{rt}/r$, the mean population P_m is obviously

$$(6)\ldots \frac{1}{t}\int_0^t P_t\,dt = \frac{P_0}{t}\int_0^t e^{rt}\,dt = \frac{P_0\,(e^{rt}-1)}{rt} = P_0\left(1 + \frac{rt}{2!} + \frac{r^2t^2}{3!} + \text{etc.}\right)$$

a formula which is suitable for determining the mean from the initial population. This expression may be put also in the form, see (5)

$$(7)\ldots\ldots (P_t - P_0) / rt\ ;\ \text{ or } P_0 r_t / rt$$

by means of which, when the rate is constant and known, the mean population can be calculated, either from the absolute increase for a given period, or from the ratio of the increase for a given period to the initial population for that period.

[1] Partial, because the values of t and not of the logarithms of t are not used as the abscissæ.

[2] 9 is used instead of $\bar{1}$.

5. **Error of the arithmetical mean : rate constant.**—The arithmetical mean of the population at the beginning and end of any finite period differs, of course, from the true mean. The magnitude of this difference is sometimes required. From (2a) and (6) we obtain—

$$(8)\ldots\ldots P_m = \tfrac{1}{2}(P_0 + P_t) - P_0\left(\frac{r^2t^2}{2.3!} + \frac{2r^3t^3}{2.4!} + \frac{3r^4t^4}{2.5!} + \text{etc.}\right)$$

which may also be written—

$$(8a)\ldots\ldots P_m = \tfrac{1}{2}(P_0 + P_t) - P_t\left(\frac{r^2t^2}{2.3!} - \frac{2r^3t^3}{2.4!} + \frac{3r^4t^4}{2.5!} - \text{etc.}\right)$$

When expressed in terms of the arithmetical mean itself, the odd powers of r and t disappear, thus

$$(8b)\ldots\ldots P_m = \tfrac{1}{2}(P_0 + P_t)\left(1 - \frac{r^2t^2}{2.3!} + \frac{2r^4t^4}{2.5!} - \frac{17r^6t^6}{2^2.7!} - \text{etc.}\right)$$

This last is the most convenient formula. The values of the coefficients are $\frac{1}{12}$, $\frac{1}{120}$, $\frac{17}{20160}$, etc.

Remembering that the maximum value of r is about 0.03, all these series converge with sufficient rapidity.

6. **Empirical expression for any population-fluctuation.**—If the population of a country be determined at $n + 1$ different dates, then a curve of the n^{th} degree can be arbitrarily drawn, passing through the graph of the coordinates. In the absence of any information as to the magnitude of the population between the given dates, the ordinate to the curve drawn from the terminal of the abscissa corresponding to the date may be assumed to be a probable value for the population at that date.

The curve in question may be written[1]—

$$(9)\ldots\ldots P_t = P_0\,(1 + at + bt^2 + ct^3 + \text{etc.})$$

which, for purposes of practical calculations or computational check, may be found convenient in the form :—

$$(9a)\ldots\, P_t = P_0\,\{1 + t\,[a + t\,(b + tc + \text{etc.})]\}$$

7. **Mean population for any period : rate not constant.**—Using the same notation as in II., 4, equation (6), we have—

$$(10)\ldots\, P_m = \frac{1}{t_2 - t_1}\int_{t_1}^{t_2} P_t\,dt =$$

$$P_0\left\{1 + \frac{a}{2}(t_2+t_1) + \frac{b}{3}(t_2^2+t_2t_1+t_1^2) + \frac{c}{4}(t_2^3+t_2^2t_1+t_2t_1^2+t_1^3) + \text{etc.}\right\}$$

Since, in the majority of calculations, t_1 is 0, in which case t_2 becomes simply t, we may write the result thus :—

$$(10a)\ldots\, P_m = P_0\,(1 + \tfrac{1}{2}at + \tfrac{1}{3}bt^2 + \tfrac{1}{4}ct^3 + \text{etc.})\ ;\ \text{or}$$

$$P_0\left\{1 + t\left[\frac{a}{2} + t\left(\frac{b}{3} + t\frac{c}{4} + \text{etc.}\right)\right]\right\}$$

the latter form being sometimes the more convenient for practical calculation.

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[1] See equation (45a) hereinafter. The fitting-efficiency of equation (9) is not equal to that indicated later, but it is more convenient to use.

8. **Change, with change of epoch, of the coefficients expressing rate.**—If the coefficients a, b, etc., have been determined for P_0 at a particular date, and it be desired to make the population, P_t the origin P'_0 for new computations, so that—

$$(11)\quad P'_\tau = P'_0\,(1 + \alpha\tau + \beta\tau^2 + \gamma\tau^3 + \text{etc.})$$

in which τ denotes the interval of time after the new epoch; that is to say, $P'_\tau = P_{t+\tau}$ and $P'_0 = P_t$

On putting $P_t \,/\, P_0 = 1 + r'$, we shall then have

$$(12)\ldots\ldots \begin{cases} \alpha = (a + 2bt + 3\,ct^2 + 4\,dt^3 + \text{etc.})/(1 + r') \\ \beta = (b + 3\,ct + 6\,dt^2 + \text{etc.})/(1 + r') \\ \gamma = (c + 4dt + \text{etc.})/(1 + r') \end{cases}$$

which is perhaps the best form for computation. If the quantity enclosed in brackets in equation (9) be denoted by $y = \phi t$, the several quantities in the brackets in (12) are dy/dt; $(d^2y/dt^2)/2\,!$; $(d^3y/dt^3)/3\,!$; etc., and the coefficients can be written out by a reference to Pascal's triangle. They are, of course, simple "figurate numbers" of the second, third, fourth, etc., orders.

That the coefficients must be altered when a new origin for t is selected, exposes one of the inherent limitations of the empirical equation.

9. **Error of the arithmetical mean: rate not constant.**—The arithmetical mean will always be in excess with either a uniform or a growing rate of increase. From (9) and (10a) we obtain—

$$(13)\ldots\ldots P_m = \tfrac{1}{2}\,(P_0 + P_t) - P_0\left(\frac{bt^2}{2.3} + \frac{2ct^3}{2.4} + \frac{3dt^4}{2.5} + \text{etc.}\right)$$

which may also be readily expressed in terms of the mean itself, as in (8b), thus—

$$(13a)\ldots P_m = \tfrac{1}{2}(P_0 + P_t)\left(1 - \frac{b}{3!}t^2 + \frac{2ab-6c}{4\,!}t^3 - \frac{5a^2b-10b^2-15ac-36d}{5\,!}t^4 - \text{etc.}\right)$$

This, however, is more tedious to use than (13).

10. **Expression of the coefficients in the empirical formula for rate in terms of the constant rate.**—If in equation (9), viz.:—

$$P_t = P_0\,(1 + at + bt^2 + ct^3 + \text{etc.})$$

$$a = r;\quad b = r^2/2\,!;\quad c = r^3/3\,!;\ \text{etc.}$$

the equation would express a constant rate, that is to say, it would be simply another form of equation (2a): and if a, b, c, etc., have not these values, the rate of increase is variable.

By substituting the corresponding values of r in (13a), it may easily be seen to be identical with (8b); and similarly as regards (13) and (8).

11. **Investigation of rate is complete only when its variations are ascertained.**—Reverting to II., 1, equation (1) may be written—

$$(14)\ldots\ldots\delta P = P\rho\delta t = P\phi(t)\delta t$$

which may be regarded as the fundamental differential form for increase of population, the final form being required, since the rate ρ is rarely if ever, constant, even for short periods of time. Hence in its theoretical form, an investigation of the fluctuations of population cannot be complete till all variations of its rate of growth are definitively ascertained, in other words, $\phi(t)$ must be ascertained.

12. **Rate is a function of elements that vary with time.**—The rate at which population increases is dependent upon elements external to and beyond the control of man, as well as upon elements within him, more or less under control. Both change with the lapse of time. In Fig. 3, § 32, hereinafter, examples are given shewing the curve of population of different countries, and in Fig. 4, of the same section, the corresponding logarithmic homologues of the populations. As already pointed out, the latter would be straight lines, if the rates of increase were constant. Hence, in the sense that it is dependent upon elements that vary with time, and may thus be directly related to the latter, the rate $\rho = \phi(t)$ may be investigated as a function of the elapsed time.

13. **Factors which secularly influence the rate of increase.**—Where not otherwise expressed, the rate of increase will be assumed to refer to total population. Let us consider primarily a community which grows by natural increase alone. This increase will be profoundly affected by four types of things, viz.:—

(i.) The material natural resources of the occupied territory.
(ii.) The various cosmic energies which facilitate man's development.
(iii.) Knowledge which increases the power of utilising natural resources.
(iv.) Sociological and other analogous standards, which react upon human activities, particularly upon man's productiveness, and the magnitude and character of his consumption of what he has produced.

Regarding (i.), it may be said that the *natural resources* of the territory occupied may be either *actual* or *potential*. Even without human intervention, a territory may be prodigal of those forms of animal and vegetable life, for example, which provide immediately for human wants. Its climate and meteorology may be propitious. It may possess large stores of readily available wealth, or of energy convertible into wealth.

Or yet, again, though in the state of Nature infertile, it may respond to well-directed efforts to make it so. It may have large hidden resources which can be recognised, and can become available only through a considerable development of scientific and technical knowledge, and through practical ability in applying the same. Lastly, it may contain types of wealth, as for example mineral wealth generally, which, though valueless *per se* to sustain life, may be made contributory to the growth of population through the part they play in the world-economy.

All these may be summed up under two headings, viz.:—

(i.) Natural fertility and resources of the territory independent of human action.

(ii.) Wealth or resources dependent on human action.

Both, however, are potentialities rather than actualities in regard to population: how they eventuate in respect thereto depends upon other and very subtle factors inhering in that order of things which concerns the general sociological and economic beliefs and in the traditions and activities of the people. For example, the general attitude of a people in respect to the question of *fecundity* and the prevailing view as to what should constitute a reasonable *standard of living*, profoundly affect the rapidity of the increase of the population, and the reaching of the time when natural limitations of fecundity operate severely.

There is still another factor of an analogous nature that plays a part, the significance of which is each year becoming more manifest, viz.:— The attitude of a people toward the development of the intellectual powers of man, and toward the application of such powers to the availment of the resources of Nature. Indeed, in general, the great advantages of the human being over the larger mammals is due to the efficiency in this direction of his intellectual endowment, and his power by systematising to store and apply acquired knowledge.

If we denote natural fertility or wealth of resources of the territory, say, by w; what may be called its geographical and climatic advantages by g; its other available resources when better scientific knowledge is applied, or even when new wants are created by advancing civilisation, by u; the factors expressing themselves in the matter of fecundity by f; through standard of living, including hygiene, by l; through intellectual knowledge and its range, energy, and wisdom of application by i; then we must regard the increased population as really a function of all these, that is to say—

$$(15)\ldots\ldots P = P_0\,\phi\,(w, g, u, f, l, i, \ldots. t)$$

The influences of these elements are, in general, secular in character, *i.e.*, they produce slow changes, some being manifest in the years of a decade, others only in many decades. They are all determining factors of the possibilities of population, but do not necessarily express its actuality.

Their specific character is such that ordinarily they produce gradual and more or less remote effects, rather than effects which are instantaneous and immediately of great magnitude. Such effect may tend towards a constant value, may increase, or diminish, but in all cases the consequent changes will be gradual. It is to be noted, however, that some of the factors may acquire for a short time an importance which, locally at any rate, may lead to rapid changes.

Factors of the kind considered are probably either non-periodic, or if periodic their period is secular.

A general solution, if it were possible, would presuppose that the way in which w, g, u, f, l and i, varied with elapsed time was determinable. This variation, however, is not susceptible of exact definition: nevertheless, the *form* of the functions expressing their effect on the rate of increase ρ is not always wholly indeterminable.

14. **Variations which depend on natural resources, irrespective of human intervention.**—This may include both periodic and non-periodic elements. The natural wealth of a territory, as unaffected by the intervention of man, is, in general though not invariably, a maximum initially,[1] though its values may oscillate between very wide limits, owing to variations of meteorological or climatological factors. Where natural wealth is of a type that is subject to steady decline, its effect on the rate of increase may be represented for all practical cases probably by a very simple function of the elasped time.

15. **Variations of rate of long periods.**—Any periodicity in meteorological and other factors, affecting the natural wealth of a territory, however much their influence may be masked by other factors, will in most cases cause a collateral periodicity in rate of increase. This can be represented by such a formula as the following, viz. :—

$$(16)\ldots\rho_t/\rho_0 = 1 + [a_0 + a_1 \sin(\alpha_1 + \frac{t}{T_1}) + a_2 \sin(\alpha_2 + \frac{t}{T_2}) + \text{etc.}] + Q$$

in which T_1, T_2, etc., will represent the lengths of the various periods to which the elapsed time t is related; α_1, α_2, etc., are intervals determining the epochs of T_1, T_2, etc.; and finally a_1, a_2, etc., are the amplitudes of the variation from the mean value. Thus necessarily—

$$(17)\ldots\ldots a_0 = -(a_1 \sin \alpha_1 + a_2 \sin \alpha_2 + \text{etc.})$$

and Q will of course represent the effect of the other elements influencing the rate of increase to which reference will be made later. Equation (16) is specially suitable for representing fluctuations of long period, which are expressible in terms of a sine series.

[1] Examples could be drawn in recent times from America or Australia. It may, however, even in regions which nevertheless can be made habitable, be actually zero, as for example, in the Sahara, in Arizona, and in some parts of Australia.

16. **Representation of periodic elements in non-periodic form.**—Where T is *exceedingly long* as compared with t, the numerator of the expression (16) may take a much more simple form, available probably for all practical cases. For putting—

$$(18)\begin{cases} A_1 = \Sigma\,[(a_i \cos\,\alpha_i\,)/T_i\,]\,; & A_2 = -\frac{1}{2\,!}\,\Sigma\,[(a_i \sin\alpha_i\,)\,/\,T_i^2\,]\,; \\ A_3 = -\frac{1}{3\,!}\,\Sigma\,[(a_i \cos\alpha_i\,)/T_i^3]; & A_4 = +\frac{1}{4\,!}\,\Sigma[(\,(a_i \sin\alpha_i\,)/T_i^4]; \\ \text{etc., etc.,} & ;\ \text{etc., etc.} \end{cases}$$

the limits of the summation being from $i = 1$ to $i = n$, and n being the number of periodic terms. Then remembering that

$$a_0 + \Sigma\,(a_i\ \sin\alpha_i\,) = 0$$

with the same limits, we can express (16) in the form

$$(19)\ldots\rho_t/\rho = 1 + a_0 + a_1 \sin\,(\alpha_1 + \frac{t}{T_1}\,) + \text{etc.} = 1 + A_1 t\ + A_2 t^2 + \ldots + \text{etc.}$$

which, with (18), connects the coefficients with the amplitude and epoch of the periodic fluctuations.

The values of A_1, A_2, etc., may be either positive, negative, or zero.

17. **Influence of natural resources disclosed by advancing knowledge.**—Turning now to the question of the various terms in Q, viz., those representing in equation (15) the effect of u, f, l, and i on the rate of increase, we remark first of all that increased scientific knowledge, especially in physics and chemistry, suggests that possibly the available resources of Nature are practically without limit, (that is $u = \infty$). This being so, the rate of increase may be regarded as dependent, not so much upon Nature's limitations as upon the extent and character of our knowledge, and of our energy and wisdom in applying it; that is, in the formula, it depends upon i, not upon u. We shall find, however, that Nature's limitations are very real, for rates of increase of population which characterise many countries at the present time cannot be maintained for several thousand years.

18. **Influences of resources dependent upon human intervention.**—There is a narrower sense, however, in which u may represent specific and finite quantities, which can be sufficiently indicated by two or three illustrations. Territories like portions of the Sahara in Africa, and of Arizona in America, apparently hopeless waste, may in response to the application of artesian water, become fertile and habitable. In ordinary agriculture, land, practically valueless in the state of Nature, may become valuable by the application of suitable fertilisers. The infertility of land which is due to the absence of the necessary micro-organisms, may, when once such organisms are introduced, quite disappear, and the potential wealth in the territory existing may have been quite undreamt of. Or yet again, the value to man of a natural product, utilisable in

the natural state, or after being treated technically, may be wholly unknown ; the discovery of its real value may so change the economic conditions of a territory as to greatly facilitate increase of population. In these and many other similar ways, natural resources reacting to man's operation may be found to be very great, though at first apparently non-existent. It would obviously therefore be very difficult to assign a form to the function which is in any way to represent the effect of natural resources.

19. **Effects of migration.**—Migration operates in several ways on the rate of increase of population, viz. : (i.) By the actual addition or withdrawal of the migrants ; (ii.) by the change of the constitution of the population, thus affecting its rate of fecundity ; (iii.) by consequential economic changes which favour or impair the rate of increase. A complete expression for its effects would therefore be elaborate in form. Since, however, the community changed by migration tends to adjust itself to the economic condition of the country, the real elaboration into each component element is unnecessary, and the resultant of all the elements operating may take a relatively simple form.

Migration itself is of two forms—periodic and non-periodic. The population of countries, for example, which at certain seasons are visited by large numbers of tourists, or from which large numbers depart, may be taken as affording illustrations of periodic migration. The rate of influx or efflux is usually slow initially ; it then increases, becoming a maximum ; when it declines much in the same way. In form, the curve of absolute increase or decrease is somewhat similar to the probability curve, but the curve is probably rarely symmetrical with respect to the maximum ordinate.

Non-periodic migration may, in addition to the effect of its absolute amount, change the final rate of increase or leave it as it was originally. Although both periodic and non-periodic migration may be actually discontinuous, no material error will ordinarily be committed if it be assumed to be continuous, provided that in amount it be negligibly small for the part of the year when it has actually ceased. So that there is no serious objection to the use of an essentially continuous function.

20. **Simple variation of rate, returning asymptotically to original value.**—Non-periodic migration of population, frequent in new countries, may produce a simple variation of rate which ultimately disappears. Owing to the reputation the territory acquires in respect of some real or supposed advantage, immigration sets in, increasing in rate till a maximum is reached, and declining again till the original rate is restored. For the territory or territories from which the emigration takes place, the converse effect may be true. If the rate can be ascertained at several periods, the total effect on the population can then be deduced with fair accuracy.

The simplest variation of this type, and one which will probably represent most instances with sufficient precision, may be expressed in the form—

$$(20)\ldots\ldots\rho_t/\rho_0 = 1 + \eta t^{m-nt}$$

η being positive for cases of immigration, and negative for those of emigration. This form would be suitable for deductions as to population based on the determination of rate of increase at various times.

By suitably selecting the unit of t, the parameter η and the index-numbers m and n, equation (22) may be made to represent the very different circumstances which may obtain at the commencement, and during the development and passing away of the effect of migration on the original rate of increase. For example, it will express that type of migration in which the increments per unit of time to the rate of increase, though initially slow, grow and decrease with continually changing velocity, till the original rate is restored ; or, on the other hand, it will express that type where the migration effect on the rate is sudden. This is illustrated by the curves in Fig. 1., viz. :—

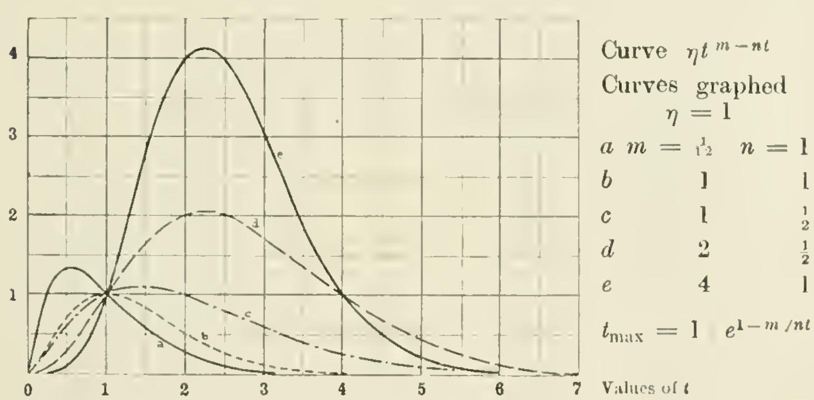


Fig. 1.

Curves $y = \eta t^{m+nt}$

in which the parameter η is unity throughout. The possible varieties of change of rate of increase are obvious from the figure, when it is remembered also that the horizontal proportions can be maintained, and the vertical changed at pleasure by simply altering the value of η.

21. **Examination of exponential curves expressing variation of rate.**—The curve of equation (20) demands special consideration. For brevity put R for $(\rho_t - \rho_0) / \rho_0$, then we can re-express (20) in the form[1]

[1] An expression of still greater fitting power is $y = At^m e^{nt^r}$. See a paper on the curve by G. H. Knibbs. Journ. Roy. Soc. N.S. Wales, Vol. XLIV., pp. 341-367.

(20a) .... $R = \eta t^{m - nt}$

therefore

(21)....$dR / dt = \dfrac{\eta t^{m-nt}}{t} \{m - nt\,(1 + \log_e t)\}$

(21a)...... $\dfrac{dy}{dt} = \eta t^{m+nt}(n \log_e t + \dfrac{m}{t} + n)$

(21b)...... $\dfrac{d^2y}{dt^2} = \eta t^{m+nt} \{(n \log_e t + \dfrac{m}{t} + n)^2 + \dfrac{nt - m}{t^2}\}$

and hence the value of t, which gives the maximum value for R, is found by solving the equation—

(22)...... $m/n = t_{\max}\ (1 + \log_e t_{\max})$

For the maximum to correspond to a value of t less than unity and greater than $^1/e$, the equation will be of the form t^{m-nt} (m and n being positive) ; or less than $^1/e$ the equation will be of the form $t^{-(m+nt)}$ This equation can be solved by inspection, by means of the following table :—

TABLE I.

Argument t. Values of $t \log_e t$, $t\,(1 + \log_e t)$, and $t\,(-1 + \log_e t)$

t	$t \log_e t$	$t\,(1+\log_e t)$	$t\,(-1+\log_e t)$	t	$t \log_e t$	$t\,(1+\log_e t)$
0.1	0.2303*n*	0.1303*n*	—	1	0.0000	1.0000
0.2	0.3219*n*	0.1219*n*	—	2	1.3863	3.3863
0.3	0.3612*n*	0.0612*n*	—	3	3.2958	6.2958
0.4	0.3665*n*	0.0335	—	4	5.5452	9.5452
0.5	0.3466*n*	0.1534	—	5	8.0472	13.0472
0.6	0.3065*n*	0.2935	—	6	10.7506	16.7506
0.7	0.2497*n*	0.4503	—	7	13.6214	20.6214
0.8	0.1785*n*	0.6215	—	8	16.6355	24.6355
0.9	0.0948*n*	0.8052	—	9	19.7750	28.7750
1.0	0.0000	1.0000	9.0000	10	23.0259	33.0259
1.1	0.1048	1.2048	9.0048	11	26.3768	37.3768
1.2	0.2188	1.4188	9.0188	12	29.8189	41.8189
1.3	0.3411	1.6411	9.0411	13	33.3443	46.3443
1.4	0.4711	1.8711	9.0711	14	36.9444	50.9444
1.5	0.6082	2.1082	9.1082	15	40.6208	55.6208
1.6	0.7520	2.3520	9.1520	16	44.3614	60.3614
1.7	0.9021	2.6021	9.2021	17	48.1646	65.1646
1.8	1.0580	2.8580	9.2580	18	52.0267	70.0267
1.9	1.2195	3.1195	9.3195	19	55.9443	74.9443
2.0	1.3863	3.3863	9.3863	20	59.9146	79.9146
2.1	1.5581	3.6581	9.4581	21	63.9350	84.9350
2.2	1.7346	3.9346	9.5346	—	—	—
2.3	1.9157	4.2157	9.6157	—	—	—
2.4	2.1011	4.5011	9.7011	—	—	—
2.5	2.2907	4.7907	9.7907	—	—	—
2.6	2.4843	5.0843	9.8843	—	—	—
2.7	2.6818	5.3818	9.9818	—	—	—
2.8	2.8829	5.6829	10.0829	—	—	—
2.9	3.0877	5.9877	10.1877	—	—	—
3.0	3.2958	6.2958	10.2958	—	—	—

NOTE.—The *n* denotes that the quantity is negative. In the column for $t \log_e t$ and $t\,(-1 + \log_e t)$, the whole number 9 has been used in preference to the more awkward form $\bar{1}$: in these cases the values given therefore exceed the true values by 10.

The suitability of the assumption of a curve of the type in question may be fairly well ascertained in the following way. For $t = 1$, $R = \eta$ in (20a) (the unit of t may be 1 week, 1 month, 1 quarter, or 1 year, say, according to the character of the migration under review).

Taking the logarithm of both sides of (20a) we have—

$$(23)\ldots\ldots \log R = \log \eta + (m - nt) \log t$$

From the observed values of R, the values of $\mathfrak{R}$

$$(24)\ldots\ldots \mathfrak{R} = (\log R - \log \eta) / \log t = m - nt$$

may be formed. These are plotted as ordinates, with the corresponding values of t as abscissæ; then if the points $\mathfrak{R}$ lie on a straight line, m will be the intercept on the axis of ordinates, and n will be the tangent of the negative angle which the line of points makes with the axis of abscissæ. If they do not lie on a straight line, the assumption is invalid.

If, moreover, we have the epoch at which the rate was a maximum, we have also from (22) the ratio of m/n, and obviously the two should be in agreement. This is a further test of the validity of the assumption.

22. **Determination of constants of such exponential curves.**—The constants η, m and n in equation (20) may be found from three observations at any suitable intervals, say at the times t_1, t_2, and t_3, the commencements of the fluctuation being therefore also known. If the value of R for $t = 1$ is not known, put—

$$(25)\ \left\{\begin{array}{l} S_{2.1} = \log R_2 - \log R_1\ ;\ S_{3.2} = \log R_3 - \log R_2 \\ u_{2.1} = \log t_2 - \log t_1\ ;\ v_{2.1} = t_2 \log t_2 - t_1 \log t_1 \\ u_{3.2} = \log t_3 - \log t_2\ ;\ v_{3.2} = t_3 \log t_3 - t_2 \log t_2 \end{array}\right.$$

then we shall have—

$$(26)\ldots\ldots m = (S_{3.2}v_{2.1} - S_{2.1}v_{3.2}) / (u_{3.2}v_{2.1} - u_{2.1}v_{3.2})$$

and

$$(27)\ldots\ldots n = (S_{3.2}u_{2.1} - S_{2.1}u_{3.2})/(u_{3.2}v_{2.1} - u_{2.1}v_{3.2})$$

The values of m and n being found, η is best found from.

$$(28)\ldots\ldots \log \eta = \log R - (m - nt) \log t.$$

the suffixes of R and t being identical.

If, however, the rate for $t = 1$ be known, then η is $R_{t=1}$; and, see (24), the suffixes of $\mathfrak{R}$, R and t being identical, we shall have—

$$(29)\ldots\ldots m = (\mathfrak{R}_2 t_3 - \mathfrak{R}_3 t_2)/(t_3 - t_2)$$

and

$$(30)\ldots\ldots n = (\mathfrak{R}_2 - \mathfrak{R}_3)/(t_3 - t_2)$$

a solution much less tedious than the former. The values of t_2 and t_3 must be well selected.

Obviously, if more than three values of R are taken, the application of the test indicated by equation (24) is necessary.

23. **Case of total non-periodic migration represented by an exponential curve.**—Where migration adds or subtracts its quota to the population only temporarily, as in cases of temporary migrations to or from a country, the exponential curve of equation (20) will often represent with exactitude, not merely the variations of the rate of increase, but of the absolute population. The complete expression put into non-periodic form would thus be, for the case in question—

$$(31)\ldots\ldots P_t = P_0 \{ e^{pt} + \eta\,(qt)^{m\,-\,nt} \}$$

the factor q depending upon the value of the unit of t used in (20) (*i.e.*, if η, m and t be determined for months, then $q = 12$, t in e^{pt} being in years). This formula would represent a single migration effect, vanishing asymptotically. If the migration be itself proportioned to the magnitude of the population at each instant, as may often be the case, then the preceding equation (31) will become—

$$(31a)\ldots\ldots P_t = P_0\, e^{pt}\ \{1 + \eta\,(qt)^{\,m\,-\,nt}\}$$

In using either (31) or (31a), it is of course necessary that P_0 be the population at the commencement of the migration effect; *i.e.*, the origin of t must be identical in both parts of the complete expression.

Yet again, if the expression represents only the variation of the rate, we shall have—

$$(31b)\ldots\ldots P_t = P_0\, e^{pt\,(1\,+\,\eta t^{m\,-\,nt})}$$

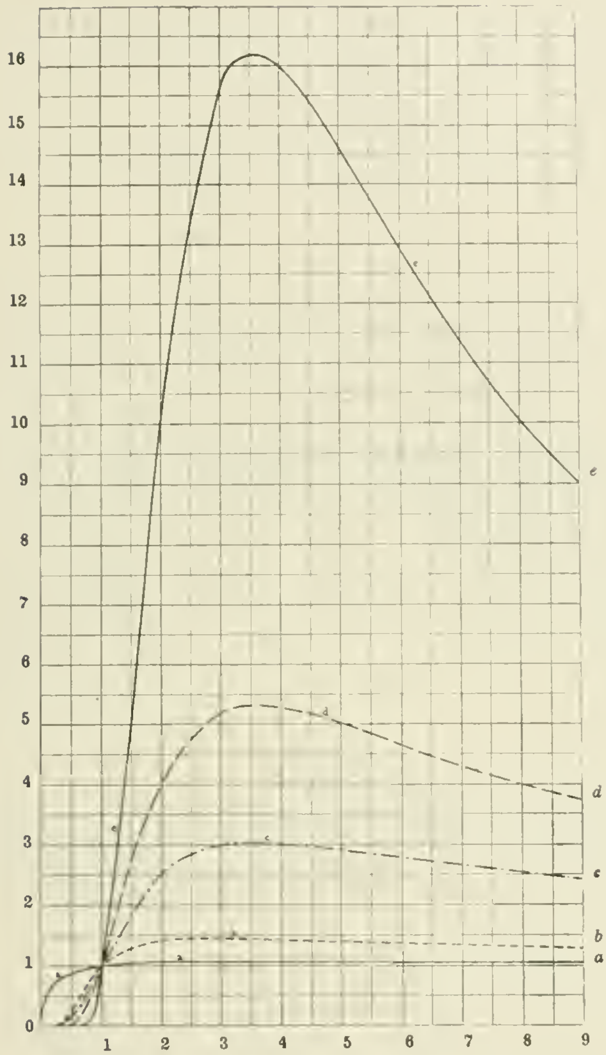
which, however, will be considered in a more general form hereinafter.

24. **Simple variation of rate, returning asymptotically to a particular value.**—A variation of rate may tend to return to some new value, greater say than the original. Such a variation can be expressed in the following way, viz.:—

$$(32)\ldots\ldots \rho_t/\rho_0 = 1 + \eta t^{\,\pm\,\frac{1}{m\,+\,nt}}$$

m and n being positive.

Since this curve becomes asymptotic to a line parallel to the t axis at the distance η, and has the ordinate value η for $t = 1$, the unit by which t is measured must give an abscissa of unity for the first value η of the ordinate. This somewhat limits the convenience of its application. Some of the forms of the curve are illustrated by Fig. 2.



Curves $\eta t^{\frac{1}{m + nt}}$

Curves graphed
$\eta = 1$

a	$m = 4$;	$n = 4$
b	0	1
c	$\frac{1}{4}$	$\frac{1}{4}$
d	$\frac{1}{6}$	$\frac{1}{6}$
e	$\frac{1}{10}$	$\frac{1}{10}$

$t_{max} = e^{1 + \frac{m}{nt}}$

Fig. 2.

25. **Examination of the preceding curve.**—As in section (21), put

$$(32a)\ldots\ldots R' = \eta t^{\frac{1}{\pm m + nt}}$$

then

$$(33)\ldots\ldots d\,R' / dt = \eta t^{\frac{1}{\pm m + nt}} \cdot \frac{1}{m + nt}\left(\frac{1}{t} - \frac{n \log_e t}{\pm m + nt}\right)$$

and consequently the value of t which gives the maximum value for R' is found by solving the equation $\frac{1}{t} = \frac{n \log_e t}{\pm m + nt}$ which leads to—

$$(34)\ldots\ldots \frac{\pm m}{n} = t\,(\log_e t - 1)$$

For a maximum to correspond to a value of t greater than e, the equation will be of the form $t^{\frac{1}{m+nt}}$ (m and n being positive); or less than e, equation will be of the form $t^{-\frac{1}{m+nt}}$
This may be solved for the series of values already given in Table I. for $t\,(\log_e t - 1)$: see section 21 hereinbefore.

Similarly to the preceding case we take the logarithm of both sides of (32a), we have—

$$(35)\ldots\ldots \log R' = \log \eta + \log t \,/\, (\pm\, m + nt)$$

Hence as before, finding $\mathfrak{R}'$ from observed values of R' we have—

$$(36)\ldots\ldots \mathfrak{R} = \log t \,/\, (\log R' - \log \eta) = \pm\, m + nt$$

which enables us to examine the validity of the assumption, since it is the equation of a straight line of which the values of $\mathfrak{R}'$ and t are respectively ordinates and abscissæ.

For the point of inflexion the second differential will be required: the sign of m being positive, it is—

$$(33a)\ldots \frac{d^2R^2}{dt^2} = \frac{\eta t^{\frac{1}{m+nt}}}{(m+nt)^2}\left\{\left[\frac{1}{t} - \frac{n}{m+nt}\log_e t\right]^2 - \left[\frac{1}{t} - \frac{n}{m+nt}\log_e t\right] - \left[\frac{m}{t^2} + \frac{2n}{t} - \frac{n^2}{m+nt}\log_e t\right]\right\}$$

26. **Determination of the constants of the curve.**—In this case the rate for $t = 1$ is known, and $\eta = R'_t = 1$; thus formula (29) holds when $\mathfrak{R}$ is changed for $\mathfrak{R}'$, and similarly in regard to (30) changing the sign, that is—

$$(37)\ldots\ldots m = (\mathfrak{R}'_2 t_3 - \mathfrak{R}'_3 t_2) \,/\, (t_3 - t_2)$$

$$(38)\ldots\ldots n = (\mathfrak{R}'_3 - \mathfrak{R}'_2) \,/\, (t_3 - t_2)$$

The test of (36) is necessary if there be more than three values of $\mathfrak{R}'$.

For the case of immigration η is positive, for emigration negative.

27. **Total non-periodic migration resulting in permanent increase but returning to original rate.**—Where the migration effect on total population adds or subtracts its quota, but leaves the original rate practically undisturbed, the result may be expressed similarly to (31), *i.e.*,

$$(39)\ldots\ldots P_t = P_0 \left\{e^{pt} + \eta\,(qt)^{\pm\frac{1}{m+nt}}\right\}$$

and if as supposed in section 23 the migration be itself influenced at every moment by the magnitude of the population, (39) will become—

$$(39a)\ldots\ldots P_t = P_0\, e^{pt} \left\{1 + \eta (qt)^{\pm\frac{1}{m+nt}}\right\}$$

28. **The utility of the exponential curve of migration.**—Formulæ (20) to (31b) are serviceable, when the population has to be determined by taking into account the rate of migration determined only at several suitable occasions, the intermediate migration being supposed to conform to the exponential curve assumed to represent all values intermediate to those determining it, and all future values so long as it is applied.

29. **Fluctuation of annual periodicity.**—The instantaneous rate of increase of the population of any country, at least where the population is at all numerous, must, during the course of the year, indicate a yearly period, since both the migration rate and the birth and death rates have, in general, a characteristic annual fluctuation. There is sometimes a difference, however, between the migration fluctuation, and that due to births and deaths, for the former, owing to local circumstances, is sometimes confined to a part of the year only, while the two latter extend over the entire year. The scheme of expressing long periodic fluctuations has already been indicated, viz., in equations (16) to (19). Continuous fluctuations of short periods may with advantage be put in the form—

$$(40)\ldots\rho_t/\rho_0 = 1 + a_0 + a_1 \sin (\alpha_1 + \mu_1 t) + a_2 (\sin \alpha_2 + \mu_2 t) + \text{etc.}$$

where μ_1 and μ_2 are whole numbers or proper or improper fractions, defining definitely ascertained periods, and where, as before, we must necessarily have—

$$(41)\ldots\ldots a_0 = -\Sigma\, a \sin \alpha\ ;$$

see section (17); or yet again, if the true period is not known and a curve known by experience is to be empirically reproduced, then we may put

$$(42)\ldots\rho_t/\rho_0 = 1 + a_0 + a_1 \sin (\alpha_1 + \frac{2\pi}{n} t) + a_2 \sin 2\, (\alpha_2 + \frac{2\pi}{n} t) + a_3\, sin\, 3(\alpha_3 + \frac{2\pi}{n} t) + \text{etc.},$$

the unit of t being the period (*e.g.*, one year) embracing all the fluctuations to be reproduced in the period following.

30. **Discontinuous periodic variations of rate.**—We may assume that the continuous rate is any function of t, *i.e.*, $\rho_t = \phi(t)$ say. Suppose that superimposed on this curve, there is a migration effect existing for parts of the year only, reappearing at the corresponding times in each following year. Let us suppose further that in the intervals, there is no variation of rate through migration, the fluctuation being fully expressed by $\phi(t)$ above. Then, provided that suitable values are given to the constant a_0 to the amplitudes a_1, a_2, etc., and to the epochal angles α_1, α_2, etc., the fluctuation of rate may be represented by such an expression as—

$$(43)\ldots\ldots\rho_t/\rho_0 = \phi(t) \pm \sqrt{\{a_0 + a_1 \sin (\alpha_1 + \frac{2\pi}{n} t) + \text{etc.}\}}$$

the + sign denoting immigration effects, and the — sign emigration effects. For the final term will have no *real* values when the quantity under the radical sign becomes negative: a_0 must of course satisfy the conditions expressed by equation (17) hereinbefore.

Similarly, fluctuations of other character may be represented by—

$$(43a)\ldots\ \rho_t/\rho_0 = \phi(t) \pm \sqrt{\{a_0 + a_1 \sin (\alpha_1 + t/T_1) + \text{etc.}\}}$$

or again by—

$$(43b)\ldots\rho_t/\rho_0 = \phi(t) \pm \sqrt{(at + bt^2 + ct^3 + \text{etc.})}$$

Since only real values can have any meaning the expressions under the radical sign in (43), and (43a) and (43b) are discontinuous, the discontinuity extending from each value of t where the value of the expression changes from + to —, to where it changes from — to + again.

31. **Empirical expression for secular fluctuations of rate.**—For the purpose of prediction it is usual to deal either with mean population or the population at a particular date, say the end of the year. The fluctuations of rate may be empirically determined from past records and put in the exponential form, viz.,

$$(44)\ldots\ldots\rho_t/\rho_0 = 1 + \eta t^{k + mt + nt^2 + \text{etc.}}$$

η, k, m, n, etc., being integral or fractional, positive or negative. Or again, it may be expressed in the form—

$$(45)\ldots\ldots\rho_t/\rho_0 = 1 + \alpha t + \beta t^2 + \gamma t^3 + \text{etc.}$$

or yet again in the form—

$$(45a)\ldots\ldots\rho_t/\rho_0 = 1 + \alpha t^p + \beta t^q + \gamma t^r + \text{etc.}$$

in which p, q, r, etc., are in ascending order of magnitude, but not restricted to integral values. The fitting efficiency of this latter form is much greater than where the indices are restricted to integral values,[1] but the determination of the constants α, β, γ, etc., and p, q, r, etc., are not so convenient.

32. **Growth of various populations.**—Populations increase when the additions by birth and immigration together exceed the deductions through death and emigration together. The rate of increase differs greatly as between country and country, and differs from decade to decade, so that *it cannot be regarded as in any sense uniform* even for short periods of time. This is evident from Fig. 3, in which the growth of the populations of a larger number of countries is shewn by their progression every decade, and is still more obvious in Fig. 4 (shewing their logarithmic homologues) by the changes in the slope of the lines. In the following table, the populations, given in millions and decimals of a million, are those shewn on Fig. 3.

[1] Obviously, since both the coefficients and indices are at our disposal, it is easy to see that attempts to apply (45) to the curve $y = at^p$, where p is a proper or improper fraction, are invalid. It is also invalid for the curve $y = at^p + bt^{p+q} + \text{etc.}$

The Populations of Various Countries from 1790 to 1910.

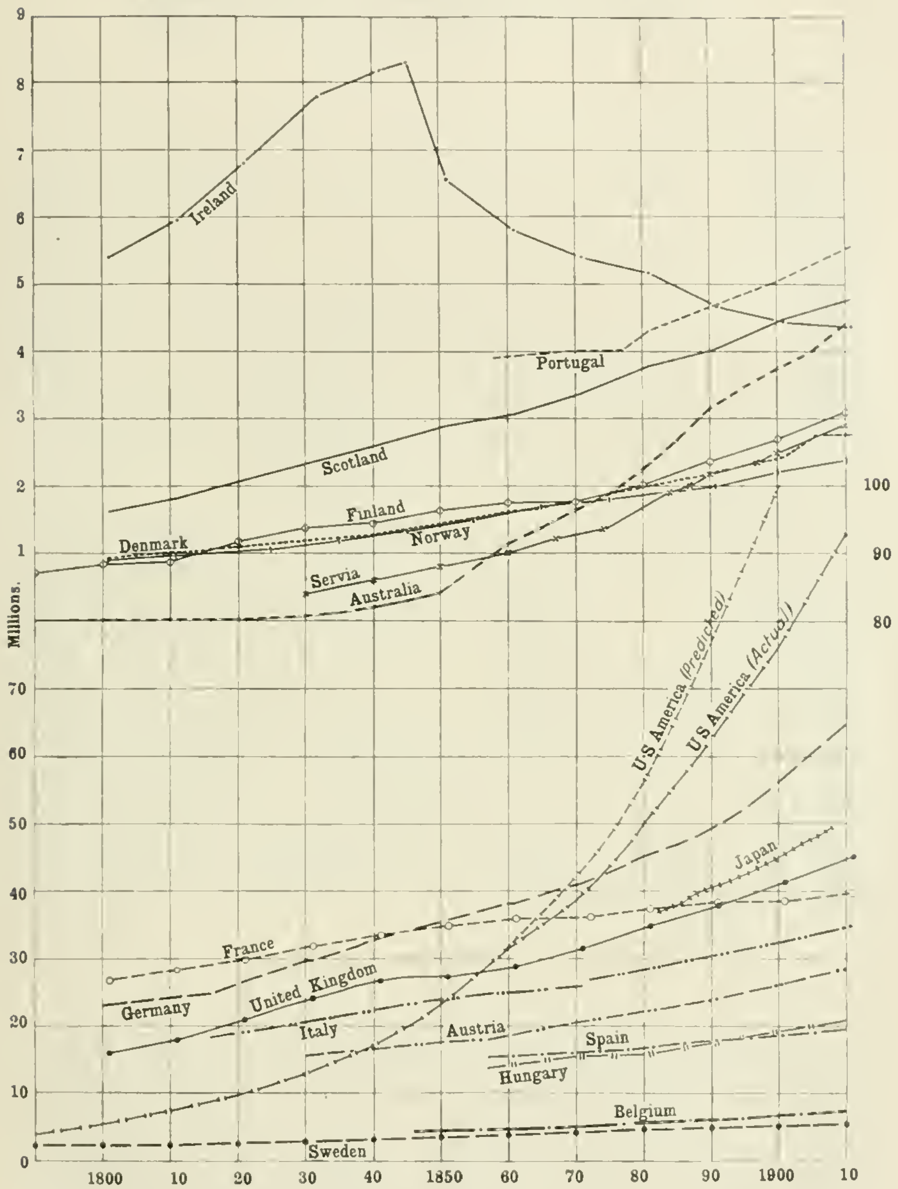


FIG. 3.

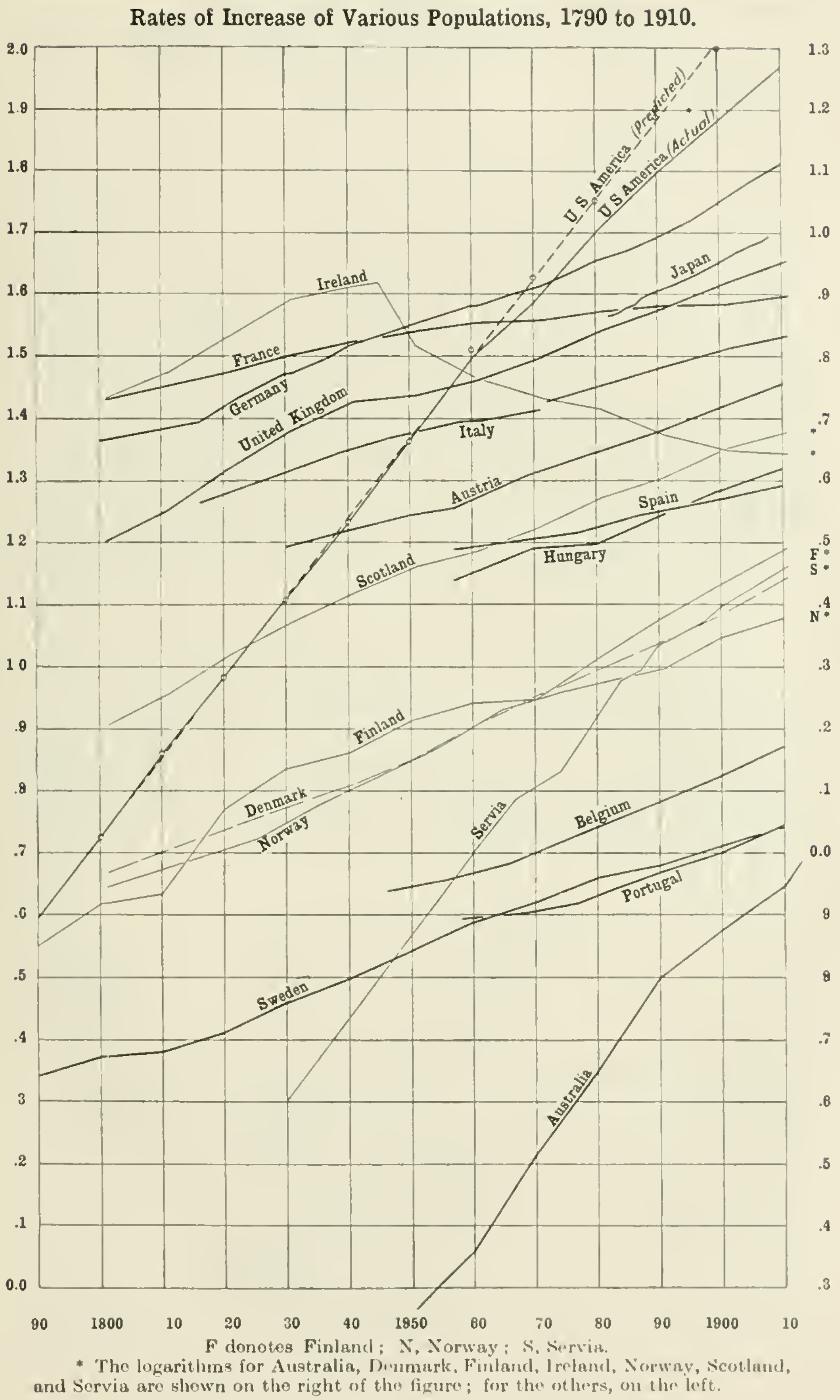
The scale for the lower part of the figure denotes ten times the numbers of the scale for the higher part. The predicted population for the United States was based on the assumption that the rate for 1790 to 1800 would be maintained constant. On the scale of the figure this curve substantially agrees with the prediction by Elkanah Watson in 1815.

Table II.—Populations in Millions, of Various Countries.

Country.	Years. 1790-9.		1800-9.		1810-9.		1820-9.		1830-9.		1840-9.		1850-9.	
Commonwealth	0	.002	0	.005	0	.01	0	.03	0	.07	0	.19	0	.41
United Kingdom	..	..	1	15.90	1	17.91	1	20.89	1	24.03	1	26.71	1	27.37
Scotland	..	..	1	1.61	1	1.81	1	2.09	1	2.36	1	2.62	1	2.89
Ireland	..	..	1	5.40	1	5.94	1	6.80	1	7.77	1	8.18	1	6.55
Austria	..	..	..	..	..	..	..	..	0	15.59	0	16.58	0	17.53
Belgium	..	..	..	..	..	..	..	..		..	6	4.34	6	4.53
Denmark	..	..	1	.93	..	..	..	..	4	1.22	0	1.28	0	1.41
France	..	..	1	26.93	..	..	1	29.87	1	31.89	1	33.40	1	34.71
Germany	..	..	0	23.18	6	24.83	2	27.04	1	29.77	0	32.79	2	35.96
Hungary	..	..	..	..	..	..	..	..		..	..	..	7	13.77
Italy	..	..	..	..	6	18.38	5	19.73	8	21.98	8	23.62	8	24.86
Norway	..	..	1	.88	..	..	5	1.05	5	1.19	5	1.33	5	1.49
Portugal	..	..	..	..	..	..	..	..		..	..	..	8	3.92
Spain	..	..	..	..	..	..	..	..		..	..	..	7	15.46
Sweden	0	2.19	0	2.35	0	2.40	0	2.58	0	2.88	0	3.14	0	3.48
Finland	0	.71	0	.83	0	.86	0	1.18	0	1.37	0	1.45	0	1.64
Servia	..	..	..	..	..	..	..	..	0	.40	..	..	..	..
U.S. America	0	3.93	0	5.31	0	7.24	0	9.64	0	12.87	0	17.07	0	23.19

Country.	Years. 1860-9.		1870-9.		1880-9.		1890-9.		1900-9.		1910-9.	
Commonwealth	0	1.15	0	1.65	0	2.23	0	3.65	0	3.77	0	4.43
United Kingdom	1	28.93	1	31.49	1	34.88	1	37.73	1	41.46	1	45.22
Scotland	1	3.06	1	3.36	1	3.74	1	4.03	1	4.47	1	4.76
Ireland	1	5.80	1	5.41	1	5.17	1	4.70	1	4.46	1	4.39
Austria	9	20.39	..	..	0	22.14	0	23.90	0	26.15	0	28.57
Belgium	6	4.83	..	..	0	5.52	0	6.07	0	6.69	0	7.42
Denmark	0	1.60	0	1.78	0	1.97	0	2.17	1	2.45	1	2.78
France	1	35.84	2	36.10	1	37.41	1	38.13	1	38.45	1	39.60
Germany	1	38.14	1	41.06	0	45.23	0	49.43	0	56.37	0	64.93
Hungary	7	1.22	0	15.51	0	15.74	0	17.46	0	19.25	0	20.89
Italy	2	25.00	1	25.96	1	28.46	1	30.46	1	32.48	0	34.67
Japan	..	..	..	..	2	36.70	0	40.45	0	44.83	0	50.50
Norway	5	1.70	5	1.82	..	..	1	1.99	0	2.22	0	2.39
Portugal	8	4.00	7	4.16	1	4.31	0	4.66	0	5.02	1	5.55
Spain	..	..	7	16.43	7	17.55	7	18.32	0	18.61	0	19.59
Sweden	0	3.86	0	4.17	0	4.57	0	4.78	0	5.14	0	5.52
Finland	0	1.75	0	1.77	0	2.06	0	2.38	0	2.71	0	3.12
Servia	0	1.00	4	1.35	4	1.90	0	2.16	0	2.49	0	2.91
U.S. America	0	31.41	0	38.56	0	50.16	0	62.62	0	76.21	0	93.35

33. **Rate of increase of various populations.**—Fig. 3 and the accompanying table reveal directly only the relative magnitude of the populations, but not their exact *rate* of growth. The latter is displayed on Fig. 4, in which (the scale being constant) the *steepness of slope of the line represents the rapidity of the rate of increase.* As before mentioned, this rate is very irregular from decade to decade, as would be revealed by dividing the population at the end of each decade by that at the beginning thereof and comparing the numbers ; *i.e.*, by finding and comparing, for example, the values of P_n/P_0 giving those of $1 + r$. The rates tabulated hereunder are the *annual rates* which, if maintained constant, would produce the populations at the end of the decades ; that is, they are the values of r found from $\log (1 + r) = (\log P_n - \log P_0)/n$, where n is the intervening number of years.



F denotes Finland; N, Norway; S, Servia.

* The logarithms for Australia, Denmark, Finland, Ireland, Norway, Scotland, and Servia are shown on the right of the figure; for the others, on the left.

FIG. 4.

Table III.—Annual Rate of Increase per 10,000 of Population of Various Countries.

Country.	Approximate Decade.											
	1790 to 1799	1800 to 1809	1810 to 1819	1820 to 1829	1830 to 1839	1840 to 1849	1850 to 1859	1860 to 1869	1870 to 1879	1880 to 1889	1890 to 1899	1900 to 1909
C'wealth ..	976	829	1124	764	1052	785	1095	370	308	351	180	163
U. K'dom*	—	120	155	141	106	24	56	85	103	79	95	87
Scotland*	—	118	145	122	105	99	57	94	108	75	104	63
Ireland* ..	—	96	136	134	52	-225	-122	—70	—45	—96	—53	—16
Austria ..	—	—	—	—	62	56	152	41‡	41‡	77	90	89
Belgium† ..	—	—	—	—	—	43	64	96**	96**	96	98	104
Denmark*	—	—	—	—	80§	97	127	107	102	97	122	127
France* ..	—	52‡	52‡	66	46	39	32	7	36	19	8	30
Germany ..	—	43(*a*)	143	114	108	77	66	74	108	89	132	142
Hungary ..	—	—	—	—	—	—	—	86**	17	104	64	51
Italy ..	—	—	—	79(*b*)	83(*c*)	72	41**	42	92	68	64	58
Norway* ..	—	74(*d*)	74(*d*)	126	112	114	133	69	56(*a*)	56(*a*)	122	74
Portugal ..	—	—	—	—	—	—	—	20	58(*c*)	87	75	92
Spain ..	—	—	—	—	—	—	—	30‡	30‡	66	33(*c*)	67
Sweden ..	71	21	73	111	87	103	104	77	92	45	73	71
Finland ..	157	36	321	150	57	124	65	12	153	145	131	142
Servia ..	—	—	—	—	—	—	—	190**	348	216	143	157
Japan* ..	—	—	—	—		—	—	—	—	116	110	118
U. States ..	306	315	291	293	286	311	309	206	267	224	198	205

* Add 1 year to date for proper decade. † Add 6 years up to 1860 inclusive. ‡ Rate for 20 years. ** Rate for 14 years. § Rate for 6 years. (*a*) Rate for 16 years. (*b*) Rate for 9 years. (*c*) Rate for 13 years. (*d*) Rate for 24 years.

34. **The population of the world and the rate of its increase.**—In dealing with the magnitude of the population of any country and the rate of its growth, the most general comparison is that made with the entire population of the world and its rate of growth. This, however, is not well ascertained. Recently, for example, the estimate for China's population has been reduced over 100 millions. The following table gives results of different estimates :—

Table IV.—Estimates of World's Population.*

Year.	Authority.	Estimate (Millions).	Year.	Authority.	Estimate (Millions).
1660	Riccioli	1,000	1813	Graberg v. Hemsö	686
1685	Isaak Vossius	500	1816	A. Balbi	704
1740	Nic. Struyck	500	1822	Reichard	732
1672	Riccioli	1,000	1824	G. Hassel	938
1742	J. P. Sussmilch	950 to 1,000	1828	G. Hassel	850
1753	Voltaire	1,600	1828	I. Bergius	893
1761	J. P. Sussmilch	1,080	1828	A. Balbi	737
1789	W. Black	800 to 1,000	1828	Balbi*	847
1804	Malte-Brun*	640	1833	Stein	872
1804	Volney	437	1838	Fränzl	950
1805	Pinkerton	700	1838	V. Rougemont	850
1805	Fabri	700	1840	Omalius d'Halloy	750
1809	G. Hassel	682	1840	Bernoulli	764
1810	Almanach de Gotha*	682	1840	v. Roon	864
1812	Morse	766	1843	Balbi	739

Table IV.—Estimates of World's Population*—*continued*.

Year.	Authority.	Estimate (Millions).	Year.	Authority.	Estimate (Millions).
1843	H. Berghaus	1,272	1880	Behm & Wagner	1,456
1845	Michelot*	1,009	1882	Behm & Wagner	1,434
1854	v. Reden	1,135	1883	Behm & Wagner*	1,433
1859	Dieterici	1,288	1886	Levasseur*	1,483
1866	E. Behm	1,350	1891	Ravenstein*	1,467
1868	Kolb	1,270	1896	Statesman's* Year	1,493
1868	E. Behm	1,375	1903	Juraschek* [Book	1,512
1870	E. Behm	1,359	1906	Juraschek*	1,538
1872	Behm & Wagner	1,377	1910	Annuaire Statistique d. l. Rep. Française*	
1873	Behm & Wagner	1,391		Juraschek*	1,610
1874	Behm & Wagner*	1,391			
1878	Levasseur*	1,439	1913	Knibbs*	1,632
1878	Levasseur	1,439	1914	Knibbs	1,649

* These will be found on the graph, Fig. 5.

This table shews, for the period 1804 to 1914, rates of annual increase ranging between 0.0015 and 0.0121[1] and averaging about 0.00864.

We may obtain some idea of the present rate of growth by taking the weighted mean of the rate for the known countries; that is, each rate of increase is weighted according to the population. In this way, it is found for the quinquennium 1906 to 1911, and for the group of countries in the Table V. hereinafter, that the general result is a rate of increase of 0.01159 per annum, or 1.159 per cent. of the population.

Table V.—Annual Increase per 10,000 Population for the quinquennium 1906-1911.

Country.	Rate	Years†	Country.	Rate.	Years†
Ireland	— 6	..	Switzerland	+ 121	57.6
France	+ 16	436	Netherlands	122	57.2
Jamaica	28	248	Denmark	126	55.4
Scotland	55	126	German Empire ..	136	51.3
Norway	66	105	Finland	143	48.8
Belgium	69	101	Rumania	148	47.2
Italy	80	87	Servia	155	45.1
Sweden	84	82.9	Chile	156	44.8
Hungary	84	82.9	United States	182	38.4
Austria	86	80.9	Commonwealth ..	203	34.5
Spain	87	80.0	New Zealand	256	27.4
England and Wales ..	104	67.0	Canada	298	23.6
Japan	108	64.5			
Ceylon	120	58.1	Weighted Average* ..	+ 115.9	60.1

* Weighted average according to population. † Years necessary for the population to be doubled in value at the rate indicated.

The number of years n in which a population, increasing at the rate r, is doubled, may be very readily computed thus :—

$$(1+r)^n = 2\,; \text{ therefore } n \log_e (1+r) = \log 2 = 0.693147$$

$$\text{consequently } n = \frac{0.69315}{\log_e (1+r)} = \frac{0.69315}{r\left(1 - \frac{r}{2} + \frac{r^2}{3} - \ldots\right)}$$

but when r is very small we may neglect powers higher than the second (that is $\frac{1}{2}r$ in the brackets); hence

$$n = \frac{0.69315}{r}\left(1 + \tfrac{1}{2}r\right), \text{ sensibly, } = \frac{0.693}{r} + 0.347\cdot$$

[1] On taking the mean of Levasseur and Behm & Wagner, and again of Levasseur and Ravenstein.

Either this rate of increase must be enormously greater than has existed in the past history of the world or enormous numbers of human beings must have been blotted out by catastrophes of various kinds from time to time. For, putting the present population at 1,649,000,000, at the average rate of increase this number would be produced from a single pair of human beings in about 1782 years,* that is to say, since A.D. 132, or since Salvius Julianus revised under Hadrian the Edicts of the Prætors. Even the rate given by the world-populations 1804 and 1914, viz. (0.0086) gives only 2397 years, carrying us back only to B.C. 483, or since the days of Darius I. of Persia.

The profound significance of this fact, accentuated also by the extraordinary increase in the length of life (expectation of life at age 0), which has revealed itself of recent years, is obvious when the correlative food requirements are taken into account. The resources of Nature will have to be exploited in the future more successfully than in the past to maintain this rate of increase (0.01159), which doubles the population every 60.15 years, and would give for 10,000 years the colossal number 22,184, with 46 noughts (10^{46}) after it.

This number is so colossal that it is difficult to appreciate its magnitude. Assuming the earth to be a globe of 3960 miles radius, of a density 5.527 compared with water, that water weighs about $62\frac{1}{2}$ lbs. per cubic foot, and that a human being weighs on the average, say, 100 lbs. (7 st. 2 lbs.), the actual mass of the earth would be equivalent only to, say, 132,265 x 10^{18} persons; that is, it would require 16,771 x 10^{23} times as much "matter" as there is in the earth. Or, to consider it as a question of surface, allowing $1\frac{1}{2}$ square feet per person, the earth's entire surface area would provide standing room for only 36,625 x 10^{11} persons. That is, the population would be 60,570 x 10^{30} times as great as there would be standing room if the whole earth's surface were available. It is evident from this that the rate of increase of human beings must have been more approximate to the rate for France at the present time, if the earth has been peopled for 10,000 years: the French rate, 0.0016, would require 12,842 years to give the present population from a single pair. This rate, however, would give a population of only 17.55 millions in 10,000 years.

The foregoing analysis of the effect of the rate of increase, with which we are familiar, *establishes* the fact that the rate must have passed through great changes, and could not have been maintained for any long period, either at its present average, or that characteristic of the last century. (See II. § 12, 13, 14 and 15.) It is not improbable that the rate of the last quinquennium will not be long maintained; and it is

* Thus dividing by 2, we have 824,500,000 = $(1.01159)^n$ where n is the number of years, that is, $n = 1{,}782$.

certain that however great human genius or effort may be, in enlarging the world's food supplies, that rate cannot possibly be maintained for many centuries. The contention of Malthus is thus placed beyond question, from a different point of view.

The analysis also *suggests* that there are probably great oscillations of the rate of increase, but since accurate records date back for so comparatively short a time, no general indication of their character can be given.

THE WORLD'S POPULATION, 1806-1914.

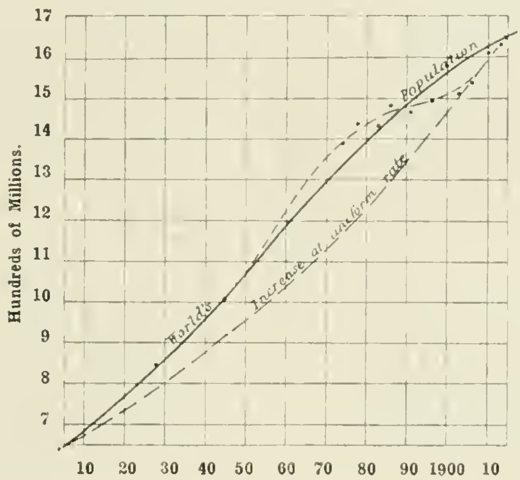


FIG. 5.

In Fig. 5 some of the estimates are shewn by black dots. The firm line drawn among these dots is intended to represent the probable development of the world's population. The thin broken line among the dots, though adhering more closely to the various estimates, is, however, of doubtful probability. The lower broken line represents a population increasing at a uniform rate from 640 millions in 1804 to 1649 millions in 1914; *i.e.*, 110 years. From the figure it is evident that the rate of increase in the early part of last century has fallen off, and the world's population increase will continue at a less rapid rate. Thus it is beyond question that there have been oscillations of rate, but their period has not yet been determined, and is perhaps not determinable, owing to lack of data. One thing is assured, viz., that the present rate of increase cannot be maintained for any lengthy period.

III.—DETERMINATION OF CURVE-CONSTANTS AND OF INTERMEDIATE VALUES WHEN THE DATA ARE INSTANTANEOUS VALUES.

1. **General.**—The data of statistics are usually to hand in two essentially different forms, viz., (a) *instantaneous values* or numbers which are true at a given moment ; as, for example, the population of a country at a given instant ; and (b) *group values* or numbers belonging to some particular interval of time, as the number of births per month, or per annum, for a population of given magnitude. Some indications have already been given of suitable formulæ for instantaneous values, and in one or two instances the mode of deducing their constants was also furnished. We proceed to consider the solution for the constants of equations which are appropriate for representing instantaneous values. In mathematical language, if $y = f(x)$, then having chosen the form of the function, it remains to determine its constants from the data. In the case of group values, the equations must denote the value of the integral of the function between given limits, and the problem has special features, the study of which will be undertaken later (IV.) There are a considerable number of cases of importance, some of which are *aperiodic*, and others *periodic*.

2. **Determination of constants where a fluctuation is represented by an integral function of one variable.**—When, as is ordinarily the case, the data consist of values corresponding to equal intervals of time, as, for example, the population at the end of each quarter, at the end of each year, or at the end of each ten years, etc., the fluctuation may be empirically represented by the equation.

$$(46)\ldots\ldots y \text{ (or } \frac{1}{y}) = a + bx + cx^2 + dx^3 + \text{etc.},$$

in which, in the above illustration, x represents *time*. In this case the number of constants to be determined will depend upon the number of instants for which we have data. Two classes of cases arise, viz., (i.) cases in which the data furnish the initial value ; (y_0), that is, a in the equation above, and (ii.) cases in which the initial value is not furnished, but is for a unit interval of time before the first result available. In other words, in the equation above, we require a series of solutions for the cases where a has a fixed value, including zero, and when it is undetermined ; or what is the same thing, when we have either y_0 or y_1 as the

initial datum. If we have the value of y_0, then n subsequent points will require an integral equation of the nth degree. If not, n points, including y_1, necessitate an integral equation of the $(n-1)$th degree.[1]

If k denote the common interval of time (represented by distance between the ordinates), the values of y in the preceding expression are :—

$$(47)\ldots y_0=a\,;\; y_1=a+bk+ck+\text{etc.}\,;\; y_2=a+2bk+4ck^2+\text{etc., etc.}$$

If a be *known*, then by subtracting a from the values of y we have a series of equations identical with the above in which—

$$(48)\ldots.y_0=0\,;\; y_1=bk+ck+\text{etc.}:\; y_2=2bk+4ck^2+\text{etc.}$$

We deal first with the cases where a is known and assume that the *ordinates* y_1, y_2, etc., are the values computed from the axis X, *so taken that* $a=0$. Then the following formulæ, in which y_1 is denoted by i, y_2 by ii., etc., may be readily deduced :—

Formulae when $y_0 = 0 = a$.

Formula.	Data.	Value of b.	Value of c.
(49)	y_1	$\frac{1}{k}$ (i.)	
(50)	y_1 and y_2	$\frac{1}{2k}$ (4i.—ii.)	$\frac{1}{2k^2}$ (—2i.+ii.)
(51)	y_1 to y_3	$\frac{1}{6k}$ (18i.—9ii.+2iii.)	$\frac{1}{2k^2}$ (—5i.+4ii.—iii.)
52)	y_1 to y_4	$\frac{1}{12k}$ (48i.—36ii.+16iii.—3iv.)	$\frac{1}{24k^2}$ (—104i.+114ii.—56iii.+11iv.)
(53)	y_1 to y_5	$\frac{1}{60k}$ (300i.—300ii.+200iii.—75iv.+12v.)	$\frac{1}{24k^2}$ (—154i.+214ii.—156iii.+61iv.—10v.)

Formula.	Data.	Value of d.	Value of e.
(51a)	y_1 to y_3	$\frac{1}{6k^3}$ (3i.—3ii.+iii.)	
(52a)	y_1 to y_4	$\frac{1}{12k^3}$ (18i.—24ii.+14iii.—3iv.)	$\frac{1}{24k^4}$ (—4i.+6ii.—4iii.+iv.)
(53a)	y_1 to y_5	$\frac{1}{24k^3}$ (+71i.—118ii.+98iii.—41iv.+7v.)	$\frac{1}{24k^4}$ (—14i.+26ii.—24iii.+11iv. 2v.)
(53b)	y_1 to y_6	Value of $f=\frac{1}{120k^5}$ (+5i.—10ii.+10iii.—5iv.+1v.)	

Instead of using the value of the ordinates it is often convenient to form the successive differences, and then the coefficients b to f can be expressed very briefly in terms of the leading differences of the ordinates, corresponding to the values 0, k, $2k$, etc., of the abscissa. In the following results, D_1, D_2, etc., represent the successive *leading differences*, that is, remembering that $y_0=0$: $D_1=y_1$; $D_2=y_2-2y_1$; $D_3=y_3-3y_2+3y_1$; etc.; etc.

[1] See II., § 6, formulæ (9) to (13a).

Formula.	Data.	Value of b.	Value of c.
(54)	y_1	$\frac{1}{k}\ D$	
(55)	y_1 & y_2	$\frac{1}{2k}\ (2D_1 - D_2)$	$\frac{1}{2k^2}\ D_2$
(56)	y_1 to y_3	$\frac{1}{6k}\ (6D_1 - 3D_2 + 2D_3)$	$\frac{1}{2k^2}\ (D_2 - D_3)$
(57)	y_1 to y_4	$\frac{1}{12k}\ (12D_1 - 6D_2 + 4D_3 - 3D_4)$	$\frac{1}{24k^2}\ (12D_2 - 12D_3 + 11D_4)$
(58)	y_1 to y_5	$\frac{1}{60k}\ (60D_1 - 30D_2 + 20D_3 - 15D_4 + 12D_5)$	$\frac{1}{24k^2}\ (12D_2 - 12D_3 + 11D_4 - 10D_5)$.

Formula.	Data.	Value of d	Value of e	Value of f
(56a)	y_1 to y_3	$\frac{1}{6k^3}\ D_3$		
(57a)	y_1 to y_4	$\frac{1}{12k^3}\ (2D_3 - 3D_4)$	$\frac{1}{24k^4}\ D_4$	
(58a)	y_1 to y_5	$\frac{1}{24k^3}\ (4D_3 - 6D_4 + 7D_5)$	$\frac{1}{24k^4}\ (D_4 - 2D_5)$	$\frac{1}{120k^5}\ D_5$

Secondly, when a is not known, and the ordinates y_1, y_2, etc., are distant k, $2k$, etc., from the Y-axis, we may readily extrapolate a by means of the differences. For the coefficients are simply the numbers of Pascal's triangle (the binomial coefficients) with the first omitted. Thus, the small Roman figures denoting suffixes only, we have—

(59)....a = 2i. − ii. ; or 3i. − 3ii. + iii. ; or 4i. − 6ii. + 4iii. − iv. ; or 5i. − 10ii. + 10iii. − 5iv. + v. ; or 6i. − 15ii. + 20iii. − 15iv. + 6v. − vi.

for two, three, etc., ordinates given. When a is found, the problem resolves itself into that for which solutions have already been given, or it may be directly solved. For five ordinates given, *not* including a, we have, for example :—

Formulæ.

(60)......a = 5i. — 10ii. + 10iii. — 5iv. + v.

(61)......$b = \frac{1}{12k}$ (— 77i. + 214ii. — 234iii. + 122iv. — 25v.)

(62)......$c = \frac{1}{24k^2}$ (71i. — 236ii. + 294iii. — 164iv. + 35v.)

(63)......$d = \frac{1}{12k^3}$ (— 7i. + 26ii. — 36iii. + 22iv. — 5v.)

(64)......$e = \frac{1}{24k^4}$ (i. — 4ii. + 6iii. — 4iv. + v.)

The values of the coefficients in terms of the leading differences (D) are :—

$$(65)\ldots\ldots a = y_1 - D_1 + D_2 - D_3 + D_4$$

$$(66)\ldots\ldots b = \frac{1}{12k}\left(12D_1 - 18D_2 + 22D_3 - 25D_4\right)$$

$$(67)\ldots\ldots c = \frac{1}{24k^2}\left(12D_2 - 24D_3 + 35D_4\right)$$

$$(68)\ldots\ldots d = \frac{1}{12k^3}\left(2D_3 - 5D_4\right)$$

$$(69)\ldots\ldots e = \frac{1}{24k^4}\,D_4$$

3. **Evaluation of the differences from the coefficients.**—When the coefficients of an integral function, viz., one of the form (46), are known, and it is desired to ascertain the values of the ordinates y_0, y_1, y_2, etc., the common interval between which is k, they may be rapidly computed from differences, viz., from $x=0$ and $y=a$, together with the following leading differences :—

Factor into numerical coefficient below—

	Differences.	bk +	ck^2 +	dk^3 +	ek^4 +	fk^5.
	D_1	1	1	1	1	1
	D_2		2	6	14	30
(70)	D_3			6	36	150
	D_4				24	240
	D_5					120

For equations of less degree than the fifth the table still serves since f, e, etc., may be put equal to 0.

4. **Subdivision of intervals.**—When the ordinates are to hand for a series of intervals, those for a subdivision of these into m parts may readily be determined by computing a new series of lesser leading differences, d say, using those, D say, of the original intervals, as a basis, as follows :—

	Difference.	$\frac{D_1}{m}$	$\frac{D_2}{m^2}$	$\frac{D_3}{m^3}$	$\frac{D_4}{m^4}$	$\frac{D_5}{m^5}$
(71)	$d_1 =$	1	$-\frac{m-1}{2}$	$+\frac{2m^2-3m+1}{6}$	$-\frac{6m^3-11m^2+6m-1}{24}$	$+\frac{24m^4-50m^3+35m^2-10m+1}{120}$
	$d_2 =$		1	$-(m-1)$	$+\frac{11m^2-18m+7}{12}$	$-\frac{10m^3-21m^2+14m-3}{12}$
	$d_3 =$	..	..	1	$-\frac{3m-3}{2}$	$+\frac{7m^2-12m+5}{4}$
	$d_4 =$	..	..	..	1	$-2(m-1)$
	$d_5 =$	..	..	..	..	1

That is, we divide the nth difference by m^n, and this factor is multiplied into the expression opposite d with the proper suffix. The sum of the terms gives the leading difference in the corresponding d in the first column.[1]

When an interval is divided into 2, 4, 8 or 16, etc., parts, the ordinates may be found by successive "*interpolations into the middle.*"

5. **Evaluation of constants of periodic fluctuations.**—The general empirical formula for a periodic curve which may be made to fit given data is—

$$(72)..y\ (\text{or}\ \frac{1}{y}) = a + b \sin(\beta + x) + c \sin 2(\gamma + x) + d \sin 3(\delta + x) + \text{etc.}$$

in which the number of terms to be taken depends upon the given data, and is sufficiently illustrated hereunder.

When the values of y are given only for the beginning of the recurring period of the total fluctuation and at the end of the first half period, we have—

$$(73)......y = a + b \sin(\beta + x)$$

$$(74)......a = \tfrac{1}{2}(y_0 + y_1)\ ;\quad b \sin \beta = \tfrac{1}{2}(y_0 - y_1)$$

Hence if any definite value be assigned to b, β becomes determinate ; or if to β, b becomes determinate.

When there are values of y for the beginning of the total period, and for the instants one-third and two-thirds of the period, then we have, writing—

$$y_0 - a = r_0\ ;\quad y_1 - a = r_1\ ;\ \text{etc.}$$

$$(75)......a = \tfrac{1}{3}(y_0 + y_1 + y_2)\ ;\quad \tan \beta = \frac{\sqrt{3}\ r_0}{r_1 - r_2}$$

a and β being found, we have—

$$(76)......b = r_0 \operatorname{cosec} \beta$$

Using r_n throughout to denote $y_n - a$, where n is 0, 1, 2, etc., we have for four values, viz., at the beginning of a period and at one-fourth, two-fourths, and three-fourths of the period, from the beginning—

$$(77)....a = \tfrac{1}{4}(y_0 + y_1 + y_2 + y_3);\quad \tan \beta = \frac{y_0 - y_2}{y_1 - y_3}$$

and in the expression for tan β, we may write r for y.

These quantities being found, we then have—

$$(78)....b = r_0 \operatorname{cosec} \beta = r_1 \sec \beta.$$

For fifth periods, that is, for equidistant ordinates 0 to 4, the formulæ for the constants are :—

$$(79)......y = a + b \sin(\beta + x) + c \sin 2(\gamma + x)$$

[1] See Text Book Institute of Actuaries, Pt. II., Ed. 1902, p. 443.

and the solution gives—

$$(80) \ldots\ldots a = \tfrac{1}{5} \Sigma_0^4 y.$$

$$(81) \ldots \tan \beta = \frac{2 \sin 36^0 \,.\, r_0 - 2 \cos 36^0 (r_1 + r_4)}{r_2 - r_3 + 2 \cos 36^0 (r_1 - r_4)}$$

$$(82) \ldots\ldots b = \frac{\operatorname{cosec} \beta \,.\, r_0 - 2 \cos 36^0 (r_2 + r_3)}{3 + 2 \cos 72^0.}$$

$$(83) \ldots\ldots \tan 2\gamma = \frac{2 \sin 36^0 \{ r_0 (2 + 2 \cos 72^0) + 2 \cos 36^0 (r_2 + r_3) \}}{r_1 - r_4 - 2 \cos 36^0 (r_2 - r_3)}$$

$$(84) \ldots\ldots c = \operatorname{cosec} 2\gamma. \frac{r_0 (2 + 2 \cos 72^0) + 2 \cos 36^0 (r_2 + r_3)}{3 + 2 \cos 72^0.}$$

The values of sin 36°, cos 36°, sin 72°, and cos 72° are respectively :— $\tfrac{1}{4}\sqrt{(10 - 2\sqrt{5})} = 0.5877853$; $\tfrac{1}{4}(\sqrt{5} + 1) = 0.8090170$; $\tfrac{1}{4}\sqrt{(10 + 2\sqrt{5})} = 0.9510565$; and $\tfrac{1}{4}(\sqrt{5} - 1) = 0.3090170$.

For sixth periods, that is, for equidistant ordinates 0 to 5 the formulæ for the constants are :—

$$(85) \ldots\ldots\ldots a = \tfrac{1}{6} \Sigma_0^5 y.$$

$$(86) \ldots\ldots \tan \beta = \frac{\sqrt{3}\,(r_1 - r_2 - r_4 + r_5)}{r_1 + r_2 - r_4 - r_5}$$

$$(87) \ldots\ldots\ldots b = \tfrac{1}{2} \operatorname{cosec} \beta \, (r_1 - r_2 - r_4 + r_5)$$

$$(88) \ldots\ldots \tan 2\gamma = \frac{\sqrt{3}\,(r_0 + r_3)}{r_1 - r_2 + r_4 - r_5}$$

$$(89) \ldots\ldots\ldots c = \frac{\sec 2\gamma}{2\sqrt{3}} (r_1 - r_2 + r_4 - r_5)$$

The solution for twelfth periods is sometimes required as, for example, when values are to hand for the beginning of each month. Denoting as before the remainders $y_n - a$ by r_n we have—

$$(90) \ldots\ldots a = \frac{1}{12} \Sigma_0^{11} y.$$

Then making the following additions for brevity of working, viz.—

$$(91) \ldots \left\{ \begin{array}{ll} L_0 = r_0 + r_3 + r_6 + r_9 \,; & N_0 = r_0 + r_6 - r_3 - r_9 \\ L_1 = r_1 + r_4 + r_7 + r_{10} \,; & N_1 = r_1 + r_7 - r_4 - r_{10} \\ L_2 = r_2 + r_5 + r_8 + r_{11} \,; & N_2 = r_2 + r_8 - r_5 - r_{11} \\ M_0 = r_0 + r_4 + r_8 \,; & R_0 = r_0 + r_2 - r_6 - r_8 \\ M_1 = r_1 + r_5 + r_9 \,; & R_1 = r_1 + r_3 - r_7 - r_9 \\ M_2 = r_2 + r_6 + r_{10} \,; & R_2 = r_2 + r_4 - r_8 - r_{10} \\ M_3 = r_3 + r_7 + r_{11} \,; & R_3 = r_3 + r_5 - r_9 - r_{11} \end{array} \right.$$

$$(92) \ldots\ldots \tan \beta = \frac{2R_0 + \sqrt{3}R_1 - R_2 - \sqrt{3}R_3}{R_1 + \sqrt{3}R_2 + R_3}$$

$$(93) \ldots\ldots\ldots b = \tfrac{1}{12} \operatorname{cosec} \beta \, (2R_0 + \sqrt{3}R_1 - R_2 - \sqrt{3}R_3)$$

$$(94) \ldots\ldots \tan 2\gamma = \frac{N_1 + 2N_0 - N_2}{\sqrt{3}\,(N_1 + N_2)}$$

$$(95)\ldots\ldots\ldots c = \frac{1}{4\sqrt{3}} \sec 2\gamma\,(N_1 + N_2)$$

$$(96)\ldots\ \tan 3\delta = \frac{M_0 - M_2}{M_1 - M_3}$$

$$(97)\ldots\ldots\ldots d = \tfrac{1}{6} \operatorname{cosec} 3\delta\,(M_0 - M_2)$$

$$(98)\ldots\ldots \tan 4\epsilon = \frac{\sqrt{3}L_0}{L_1 - L_2}$$

$$(99)\ldots\ldots\ldots e = \tfrac{1}{4} \operatorname{cosec} 4\epsilon\, L_0$$

$$(100)\ldots \tan 5\zeta = \frac{2R_0 - \sqrt{3}R_1 - R_2 + \sqrt{3}R_3}{R_1 - \sqrt{3}R_2 + R_3}$$

$$(101)\ldots\ldots\ldots f = \tfrac{1}{12} \operatorname{cosec} 5\zeta\,(2R_0 - \sqrt{3}R_1 - R_2 + \sqrt{3}R_3)$$

6. **Constants of exponential curves.**—The case of a curve of the type

$$(102)\ldots\ldots y = 1 \pm \eta t^{-m \pm nt}$$

see equation (20). has already been sufficiently considered : its constants can be found as shewn by formulæ (23) to (30) ; and also that of the type, see equation (32)

$$(103)\ldots\ldots y = 1 \pm \eta t^{\frac{1}{\pm m \pm nt}} ;$$

see formulæ (35) to (38). In general, curves of this type may be solved by forming the equations $y' = y - 1$ and taking logarithms when we get such forms as—

$$(104)\ldots\ldots u = \epsilon + \log t\,(\pm m \pm nt) \text{ and } u = \epsilon\ \frac{\log t}{\pm m \pm nt}$$

solutions for which have already been sufficiently indicated. As this process of taking logarithms is the key to many solutions, we now refer more fully to the matter. The essence of this method of solving is that if a series of values on the axis of abscissæ be taken in *geometrical progression*, their logarithms are in *arithmetical progression*. Thus, χ being $\log x$, we have—

Quantities $= x$; kx ; k^2x ; k^3x ; etc. ;

Logarithms of same $= \chi$; $\chi + k$; $\chi + 2k$; $\chi + 3k$; etc.

Hence the problems of solution are reduced to those of the examples illustrated by formulæ (46) to (71).

7. **Evaluation of the constants of various curves representing types of fluctuation.**—The evaluation of the constants of various curves can often be effected by taking suitable ordinates to the curve and solving from their logarithms. This is illustrated in the following series of equations :—

$$(105)\ldots \mathfrak{Y} = \mathfrak{C}e^{AX^m} = \mathfrak{C}e^{AM^x} = \mathfrak{C}\mathfrak{A}^{X^m} = \mathfrak{C}\mathfrak{A}^{M^x}$$

We have on taking napierian logarithms

(106)...... $Y = AX^m + C = AM^x + C$

in which $\log \mathfrak{Y} = Y$; $\log \mathfrak{C} = C$; $\log \mathfrak{A} = A$; $\log X = x$; and $\log M = m$.

The second curve may be called the first *logarithmic homologue* of the first, and the first the first *anti-logarithmic generatrix* of the second. Subsequent curves may be similarly defined as the second logarithmic homologue, etc.

Yet again, if C be zero, we have on taking the logarithm of this last expression—

(107)......$y = a + mx$,

in which $\log Y = y$; and $\log A = a$.

This will sufficiently illustrate the matter. Several examples of solution will be given of important curves for representing fluctuation.

In the curve

(108)......$y = A + Bx^m$

If $A = 0$; then the solution is found at once from any two values of y and of x. For we have—

(109)......$\log y = \log B + m \log x$.

On Fig. 21 hereinafter, these curves are shewn by thick lines for positive values of m, and by thin lines for negative values.

If, however, A be *not* zero, then we must take three values of y for abscissae of the value x, xk, xk^2, when it may easily be shewn that—

(110).... $\dfrac{y_3 - y_2}{y_2 - y_1} = k^n$; or $n = \dfrac{\log (y_3 - y_2) - \log (y_2 - y_1)}{\log k}$

The curve

(111)......$y = B + Ce^{ax}$

can be solved by taking the values of y for x, $x + k$, $x + 2k$, for

(112)...... $\dfrac{y_3 - y_2}{y_2 - y_1} = \dfrac{C\,[e^{a(x+2k)} - e^{a(x+k)}]}{C\,[e^{a(x+k)} - e^{ax}]} = e^{ak}$

Consequently putting Y_{321} for the left-hand expression, and writing 2.3025851 for the modulus for changing common into Napierian logarithms

(113)......$a = \dfrac{2.3026}{k} \log_{10} Y_{321}$

When a is found the solutions for B and C are obvious. Curves of the equation e^x are shewn by thick lines on Fig. 22 hereinafter, and those of equation $1 / e^x$ by thin lines.

The exponential curve—

(114)......$y = A + Be^{nx^p}$

can be solved if A be zero, or if A be known, and a new series of $y' = y - A$ be formed. Thus A being zero,

(114a)......$\log y = \log B + nx^p \log e$.

Hence, as before, taking three values of y for x, xk, xk^2, the solution is—

$$(115)\ldots\ldots p = \frac{1}{\log k} \cdot \log \left\{\frac{\log y_3 - \log y_2}{\log y_2 - \log y_1}\right\}$$

$$(116)\ldots\ldots n = \frac{\log y_2 - \log y_1}{x^p(k^p - 1) \log e}$$

$$(117)\ldots\ldots \log B = \log y_1 - nx^p \log e.$$

These curves are shewn for Fig. 23 hereinafter, for various values of n *and* p.

The curve—

$$(118)\ldots\ldots y = Ax^m e^{nx^p}$$

is solved by taking four ordinates, viz., for x, xk, xk^2, xk^3, when the solution becomes[1]—

$$(119)\ldots\ldots p = \frac{1}{\log k} \cdot \log \left\{\frac{\log y_2 - 2 \log y_3 + \log y_4}{\log y_1 - 2 \log y_2 + \log y_3}\right\}$$

using common logarithms. Then M denoting $\log e$, we have also—

$$(120)\ldots n = \frac{(\log y_1 - 2 \log y_2 + \log y_3)}{Mx^p (k^p - 1)^2} = \frac{(\log y_2 - 2 \log y_3 + \log y_4)}{Mx^p k^p (k^p - 1)^2}$$

$$(121)\ldots\ldots m = \frac{(\log y_2 - \log y_1) - Mnx^p (k^p - 1).}{\log k}$$

There are obviously two other possible formulæ for m.

$$(122)\ldots \log A = \log y_1 - m \log x_1 + Mnx^p$$

the value of M being 0.4342945. Three other formulæ are also possible for A. For further formulæ see (150) to (153) later; see also Figs. 21 to 27, hereinafter, for the forms of the curve.

8. **Polymorphic and other fluctuations.**—*Monomorphic* or rather *unimodal* curves disclose a single maximum (or minimum) value. But there are fluctuations which are *polymorphic* or *multimodal*. These may be regarded as compounded of monomorphic curves. Practically their dissection is best effected by the graphic methods of analysis. In general any curve can be represented with great accuracy by either

$$(123)\ldots\ldots y = a + bx^p + cx^q + dx^r + \text{etc., or by}$$

$$(124)\ldots\ldots Y = e^{a + bx^p + cx^q + \text{etc.}}$$

where p, q, r, etc., are not restricted to integral values.

The latter curve is reduced to the former by taking the logarithm; thus, $y = \log_e Y$. To solve for the constants we must have six points besides the origin. If the value of a be known, the curve can be reduced to one passing through the origin by subtracting a. Then we take values of y for x, xk, xk^2, xk^3, etc. For the case for terms in p and q only, we can proceed as follows :—

[1] For a more complete study of the curve, see "Studies in Statistical Representation. On the Nature of the Curve," above given, viz. (118), by G. H. Knibbs, Journ. Roy. Soc., Vol. XLIV., pp. 341-367, 1910.

By writing L for bx^p and M for cx^q, and α for k^p and β for k^q; we have—

(125). . $y_1 = L + M$; $y_2 = L\alpha + M\beta$; $y_3 = L\alpha^2 + M\beta^2$; $y_4 = L\alpha^3 + M\beta^3$.

Hence by eliminating L and M from the first three and from the last three equations, we have respectively—

$$(126a)\ldots \begin{vmatrix} 1 & 1 & y_1 \\ \alpha & \beta & y_2 \\ \alpha^2 & \beta^2 & y_3 \end{vmatrix} = 0; \quad \begin{vmatrix} 1 & 1 & y_2 \\ \alpha & \beta & y_3 \\ \alpha^2 & \beta^2 & y_4 \end{vmatrix} = 0 \ldots (126b)$$

Consequently α and β are the roots of—

$$(127)\ldots\ldots \begin{vmatrix} 1 & y_1 & y_2 \\ \xi & y_2 & y_3 \\ \xi^2 & y_3 & y_4 \end{vmatrix} = 0$$

Thus the two values of ξ in the equation—

(128). . $\xi^2 - \xi(\alpha + \beta) + \alpha\beta = \xi^2(y_1y_3 - y_2^2) + \xi(y_2y_3 - y_1y_4) + (y_2y_4 - y_3^2) = 0$

are the values of k^p and k^q. And since k is known, the solution is to hand by taking logarithms.

The solution for three indices is similar. The six equations can be written—

$$(129)\ldots\ldots y_{m+1} = L\alpha^m + M\beta^m + N\gamma^m$$

and α, β, and γ; that is k^p, k^q, and k^r, are the roots of the equation.

$$(130)\ldots\ldots \begin{vmatrix} 1 & y_1 & y_2 & y_3 \\ \xi & y_2 & y_3 & y_4 \\ \xi^2 & y_3 & y_4 & y_5 \\ \xi^3 & y_4 & y_5 & y_6 \end{vmatrix} = 0$$

which may be expanded in the form—

$$(130a)\ldots\ldots A_1\xi^3 - 3A_2\xi^2 + 3A_3\xi - A_4 = 0$$

where A_1, $3A_2$, $3A_3$ and A_4 are the minors respectively of ξ^3, ξ^2, ξ and 1 in the determinant.

If the constant a is included in (123) or (124), the solution is more tedious. We must then have seven values of y. Thus—

$$(131)\ldots\ldots y_{m+1} = a + L\alpha^m + M\beta^m + R\gamma^m$$

$$(131a)\ldots\ldots y_{m+2} - y_{m+1} = L'\alpha^m + M'\beta^m + R'\gamma^m$$

the accented values being $L' = L(\alpha - 1)$; $M' = M(\beta - 1)$; etc.

Thus α, β, and γ are the roots of—

$$(132)\ldots\ldots \begin{vmatrix} 1 & y_2 - y_1 & y_3 - y_2 & y_4 - y_3 \\ \xi & y_3 - y_2 & y_4 - y_3 & y_5 - y_4 \\ \xi^2 & y_4 - y_3 & y_5 - y_4 & y_6 - y_5 \\ \xi^3 & y_5 - y_4 & y_6 - y_5 & y_7 - y_6 \end{vmatrix} = 0$$

. Writing Y_3, Y_2, Y_1, Y_0 for the minors of ξ^3, ξ^2, ξ, and 1 in the determinant, the equation becomes

$$(133)\ldots\ldots \xi^3 Y_3 - \xi^2 Y_2 + \xi Y_1 - Y_0 = 0.$$

It will be seen from the preceding examples that when, as regards their indices, the equations are not restricted to integral values, the fitting power of the curve is enormously increased. To fit seven points with integral indices we should have to have an equation of the sixth degree.[1]

Figures 6 and 7 furnish graphs for simple cases with two indices only. From these graphs, which also are for integral values of the index only, it is immediately evident that the loci of curves with fractional values must lie between the curves drawn. The forms of the curves may, of course, be modified also by varying the coefficients: hence the fitting power of expressions of the type considered obviously becomes very great when the limitation imposed by restricting the indices solely to integral values is abandoned.[2]

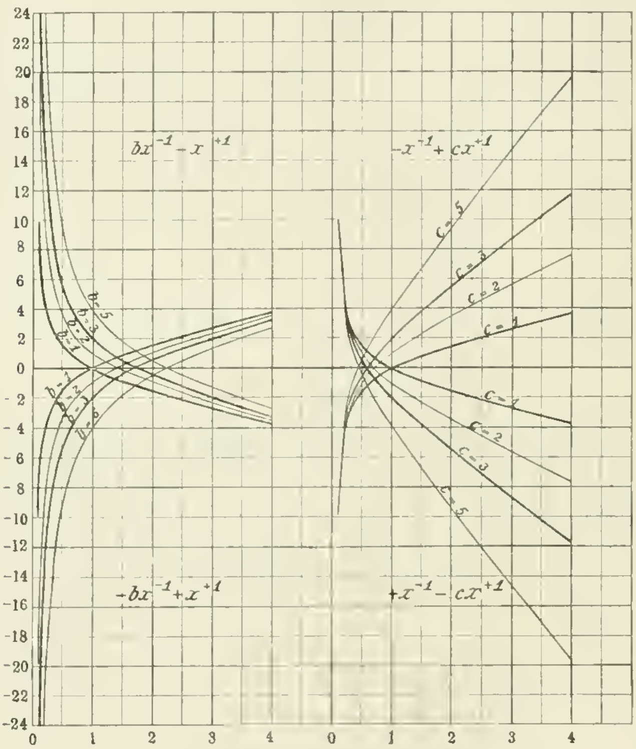


Fig. 6.

[1] See "Studies in Statistical Representation. III., Curves, their Logarithmic Homologues," etc., by G. H. Knibbs and F.W. Barford. Journ. Roy. Soc. N.S.Wales, Vol. XLVIII., pp. 473-496

[2] The limitations of the fitting power of the curve are discussed in the paper referred to in the preceding footnote. These limitations, in general, are of no moment in statistical results.

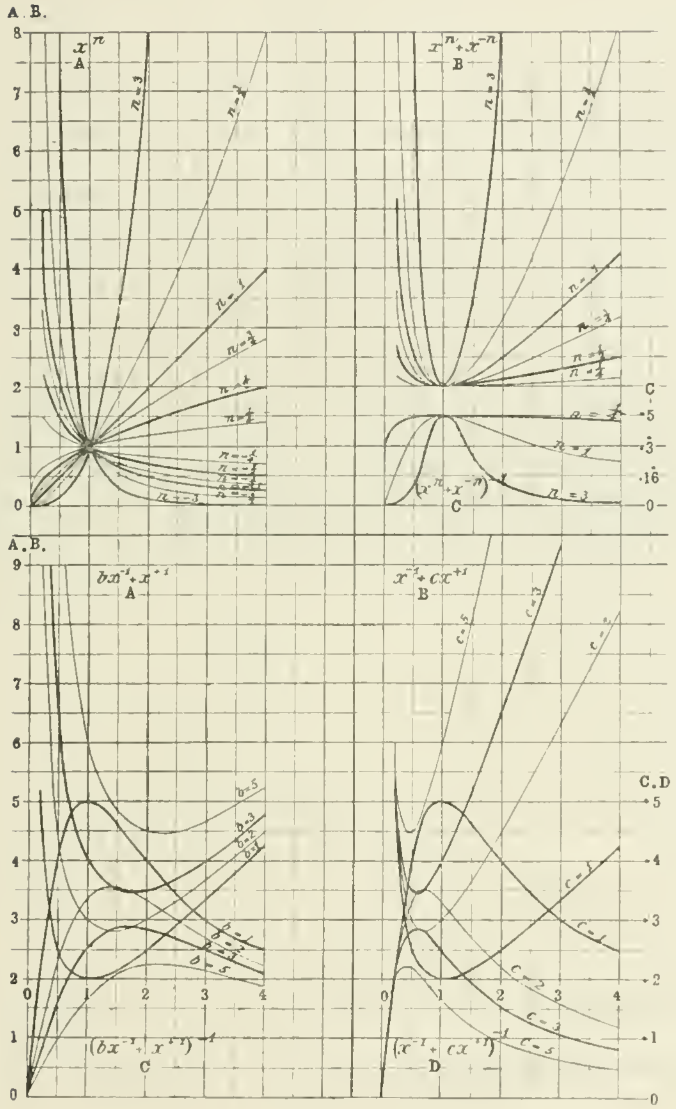


FIG. 7.

Some special cases of fluctuation will now be treated in dealing with problems treating of fluctuating elements that directly or indirectly influence the aggregate or constitution of the population.

9. **Projective anamorphosis.**—A symmetrical curve of frequency (or symmetrical distribution) may become asymmetrical by the elements being projectively varied by means of different types of projection (plane or other). This change may be called *projective anamorphosis*. Any character of a population may be regarded as subject to influence acting

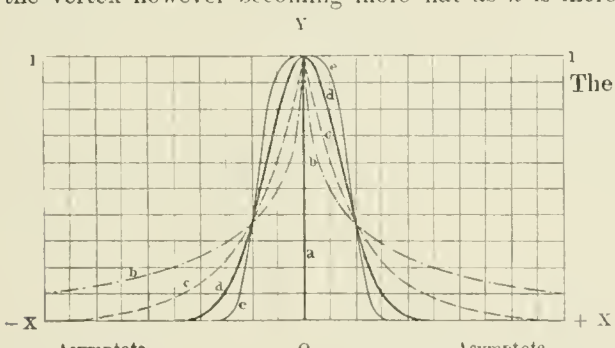
progressively (or retrogressively) with increase of the measure of the character in question, as for example, if the influence tending to increase weight (or height) acted more or less powerfully with increase of that character. This would lead to an asymmetric or *skew* frequency. Thus if a normal frequency be denoted by $y = \phi(x)$; a specialised frequency conceived to originate therefrom would be given by $y' = f(x)\,\phi(x)$. This expression may also be skew, dimorphic, polymorphic, or in fact, what we please, according to the character of $f(x)$. If $f(x) = mx$ or m/x, a symmetrical curve is converted into a skew curve. If $f(x)$ have a mode such that it is *not* identical with that of $\phi(x)$ the latter will be dimorphic. From this it is seen that the ordinates to a dimorphic curve may be the *sum* or *product* of the ordinates to two monomorphic curves. It is not proposed to elaborate just here, however, the general theory of anamorphosis by plane or other projection. It may be easily seen, however, that a skew curve may be readily derived from a symmetrical one, while retaining the general algebraical properties of the latter, by a projection. from a line parallel to the axis of the given symmetrical curve, through the curve and on to a plane passing through the axis but inclined to the plane of the given curve. This will be more fully considered hereinafter.

IV.—SPECIAL TYPES OF CURVES AND THEIR CHARACTERISTICS.

1. **General.**—When the characters of a population have a tendency to deviate in either direction equally, and the number of the population is P, the characters will be distributed as the coefficients $(\frac{1}{2} + \frac{1}{2})^m$ *i.e.*, as the numbers in Pascal's triangle, which, when m is infinite, becomes the curve

$$(134) \ldots\ldots y = Pe^{-\left(\frac{x}{k}\right)^2}\ ; \text{ or say } Pe^{-\left(\frac{x}{k}\right)^n}$$

the first form (viz., that when the power $n = 2$) being the ordinary probability curve, in which k is the modulus. This type of distribution is but one case of the more general expression which, interpreted in a certain way,[1] has a cusp for the vertex for values of n equal to or less than unity, and a curve convex upwards for all values greater than unity, the vertex however becoming more flat as n is increased.



The curves graphed are

a	$=$	axis
b		$e^{-x^{\frac{1}{2}}}$
c		e^{-x}
d		e^{-x^2}
e		e^{-x^4}

Fig. 8.

The curve $y = e^{-x^{\frac{1}{\infty}}}$ is coincident with a from the point Y to a point $y = 0.3678781$; it is then parallel to the X axis. All the curves intersect at this point.

Such a distribution is symmetrical, and takes the form in the figure hereunder, Fig. 7, in which curve 'a' shews its form for $n = 0$; 'b' for $n = \frac{1}{2}$; 'c' for $n = 1$; 'd' for $n = 2$: 'e' for $n = 4$.

When the probabilities of distribution are not equal for possible alternatives, and the probabilities of these alternatives are as p and q, the sum of p and q being unity, then the distribution will be the coefficients of $(p + q)^m$. If q and p are not equal the curve is *not* symmetrical, but is of a form like Fig. 9 hereinafter. Whether results can be made to conform to a particular type or not depends on the form of the curve, and

[1] That is, so that e^{-x^n} and $e^{-x^{n+\delta n}}$ are in the same spatial region, or on the same side of an axis, and are not allocated to different regions according to whether the number $(n + \delta n)$ is even or odd.

in particular on the position of its vertex; on whether its sides meet the axis of the variable more or less sharply or asymptotically, on whether it is *monomorphic* or *polymorphic*, or has one "mode" (*is unimodal*) or more modes than one (*is multimodal*). Various types of unimodal fluctuations, commencing and ending with zero values or otherwise, have been given by Prof. Pearson. These are intended to reproduce the group-values of statistical data, under appropriate forms of curves, by a method which has been called the method of moments, the forms of the curves being derived from the normal curve of probability. We shall later refer to these, but remark first of all that the critical elements of the curves representing distributions or fluctuations are as follows, viz.:—

(*a*) the value of the ordinate when the variable is zero;
(*b*) the values of the variable for which the ordinates become zero;*
(*c*) or, if they do not become zero, the value of the ordinate when the variable is infinity;
(*d*) the abscissa of the *mode*, or greatest ordinate, and the value of that ordinate;
(*e*) the abscissa of the ordinate which equally divides the curve area (as, for example, the abscissa which corresponds to the average value, or the *centroid vertical*);
(*f*) the distance between these two ordinates (*d*) and (*e*) (the numerator of the quantity defining the *skewness*);
(*g*) the *mean-deviation* of the curve (or denominator of the skewness);
(*h*) the abscissa of the point where the curvature changes its sign, (point of inflexion);
(*i*) the abscissa of the point of most rapid change of direction of the curve.

(*a*)......$y = f(0)$; (*b*)......$f(x) = 0$; (*c*)......$f(\infty) = k$ or 0;
(*d*)......x_m when $df(x) / dx = 0$; and $y_m = f(x_m)$;
(*e*)......x_a when the value of $\int xf(x)\,dx \div \int f(x)\,dx$ for the range of the variable up to x_a is equal to that for x_a onward;
(*f*)......$(x_a \sim x_m)$
(*g*)......$m_2 = \sqrt{[x^2 f(\chi)\,d\chi \div \int f(\chi)\,dx]}$, in which χ is measured from the position of the mean (x_a).
(*h*)......x_i when $d^2 f(x) / dx^2 = 0$;
(*i*)......x_p when $d^3 f(x) / dx^3 = 0$.

---

* The approach of statistical curves to the axis of abscissæ or to the axis of ordinates is, in general, not determined by mathematical considerations, but by a knowledge of the nature of the data itself. For example, the terminals of the curve of fertility (discussed hereinafter) deduced from ex-nuptial births, shews a diminution which may be represented very closely by the numbers 1078, 154, 22, 3½, for the ages 16, 15, 14 and 13 respectively, *i.e.*, each number is one-seventh of the number preceding it. Merely mathematically, therefore, it is more probable that these should continue for the ages 12, 11, 10, 9, etc., as 0.45; 0.064; 0.009; 0.0013, etc. Even at age 0 there would, of course, be still a positive value though small. But physiological knowledge indicates that the earliest arrival of puberty is probably over 10 years, hence 11 would be the earliest age for birth, and the ordinate must be zero.

2. **Curves of generalised probability.**—Prof. Pearson proposes to reduce forms of distribution of statistical facts under a series of seven type-forms of curves, representing what may be called curves of generalised probability,[1] and much work has been reduced on this system.

Fig. 9
Type I. (i.).

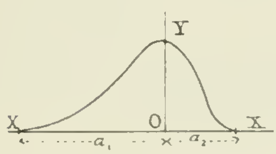


Fig. 10.
Type I. (ii.).

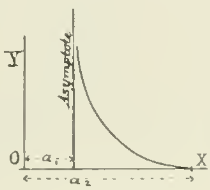
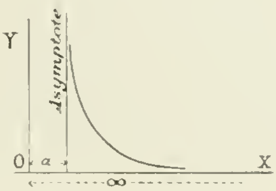


Fig. 11.
Type I. (ii.a).



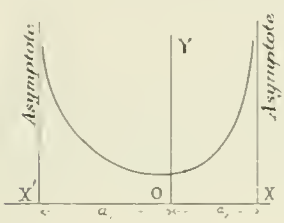


Fig. 12.
Type I. (iii.).

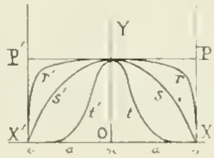


Fig. 13.
Type II. (i.).

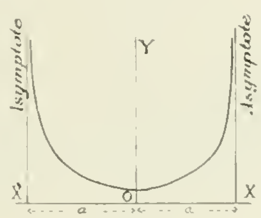


Fig. 14.
Type II. (ii.)

His first type (Type I.) is :—

$$(135)\ldots\ldots y_0 = y\left(1 + \frac{x}{a_1}\right)^{\nu a_1}\left(1 - \frac{x}{a_2}\right)^{\nu a_2}$$

which may take two other fundamental sub-forms, viz.,

$$(136)\ldots\ldots y = y_0\left(\frac{x}{a_1} - 1\right)^{-\nu a_1}\left(1 - \frac{x}{a_2}\right)^{\nu a_2}\text{; and}$$

$$(137)\ldots\ldots y = y_0\left(1 - \frac{x}{a_1}\right)^{-\nu a_1}\left(1 + \frac{x}{a_2}\right)^{-\nu a_2}$$

which are represented respectively by the forms in Figs. 9 to 12.[2] When ν, a_1 and a_2 are positive the curve meets the X axis at the distances a_1 and a_2, see the figures. The abscissa of the mode is 0 and the curve is skew.

[1] See his "Contributions to the Mathematical Theory of Evolution." Phil. Trans., Vol. 185 (1894) A, pp. 71-110; Vol. 186 (1895) A, pp. 343-414; Vol. 187 (1896) A, pp. 253-318; Vol. 191 (1898) A, pp. 229-311; Vol. 192 (1898) A, pp. 169-244; Vol. 192 (1899) A, pp. 257-330; Vol. 195 (1900) A, pp. 1-47; Vol. 195 (1900) A, pp. 79-150; Vol. 197 (1901) A, pp. 285-379; Vol. 197 (1901) A, pp. 443-459.

[2] See Phil. Trans., Vol. 186 A, pp. 364-5.

If, in the formula for Type I., a_2 be made equal to a_1, then the formula becomes that of Type II.,[1] shewn by Figs. 13 and 14, viz.—

$$(138)\ldots\ldots\; y = y_0 \left(1 - \frac{x^2}{a^2}\right)^{\nu a}$$

the *basic form* of which, when y_0 is unity, is an ellipse with semiaxes a and 1. The figure becomes a circle when ν is $\frac{1}{2}$ and a is 1. In general, any form can be deduced from the basic form which, when νa is unity, is a parabola (the quantity within the brackets) in (138). If this quantity be infinite and positive the figure becomes X′ P′ Y P X : see Fig. 13. If positive and greater than unity, it is the curve $r'r$; if unity it is the parabola $s's$; if less than unity, the curve $t't$ in Fig. 13. The abscissa of the mode is 0, and the curve is of course symmetrical.

If ν be made negative in (138) the formula becomes

$$(139)\ldots\ldots\; y = y_0 \frac{1}{\left(1 - \frac{x^2}{a^2}\right)^{\nu a}}$$

and is shewn by Fig. 14. The abscissa of the mode (of mediocrity) is at the origin.

If in the second sub-form of Type I. we make a_2 infinity, then

$$(140)\ldots\ldots\; y = y_0 \left(\frac{x}{a} - 1\right)^{-\nu a} e^{-\nu x}$$

the form of which is shewn in Fig. 11 ; that is, the curve is asymptotic to the ordinate whose abscissa is distant $+a$ from the origin, and asymptotic also to the axis OX.

Fig. 15.
Type III.

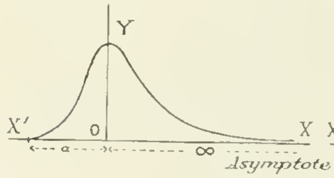


Fig. 16.
Type IV.

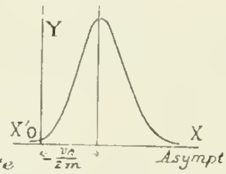
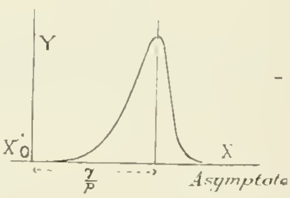


Fig. 17.
Type V.



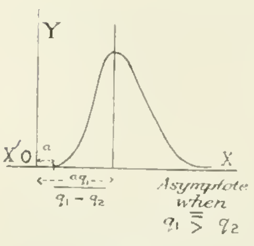


Fig. 18.
Type VI.

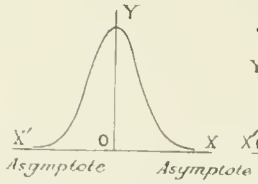


Fig. 19.
Type VII.

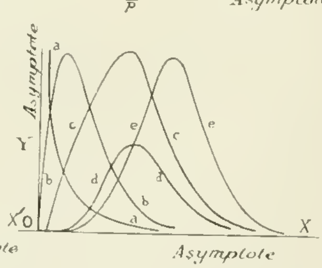


Fig. 20.
Various.

[1] Op. cit., pp. 364-5.

When in formula (135) a_2 is infinity, then its form becomes Type III., viz.,

$$(141)\ldots\ldots y = y_0 \left(1 + \frac{x}{a}\right)^{\nu a} e^{-\nu x}$$

and is of the form shewn in Fig. 15. The abscissa of the mode is at the origin, and the curve is skew.

Type IV. is of the form shewn in Fig. 16 ; its equation being :—

$$(142)\ldots\ldots y = \left(1 + \frac{x^2}{a^2}\right)^{-m} e^{-\nu \tan^{-1} x/a} \text{ ; or } = y_0 \cos^{2m}\theta . e^{-\nu\theta}$$

θ being the angle the tangent of which is x/a. The curve is asymptotic to the X axis on both branches ; its mode is at the distance $-\nu a/2m$ from the origin, and it is skew : see Fig. 16.

Type V., is of the form shewn in Fig. 17, and its equation is :—

$$(143)\ldots\ldots y = y_0 \; x^p e^{-\frac{\gamma}{x}}$$

The curve is limited on one side at the axis X, *i.e.*, for $x = 0$, and is asymptotic thereto at the other ; its mode being at the distance γ/p. The curve is skew. The *mean* is at the distance $\gamma \,/\, (p-2)$ from the origin.

Type VI. is of the form shewn in Fig. 18. Its equation is :—

$$(144)\ldots\ldots y = y_0 \,(x - a)^{q_2} \, x^{-q_1}$$

The curve is limited on one side only, viz., when $x = a$. The mode is at $aq_1 \,/\, (q_1 - q_2)$.

Type VII. is the ordinary probability curve: see Fig. 19, viz. :—

$$(145)\ldots\ldots y = y_0 \, e^{-\frac{x^2}{c}}$$

the mode being at the origin and the curve unlimited in either direction, and of course symmetrical.

Curves *a* to *e*, Fig. 20, are typical forms of the following characteristics in a population, viz. :—

(*a*) Infantile mortality, income, probates, value of houses, etc. ;

(*b*) Mortality from scarlet fever, diphtheria, etc. ;

(*c*) Pauper frequency, divorce frequency with respect to duration of marriage, frequency of scarlet fever with age, of typhus, etc. ;

(*d*) Senile mortality, mortality from enteric at different ages, frequency of marriage of wives corresponding to age of husbands at marriage, etc. ;

(*e*) Height, weight, strength frequency, anthropometric measurements, etc.

3. **The method of evaluating the constants of the curves of generalised probability.**—Two things are requisite in using the Pearson curves, viz. (i.) to select the appropriate type of curve; and (ii.) to evaluate the constants of the selected curve. The selection of a curve which can be made to fit the given group-data depends upon relationships among the moments calculated about the *mean*. These relationships determine three criteria, which, after the necessary computations have been made, indicate the appropriate selection.[1]

Solutions can also be effected by means of a combination of graphical and numerical methods. The numerical solutions can be effected by taking logarithms, that is,

$$(146)\ldots\ldots\log y = \log y_0 + \log f(x).$$

The process in detail can readily be followed from the examples in III. (See in particular § 7). In general the solution must be tentative, and it is important to notice that the type-curve selected is not valid if the data have to be altered larger amounts than they are probably in error. The principle which should be employed is the following:—The adoption of a type-curve can be regarded as satisfactory only when it represents the data within the limits of their probable errors. In other words the geometric form and the algebraic processes should be subordinated to the data and not *vice versa*.

4. **Flexible curves.**—Although the type-curves just considered fulfil their general purpose fairly well, experience shews that their "*fitting power*" is somewhat limited. To overcome this, other types are necessary, the "fitting power" of which is greater. In order to embrace as many forms as possible under cover of a single formula a curve may be so taken that its limiting forms shall include all parabolas, all hyperbolas (or parabolas with negative indices), all exponentials with positive or negative indices, and all curves of the normal probability type. Such a curve will necessarily include all intermediate forms. I have called this type of curve a *flexible curve*.

Formula (149) in the next section is a curve of the type in question. Its graph depends fundamentally upon the values of the indices m, n, and p, and its vertical scale depends upon the constant A. The mode of solving to determine its constants depends upon taking a series of values of the abscissa in geometrical ratio, and is indicated in the next section.

[1] See the article by Professor Pearson already referred to, also "Frequency Curves and Correlation," by W. Palin Elderton (C. and E. Layton, London, 1906); and "Statistical Methods with special reference to Biological Variation," by C. B. Davenport (Chapman and Hall Ltd., London, 1904).

The curves indicated on p. 57 and p. 81 of Mr. W. Palin Elderton's work do not satisfactorily represent the data, forasmuch as the curves chosen were insufficiently flexible.

5. **Determination of the constants of a flexible curve.**—The probability curve, see (134) hereinbefore, viz.,

$$(147)\ldots\ldots y = Ce^{-\frac{x^2}{k^2}} \text{ or } e^{-\frac{x^2}{k^2}+c}$$

in which $c = \log_e C$, may be put in a more general form, viz.—

$$(148)\ldots\ldots y = e^{-\frac{x^2}{f(x)} + F(x) + c}$$

that is, its modulus k and constant C may be assumed to be functions of x.

If we suppose that

$$F(x) = \alpha + \beta \log(\pm x)\,;\; f(x) = \gamma x^s\,;\; c = 0\,;$$

and write $p = 2 - s$; $n = -1/\gamma$; $\log A = \alpha$; $m = \beta$, the expression (148) can be written

$$(149)\ldots\ldots y = Ax^m\, e^{nx^p}$$

see (118) in III. 7. This curve can fit a great variety of forms, viz., such as are shewn on Figs. 21 to 27, referred to later.

In practice it is not quite satisfactory to depend on four points. A better fit can be secured by taking several, say r, series of ordinates for values of the abscissa x_a, $k_a x_a \ldots\ldots k_a^3\, x_a; \ldots\ldots\ldots x_r$, $k_r\, x_r \ldots\ldots\ldots k_r^3\, x_r$. Each set will give a value for p, say p_a, p_b, etc., and a mean (geometric, arithmetic, or other) can be taken. p say. Or writing Y_{pqr} for $\log y_p - 2\log y_q + \log y_r$, we have

$$(150)\ldots\ldots p\log(k_a . k_b \ldots\ldots k_r) = \log\; \Pi_1^r (Y_{234}/Y_{123})\,;$$

$\Pi_1{}^r$ denoting the product of r different sets of the quantities in the brackets.

The use of this mean value of p, being inconsistent with each set of four ordinates, gives for each set two solutions for n, three for m, and four for A, that is in all $2r$, $3r$ and $4r$ solutions respectively for these constants. Having found the mean value for p we use it, in solving for a mean value of n, thus :—[1]

$$(151)\ldots r\log n = \log \frac{\Pi_1^r (\log y_1 - \log y_2 - \log y_3 + \log y_4)}{\Pi_1^r \{Mx^p (k^{2p} - 1)(k^p - 1)\}}\,; \text{ or}$$

$$(151a)\ldots\ldots n = \frac{\Sigma_1^r \log(y_1 y_4 / y_2 y_3)}{\Sigma_1^r \{Mx^p (k^p + 1)(k^p - 1)^2\}}$$

[1] By comparing this with (120) it will be seen that the mean is taken of two quantities each of which give n, on the principle that if $a/b = c/d$ approximately $(a+c)/(b+d)$ is sensibly the arithmetic mean, or having two equations which give n, we assign an equal weight to each. The geometric mean, however, is taken in obtaining a mean result from the different sets. Of course $(n_1 + \ldots\ldots n_r)/r$ would also be a satisfactory value, n here denoting the value obtained by using the mean value of p. Although the two formulæ are not identical, practically there is no cogent reason for preferring one to the other.

Adopting the mean values, thus found, for p and n, we have three different values for m given by each set. Reverting to formula (121), if we give double weight to the value found from the intermediate term we get[1]

(152)......

$$m = \frac{-\Sigma_1^r(\log y_1 + \log y_2) + \Sigma_1^r(\log y_3 + \log y_4) - Mn\Sigma_1^r\{x^p(k^{2p}-1)(k^p+1)\}}{4\Sigma_1^r \log k}.$$

Mean values for p, n and m being to hand, we have for A four values from each set of ordinates, see (122) hereinbefore, the general formula being

$$(153)\ldots\ldots \log A = \log y_h - \log (k^{h-1}x) - Mn\,(k^{h-1}x)^p.$$

hence for a mean of $4r$ values of A we have

$$(153a)..\log A = \frac{1}{4r}\Big\{\Sigma_1^r \log (y_1 y_2 y_3 y_4) - 4m\Sigma_1^r (\log x + \tfrac{3}{2}\log k)$$
$$- Mn\,\Sigma_1^r[x^p(k^{3p} + k^{2p} + k^p + 1)]\Big\}$$

M denotes throughout 0.434..etc., if common logarithms are employed, or unity if Napierian.

Ignoring the coefficient A the first and second derivatives of the curve (149) are respectively

$$(154)\ldots\ dy/dx = x^{m-1}e^{nx^p}(m + npx^p);\ \text{and}$$

$$(155)..d^2y/dx^2 = x^{m-2}e^{nx^p}\{m(m-1) + npx^p(2m+p-1) + n^2p^2x^{2p}\}$$

hence the mode (maximum or minimum value) is given by

$$(156)\ldots\ldots\ldots\ x = (-m/np)^{\frac{1}{p}}$$

which becomes, for $p = 1$, simply $-m/n$. The point of inflection is given by solving the equation

$$(157)\ldots\ldots P^2 + P(2m + p - 1) + m(m - 1) = 0$$

in which P denotes npx^p; this gives:

$$(158)\ldots\ldots x_i = \left\{-\frac{2m + p - 1 \pm \sqrt{[4mp + (p-1)^2]}}{2np}\right\}^{\frac{1}{p}}$$

which, when $m = 1$ gives the value

$$(159)\ldots\ldots x_i = -\left(\frac{1+p}{np}\right)^{\frac{1}{p}} \text{ and also } x_m - (1/np)^{\frac{1}{p}}$$

for the abscissa of the mode.

[1] The principle indicated in the preceding note applies, viz., if $(a+b+c)/d$ equals $(\alpha + \beta + \gamma)/\delta$ approximately, then $(a + \alpha + b + \beta + c + \gamma)/(d + \delta)$ is sensibly the arithmetic mean.

The integral of a curve can take a number of forms as follows, viz :—

$$(160)..\int y dx = \int x^m e^{nx^p} dx =$$

$$\frac{x^{m+1}}{m+1}\left\{1 + \frac{nx^p(m+1)}{(m+p+1)\,1!} + .. + \frac{n^r x^{rp}(m+1)}{(m+rp+1)\,r!} + ..\text{etc.}\right\}; \text{ or}$$

$$(161)..\ \frac{x^{m+1}}{m+1} e^{nx}\left\{1 - \frac{npx^p}{m+p+1} + \frac{(npx^p)^2}{(m+p+1)(m+2p+1)} - ..\right.$$

$$\left. + \frac{(-1)^r\,(npx^p)^r}{(m+p+1)....(m+rp+1)} \pm \text{etc.}\right\}; \text{ or}$$

$$(162)..\ \frac{x^{m-p+1}}{np} e^{nx^p}\left\{1 - \frac{m-p+1}{npx^p} + \frac{(m-p+1)(m-2p+1)}{(npx^p)^2} - ..\right.$$

$$\left. +(-1)^r\frac{(m-p+1)....(m-rp+1)}{(npx^p)^r} \pm \text{etc.}\right\}$$

Between the limits 0 and ∞ the integral may be put into the forms of the second Eulerian integral, and is

$$(163).....\int_0^\infty x^m e^{-nx^p} dx = \frac{\Gamma\left(\frac{m+1}{p}\right)}{\left(pn^{\frac{m+1}{p}}\right)}$$

which, when $m = o$, gives

$$(164).....\int_0^\infty e^{-nx^p} dx = \Gamma\left(\frac{1}{p}\right) / \left(pn^{\frac{1}{p}}\right)$$

The abscissa, x_c say, of the *centroid vertical*, or mean of the distribution, is

$$(165).....x_c = \frac{\int x^{m+1}\,e^{-nx^p} dx}{\int x^m\,e^{-nx^p} dx} = \frac{\Gamma\left(\frac{m+2}{p}\right)}{\Gamma\left(\frac{m+1}{p}\right) n^{\frac{1}{p}}}$$

It is sometimes necessary to make the definite integral (163) when multiplied by the coefficient A, equal to unity. In such a case we must have the value of this constant the reciprocal of that given in the value of the integral mentioned, viz. (163); that is

$$(166).....\mathrm{A} = pn^{\frac{m+1}{p}} \Big/ \Gamma\left(\frac{m+1}{p}\right)$$

Simplications of these general formulæ are often possible.[1]

[1] For a fuller study of this curve, see "Studies in Statistical Representation," by G. H. Knibbs. Jour. Roy. Soc. N.S.W., Vol. XLIV., pp. 341-367; 1910.

The forms of the curves are as shewn on the Figs. 21 to 27. If n in e^{nx^p} be zero, the curve degrades to Ax^m, and we have the forms in Fig. 21, in which the capital letters shew the curves when m is positive, and the small letters when m is negative.

Fig. 21.

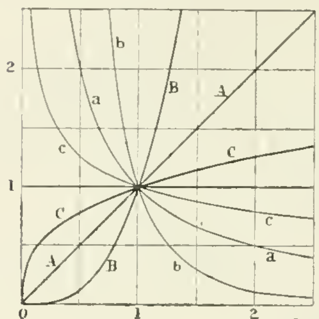
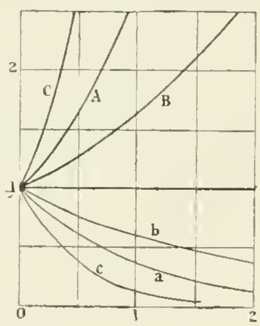


Fig. 22.



If m be zero, x^m will be unity, and if p also be unity, the curves become e^{nx}, the forms of which are shewn on Fig. 22, the upper lines denoting the values when n is positive and the lower when n is negative.

If p, however, be not unity, and p and n be positive, we shall have such forms as A, B, and C on Fig. 23. If p be negative and n positive, the forms become those shewn by the curves D, E, and F in the same figure.

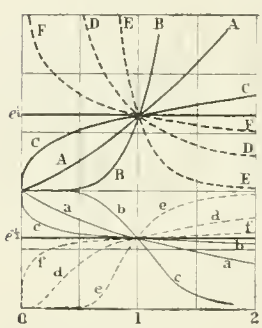


Fig. 23.

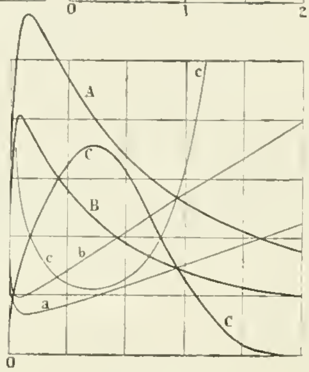


Fig. 24.

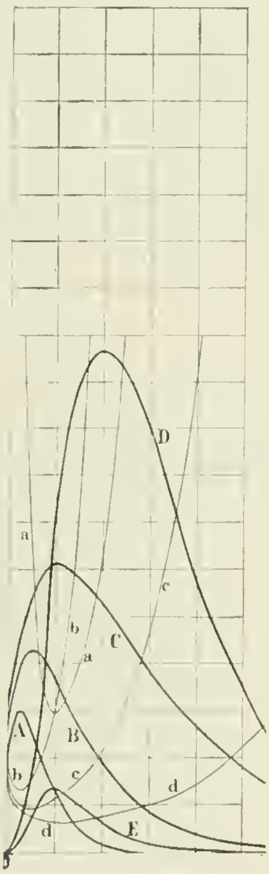
If n be negative and p be positive, the forms become a, b, and c, the reciprocals respectively of curves A, B, and C; and if both n and p be negative, the curves are such as d, e, and f, viz., the reciprocals respectively of curves D, E and F in the same figure, viz., Fig. 23.

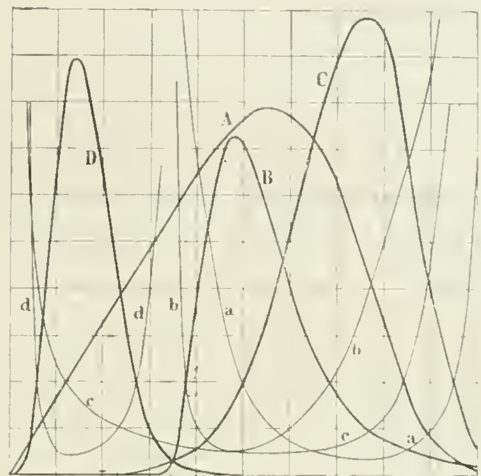
Figs. 24 to 27 give values of the curves when both m, n, and p have values other than zero, the light lines denoting the reciprocals of the curves shewn by the heavy lines, and the curves being the following, viz.:

			Values of—						Values of—		
			m	n	p				m	n	p
Fig. 24	..	A =	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{3}$	Fig. 26	..	A =	1	$-\frac{1}{4}$	6
,,	..	B =	$\frac{1}{2}$	$-\frac{1}{2}$	3	,,	..	B =	-6	$-\frac{1}{4}$	-6
,,	..	C =	$\frac{1}{2}$	-1	$\frac{1}{3}$	,,	..	C =	6	$-\frac{1}{4}$	6
Fig. 25	..	A =	$\frac{1}{2}$	-2	1	,,	..	D =	6	$-\frac{1}{4}$	1
,,	..	B =	$\frac{1}{2}$	-1	1	Fig. 27	..	A =	-1	$\frac{1}{4}$	6
,,	..	C =	$\frac{1}{2}$	$-\frac{1}{2}$	1	,,	..	B =	-6	$\frac{1}{4}$	6
,,	..	D =	2	-1	1	,,	..	C =	1	-1	1
,,	..	E =	2	-2	1	,,	..	D =	1	-2	1

In the reciprocal curves, viz., a, b, c, d, etc., the signs of m and n are changed, but not that of p.[1] These will sufficiently illustrate the possible forms of the curve.

Fig. 26.





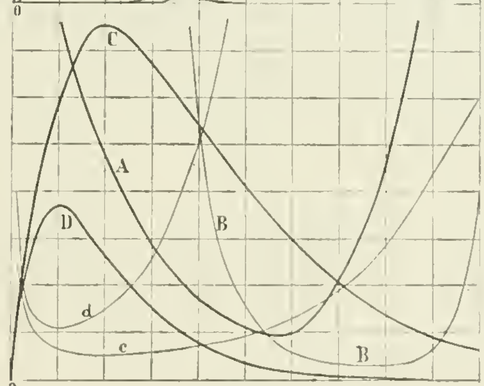


Fig. 25.

Fig. 27.

6. **Generalised probability curves derived from projections of normal curve.**=In Fig. 28 let bYa denote a normal "error" (or probability) curve, the ordinates of b and a being denoted by corresponding suffixes. If a line be drawn the distance l above OY and parallel thereto (and parallel therefore also to the plane of the curve), it may be represented by the point O′ in any plane at right angles to the plane of the curve.

[1] It may be mentioned that H. Pélabon, in dealing with the influence of temperature on chemical reactions, developed a relation in the form

$$\log y = a + b / x + c \log x :$$

which, of course, may be written in the form $y = ab^{1/x} x^c$, which is merely a simple case of formula (149). See Mém. d.l. Soc. des Sciences physiques et naturelles de Bordeaux [5]. 3, pp. 141, 257 ; 1898 : Compt. Rend. 124, pp. 35, 360, 686 ; 1897.

Let a line be drawn from any point, on the curve, viz., a, at right angles to O′. This will be the line O′Q, which, when produced to q on a line VOqU, making the angle θ with the line PO, gives the point corresponding to a. The abscissa then may be taken either as Oq or as its orthogonal projection on OP. The latter is more simple. If it be produced to q′ on a plane making the angle θ' with the axis OY, it will give a result of greater skewness, see the points a_1 and a_2 in the figure. The scheme of projection will be obvious from the figure, and need not be described in detail.

Let ξ denote any abscissa on the curve derived by projection, and x the corresponding abscissa on the original curve. Then by similar triangles we have at once the relation of x and ξ in terms of l and θ, inasmuch as

$$(167)\ldots\ldots x / l = (\xi - x) / \xi \tan\theta.$$

This gives, on writing m for $(\tan\theta) / l$,

$$(168)\ldots\ldots \xi = x / (1 - mx)\,;\quad x = \xi / (1 + m\xi)$$

from which it is at once evident that the same result may be obtained by any values of l and θ whatsoever, which give the same value of m.

Fig. 28.

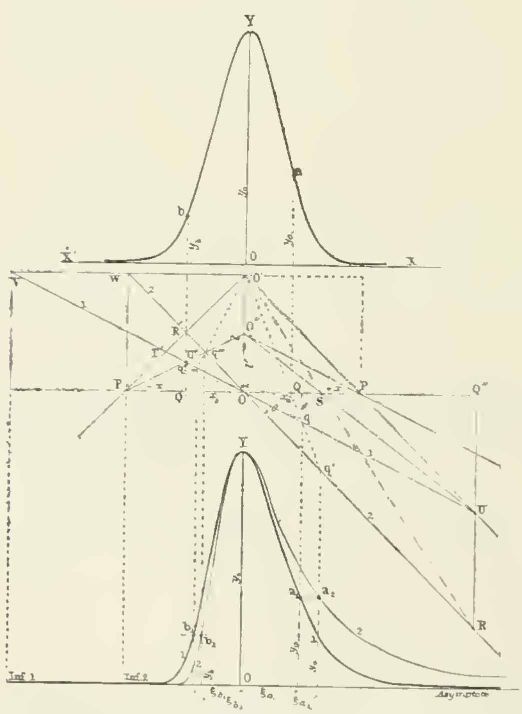


Fig. 29.

Thus, the point S with the projecting height OO″ = l' gives the point U, the orthogonal projection of which Q″ is identical with the result with the projecting height OO′ = l, viz., R, as is evident from the figure.

Fig. 29 shews by a heavy line the curve derived from the curve in Fig. 28 by projection on to the plane VOU, and by a thin line the curve similarly derived by projection on to the plane WOR in that figure.

Hence, if for x in the probability-curve equation, we substitute its numerical equivalent, we obtain

$$(169) \ldots\ldots y = 1\Big/e^{c + \frac{\xi^2}{2cm\xi + c\,\xi^2}} = 1\Big/e^{\frac{\xi^2}{\kappa(\xi+\mu)^2}};$$

κ in the second expression being cm^2, and μ being the reciprocal of m. The curve is asymmetric, since the denominator differs in value according as ξ is negative or positive. Incidentally we notice that if l be relatively large or θ relatively small, m is small, and the asymmetry is not marked; and when l is infinite or θ zero, the asymmetry vanishes, as is seen by the projection. In this last expression when ξ is negative and equal to μ, $y = 0$, so that there is a terminal of the curve on the negative side corresponding to $x = -\infty$. When $\tan\,\theta = l/x$, then $mx = 1$, and ξ is infinity; that is to say, the projecting line is parallel to the plane O through the axis. When $-\xi$ is one-half of $-x$, then the point with the same ordinate on the positive side is at infinity. This can also be seen on the figure.[1]

This indicates the limitation of the method of projecting onto a plane, namely, that if there is to be a corresponding point at a finite distance on one side of the axis, the abscissæ on the other side cannot be reduced to a greater amount than one-half. This, however, can be overcome by projection on a curved surface. Thus, if projected from the intersections with an equilateral hyperbola orthogonally on to the X axis, from a line parallel to and distant the height l from the Y axis, the Y axis O″H of the hyperbola being the distance p on the negative side, and the X axes, being identical (see Fig. 30) we have

$$(170) \ldots\ldots x = \frac{\xi}{1 - \lambda\,(p + \xi)}\ ;\ \text{or}\ \xi = x.\frac{1 - \lambda p}{1 + \lambda x}$$

λ denoting p^2/l. Hence, substituting the former expression in the ordinary probability curve equation, we obtain

$$(171) \ldots\ldots y = 1/e^{\frac{\xi^2}{c\,(1 - \lambda\,(p + \xi))^2}}$$

This gives a terminal to the curve on one side, and an asymptotic relation to the axis on the other, and may be made as skew as we please, as is evident from Fig. 30 and from Fig. 32 giving a projection so derived. A similar scheme of projection using a surface whose right section is a parabola, the abscissa of whose vertex is p (from the origin), and whose equation is $\eta = g\,(\xi - p)^2$, gives the result

$$(172) \ldots\ldots x = \frac{\xi}{1 + \gamma\,(\xi - p)^2}$$

in which γ denotes g/l: see Fig. 31. The value of ξ, therefore, is

$$(173) \ldots\ldots \xi = p + \frac{1}{2\gamma x}\,[1 \pm \sqrt{[4\gamma x\,(p - x) + 1]}\,];$$

[1] That is, when OQ′ is one-half of OP, the corresponding point on the positive side is at infinity.

This gives terminals for both branches of the curve, viz. :—

$$(174)\ldots\ldots\xi_x = p \pm \frac{1}{\sqrt{\gamma}}$$

Since both p and γ may be arbitrarily determined, the position of the terminals of the curve, in relation to the mode, may be made whatsoever we please. Although this leads to a somewhat complicated expression for ξ, it discloses the character of the curve obtained by projection. Its equation is

$$(175)\ldots\ldots y = 1/e^{\frac{\xi^2}{c\,[1+\gamma(\xi-p)^2]^2}}$$

the asymmetry of which is evident. Fig. 30 illustrates the projection on to a surface whose right section is an equilateral hyperbola, and the type of resultant curve with one asymptote is shewn on Fig. 32 : see curve a_1, b_1, . . a_1', b_1'. . ., thereon in a thin firm line, the thick curve A, B,. . . A′, B′. . . etc., being the probability curve from which it is derived. Fig. 31 is similarly an example of a projection on to a surface whose right section is a parabola, and is shewn on Fig. 32 by a broken line : see curve a_2, b_2. . . a_2', b_2'. . . etc. The scheme of projection is sufficiently evident from the figures.

Reverting to projection on a plane, it may be noted also the projections may be varied by making l a function of y instead of a constant, as, for example, $l = ky^n$, which, writing κ for $(\tan\theta)/k$, would give

$$(176)\ldots\ldots y = 1/e^{\frac{\xi^2}{c\,(1+\kappa\xi/y^n)^2}}$$

This does not lead, however, to any simple expression for y in terms of ξ only. We may notice that since $l = 0$ for $y = 0$, both branches are unlimited (that is to say, the asymptotic relation of the basic curve remains) and the curve is more distant from the X axis than is the basic curve ; the curve most closely approaches the type of that with l constant if n be less than unity. If n be negative and numerically greater than unity, we shall have ξ sensibly equal to x for very small values of y, or $x = \xi(1 - \kappa y^n)$ approximately, and the branches are unlimited.

These projections shew that though initially a *frequency* may be distributed according to the ordinary probability curve, yet the final circumstances may be such that the "frequency is altered in several of its characters," viz., its symmetry, asymptotic relations, etc.

Fig. 30.

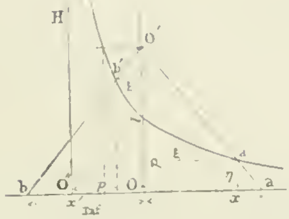
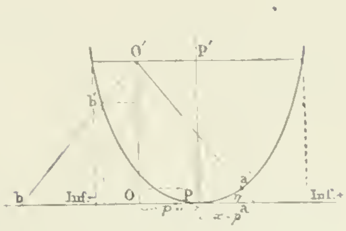


Fig. 31.



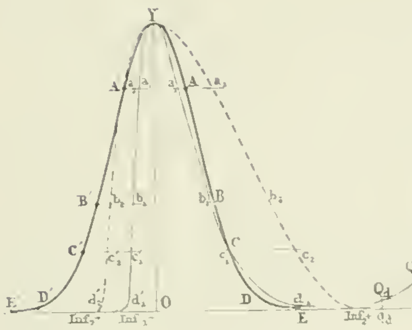


Fig. 32.

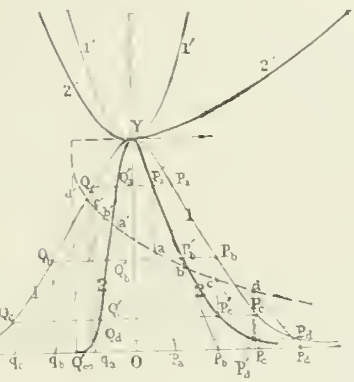


Fig. 33.

7. **Development of type-curves.**—A consideration of the form of the equations derived from projections shews that if we put as the fundamental form

$$(177)\ldots\ldots y, \text{ or } 1/y = y_0/e^{\frac{x^p}{k(1+x/a)^{ma}(1-x/b)^{mb}}}$$

we may include all cases by variations of p, m, k, a and b. When $x = 0$, the value is y_0, that is to say, the *mode is at the axis*. If a and b, each supposed to be positive, are finite, then for a negative value of x equal to a, or a positive value equal to b, we have $y = 0$, that is the branches of the curve terminate at the axis of abscissa for the negative value of $x = a$; and for the positive value of $x = b$. If b be infinite, the curve, which is skew, becomes

$$(178)\ldots\ldots y = y_0/e^{\frac{x^p}{k(1+x/a)^{ma}e^{-mx}}}$$

and if a be infinite and b finite, the curve is skew, and its equation is

$$(179)\ldots\ldots y = y_0/e^{\frac{x^p}{ke^{mx}(1-x/b)^{mb}}}$$

If both a and b are infinite, then the preceding curves (178) (179) become

$$(180)\ldots\ldots y = y_0/e^{\frac{x^p}{k}}$$

and is symmetric, but if b (or a) be negative, then the curve is

$$(181)\ldots\ldots y = y_0/e^{\frac{x^p}{ke^{nx}}}$$

This curve is asymmetric[1] and both branches are asymptotic to the axis. The reciprocals of these curves give the other forms required.

8. **Evaluation of the constants of the preceding type-curves.**—The value of y_0 is assumed to be derived from the data. When all the quantities are divided by the ordinate of the mode, viz., by y_0, we have a series of *reduced values of the ordinates*, η say. Then, as a rule, by taking the logarithm twice we can obtain the necessary solution. Thus—

$$(182)\ldots\eta = e^{f(x)}\text{; hence } \log\eta = \eta' = f(x)\text{; and } \log\eta' = \log f(x)$$

which gives a linear equation. Thus, with the necessary number of values of the ordinate and the corresponding values of the abscissa, a solution of the constants is to hand. If more than the necessary number are given, the least-square method of forming normal equations may be employed. This method will not solve, however (177), (178), or (179), where

$$(183)\ldots\log\eta' = p\log x - \left\{\log k + ma\log\left(1+\frac{x}{a}\right) + mb\log\left(1+\frac{x}{b}\right)\right\}$$

These, however, are very readily solved by expanding the logarithms, and sometimes a and b can be estimated from the graph of the curve.

9. **To determine the surface on which the projection of a normal probability-curve will result in a given skew-curve.**—From what has preceded, and from Figs. 28 to 32, it is evident that the form and equation of the curved surface, on which the projection of a normal probability-curve will furnish any given skew curve, may readily be determined. The problem more generally stated is :—Given two curves to find the surface on which the projection of one will furnish the other. On Fig. 33 let $\mathrm{Y}\ldots\mathrm{P_d}$ and $\mathrm{Y}\ldots\mathrm{Q_d}$ be the branches of a normal probability curve, and $\mathrm{YQ'_a}\ldots\mathrm{Q'_\infty}$, and $\mathrm{YP'_a}\ldots P'_\mathrm{d}\ldots$ be the branches of a skew-curve, the axis OY being identical for each. Draw radial lines from Y to the orthogonal projections on to the X axis of various points on the normal probability curve, viz., to the points $\mathrm{q_a}$, $\mathrm{q_b}$, etc., and $\mathrm{p_a}$, $\mathrm{p_b}$, etc., and from the points $\mathrm{Q'_a}$, etc., $\mathrm{P'_a}$, etc., whose ordinates to the skew-curve are identical with those of the corresponding points on the normal curve ; and draw lines parallel to the axis OY. Then the intersections a, b, etc., a′, b′, etc., are points on the projection surface. Reference to the figure

[1] p is to be understood merely as an *operator* raising the number in numerical value, but not affecting its sign.

makes the proposition obvious. Thus, the equation to the normal curve being known, that of the skew-curve can be found in the form $y = 1/e^{f(\xi)}$, as soon as the equation of the curve of the projection surface is ascertained.

In finding an equation to fit any series of groups the skew-curve may, in practice, be drawn freehand : a suitable normal probability-curve may then be drawn with the same mode and vertical height : the points on the surface found by the method indicated. In general, this will give a somewhat irregular projection-surface, which, however, may ordinarily be so modified as to conform to some geometrical form easily expressible algebraically, from which the requisite formula may then be found. From Figs. 30, 31, and 33 it will be evident how the equation may be ascertained.

10. **Reciprocals of curves of the probability-type.**—The curve $\eta = 1/y$, also of type of practical importance, may similarly be derived by projection from the normal probability-curve : thus

$$(184)\ldots\ldots\eta = 1/y = e^{\frac{x^2}{c}}, \text{ or more generally, } \eta = e^{\left(\frac{x}{k}\right)^p}$$

that is, its logarithmic homologue is the parabola $\eta' = \mu x^p$, in which $\eta' = \log \eta$, and $\mu = 1/k^p$. Thus in Fig. 33 the reciprocal of the normal probability-curve (curve 1) is shewn by the curve marked 1′, 1′, while the curve 2′, 2′, is the reciprocal of the curve marked 2, 2. The lateral scale in the figure, however, for curve 2, is four times greater than for curve 1. It will be seen that the type is somewhat similar to the curve of instantaneous rate of mortality according to age.

11. **Dissection of multimodal fluctuations into a series of unimodal elements.**—It is obvious that any multimodal fluctuation may be analysed into a series of unimodal elements ; for example, a series of the form

$$(185)\ldots\ldots y = A_0 + A_1 e^{\pm(x-a_1)/c} + \ldots A_r e^{\pm(x-a^r)/c^r} + \ldots$$

may, with a sufficient number of terms, be made to fit any continuous curve whatsoever to any assigned degree of accuracy.[1] There is no complete general solution of the problem, however, of dissection. We have already shewn that a dimorphic curve may be the sum or product of two monomorphic curves (see III., § 9, Projective anamorphosis). The difficulties of dissection, however, are not unduly great with graphic methods.

[1] See "Contributions to the Mathematical Theory of Evolution" (on the dissection of Asymmetrical and Symmetrical frequency curves, etc.), Prof. Karl Pearson, Phil. Trans., Vol. 185-A, pp. 71-110 ; 1894.

"Sui massime dello curve dimorfiche," Dr. F. de Helguero, Biometrika, Vol. III., pp. 84-98, 1904 ; and also his "Per la risoluzione dello curve dimorfiche." Biometrika, Vol. IV., pp. 230, 231 ; 1905-6.

"Sulla statura degli Italiani," R. Livi, Firenze, 1883. "Die naturliche Auslese beim Menschen," O. Ammon, Jena, 1893.

V.—GROUP-VALUES, THEIR ADJUSTMENT AND ANALYSIS.

1. **Group-values and their limitations.**—The data of population statistic are ordinarily given in the form of *group-values*. For example, in the age-distribution of a population the data are ordinarily in the form of the numbers of persons between the ages x and $x+k$, $x+k$ and $x+2k$, and so on, where k may be a month, a year, 5 years, 10 years, etc. Hence, when the number for any group of smaller limits is required, some curve must be assumed which will give the same group-values if the latter are to be regarded as correct.

In other words, if we suppose the numbers between the ages x and $x + dx$ to be $P\phi(x)\, dx$, then the number in the group between the ages x and $x + k$ is

$$(186) \ldots\ldots {}_xN_{x+k} = P \int_x^{x+k} \phi(x)\, dx$$

in which, if P denote the total population of all ages, the value of the integral between the limits 0 and the end of life, say 105 (or ∞) is necessarily unity. This is the fundamental conception of the use of group-values. Thus, omitting the coefficient P, the value of the integral between any limits, when its total value is unity, is the proportion of the whole population which lies between the limits in question.

When *group-values* are known to be *subject to error*, each group can be modified in amount so as to conform to some distribution regarded as more probable than that furnished by the crude data. Thus, if in the numbers according to age a census return gave for "ages last birthday" 29, 30 and 31, the numbers 20,000; 24,000; 18,000; we should know ordinarily that the number 24,000 was in excess, since the numbers must fall off as the ages increase unless immigration prevent. We deal primarily with the case where the groups are assumed to be correct; having either been corrected, or having been taken accurately.

2. **Adjustment of group-values.**—In cases where group-values are properly regarded as subject to appreciable error, they should either be first adjusted before the constants of mathematical formulæ representing them are determined, or the computation should be so effected as to automatically make the adjustment a minimum.

The limitations under which group-results are obtained are of two kinds. The results furnished may be either—

(*a*) actually subject to large errors; or

(*b*) insufficient in number to furnish a truly representative example.

For example, misapprehensions as to one's exact age must necessarily have the effect of causing numbers of persons to be attributed to the wrong age-group, thus diminishing some groups and increasing others.

A certain tendency to misstatement is confirmed by census-results, which reveal the fact that ages ending in 0 are characterised by excessively large numbers, and that the numbers for ages ending in 5 are also somewhat excessive, while the numbers for the adjoining years are in defect.

In the other case, limitations in the numbers available prevent one knowing exactly what would have been given had the numbers been indefinitely large. In these latter cases, however, it is often possible to surmise what the curve would have been had the numbers been large, and the actual data may be redistributed so as to conform therewith. In both instances the principle to be followed is that some groups should be so increased, while others should be so diminished as to conform to the most probable distribution which may, for convenience, be called the "*ideal distribution*."

In effecting these changes in the numbers furnished by the data for individual groups, the alterations should not only be as small as possible, but also the accumulation of the alterations (that is, their algebraic sum) should be alternately plus and minus, and should never become large in amount.

Various considerations may serve as a guide in effecting the alteration: for example, excluding the consideration of deliberate misstatement of age and tendency to uniform error in one direction, the number of cases in which the misstatement of age is one year only is, in general, larger than the number in which the misstatement is two years; and so on. Experience shews also that large positive errors are likely to be made for ages ending in 0: for example, 30, 40, 50, etc.; and lesser positive errors are likely to be made for ages ending in 5; for example, 35, 45, 55, etc., while errors of defect are to be expected in ages 29 and 31, etc., and 34 and 36, etc.

Adjustments are, as a rule, preferably made in the light of a full consideration of all the circumstances affecting the case, and not merely by purely mechanical or merely arithmetical methods.

A redistribution of values may be regarded as excellent when the curve giving the values of the groups is, in the nature of the case, probable, and when at all points it deviates from the successive values of the groups in such wise that the deviation is always relatively small, and the aggregate alternately plus and minus.

3. Representation of group values by equations with integral indices. Any curve representing a series of statistical data may be represented by the following expression, viz.,—

$$(187) \ldots\ldots y = a + bx^p + cx^q + dx^r + \text{etc.}$$

and, if p, q, r, etc., be not necessarily integral, with a small number of terms. Integrating this we shall have

$$(188) \ldots\ldots \int y\,dx = x\,(A + Bx^p + Cx^q + Dx^r + \text{etc.})$$

in which $A = a$; $B = \dfrac{b}{p+1}$; $C = \dfrac{c}{q+1}$; $D = \dfrac{d}{r+1}$; etc.

When p, q, r, etc., are the successive integers 1, 2, 3, we have for $x=0$, k, $2k$, $3k$, etc.

	Group.	Range of the Abscissæ.	Factors into numbers below a	$\frac{1}{2}bk$	$\frac{1}{3}ck^2$	$\frac{1}{4}dk^3$	$\frac{1}{5}ek^4$
(189)..	I.	$0 - k = k$	(1	1	1	1	1)
	II.	$k - 2k = k$	(1	3	7	15	31)
	III.	$2k - 3k = k$	(1	5	19	65	211)
	IV.	$3k - 4k = k$	(1	7	37	175	781)
	V.	$4k - 5k = k$	(1	9	61	369	2101)

It is easily seen that with integral indices, the above expression of $n+1$ groups can be fitted by an arbitrary equation of the nth degree. Denoting the *heights* of the groups by the small Roman letters i. to v., the heights being found by dividing the group-values by the base k, and the successive differences of height by h_0, h_1, etc., the simplest scheme of solution is to hand in the following series of equations, which are readily obtained by differencing and substitution.

4. **Formulæ depending on successive differences of group-heights.**—We give first formulae depending merely on the difference of heights, viz., the differences i.— 0 ; ii. — i. ; iii. — ii. ; etc. ; that is, if we denote the successive heights of the groups by

$$(190)\ldots\ldots h_0\ ;\ h_0 + h_1\ ;\ h_0 + h_1 + h_2\ ;\ \text{etc.,}$$

the successive differences of height will be

$$(191)\ldots\ldots h_0\ ;\ h_1\ ;\ h_2\ ;\ h_3\ :\ \text{etc.}$$

$h_0 =$ i., denoting the height of the first group from the X axis, see Fig. 36.[1]

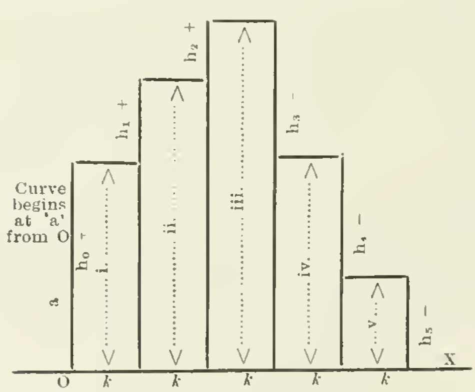


Fig. 34.

[1] These (191), are the first column of differences if the groups be divided by their base-values viz., by k.

The following, as convenient formulæ for the coefficients a, b, etc., in equation (187), can be deduced, viz. :—

For three groups :—

$$(192)..a = h_0 - \frac{1}{6}(5h_1 - 2h_2)\,;\ b = \frac{1}{k}(2h_1 - h_2)\,;\ c = \frac{1}{2k^2}(-h_1 + h_2)$$

For four groups :—

$$(193)......a = h_0 - \frac{1}{12}(13h_1 - 10h_2 + 3h_3)$$

$$(193a)....b = \frac{1}{12k}(35h_1 - 34h_2 + 11h_3)$$

$$(193b)....c = \frac{1}{4k^2}(-5h_1 + 8h_2 - 3h_3)$$

$$(193c)....d = \frac{1}{6k^3}(h_1 - 2h_2 + h_3).$$

For five groups :—

$$(194)\ ....a = h_0 - \frac{1}{2}h_1 + \frac{1}{60}(-47h_1 + 86h_2 - 51h_3 + 12h_4)$$

$$(194a)....b = \frac{1}{k}\left\{h_1 + \frac{1}{12}(33h_1 - 64h_2 + 41h_3 - 10h_4)\right\}$$

$$(194b)....c = \frac{1}{8k^2}(-17h_1 + 37h_2 - 27h_3 + 7h_4)$$

$$(194c)....d = \frac{1}{6k^3}(3h_1 - 8h_2 + 7h_3 - 2h_4)$$

$$(194d)....e = \frac{1}{24k^4}(-h_1 + 3h_2 - 3h_3 + h_4).$$

If instead of heights we use group-values, the quantities found, say a', b', c', etc., will be k times those above given, and must be reduced accordingly

5. **Formulæ depending on the group-heights themselves.**—Instead of using the difference of the group-heights, the coefficients of the equation may be expressed in terms of the successive group-heights themselves, found by dividing the group-numbers by the value of the common interval along the abscissa; that is, by dividing the integrals between the successive limits having a common interval k, by that quantity. It will be sufficient to give the results for from three to five groups. These results are :—

For three groups :—

$$(195)\ ......a = \frac{1}{6}(11\text{i.} - 7\text{ii.} + 2\text{iii.})$$

$$(195a)......b = \frac{1}{k}(-2\text{i.} + 3\text{ii.} - \text{iii.})$$

$$(195b)......c = \frac{1}{2k^2}(\text{i.} - 2\text{ii.} + \text{iii.})$$

For four groups :-

$$(196) \ldots\ldots a = \frac{1}{12}(25\text{i.} - 23\text{ii.} + 13\text{iii.} - 3\text{iv.})$$

$$(196\text{a}) \ldots\ldots b = \frac{1}{12k}(-35\text{i.} + 69\text{ii.} - 45\text{iii.} + 11\text{iv.})$$

$$(196\text{b}) \ldots\ldots c = \frac{1}{4k^2}(5\text{i.} - 13\text{ii.} + 11\text{iii.} - 3\text{iv.})$$

$$(196\text{c}) \ldots\ldots d = \frac{1}{6k^3}(-\text{i.} + 3\text{ii.} - 3\text{iii.} + \text{iv.})$$

For five groups :—

$$(197) \ldots\ldots a = \frac{1}{60}(137\text{i.} - 163\text{ii.} + 137\text{iii.} - 63\text{iv.} + 12\text{v.})$$

$$(197\text{a}) \ldots\ldots b = \frac{1}{12k}(-45\text{i.} + 109\text{ii.} - 105\text{iii.} + 51\text{iv.} - 10\text{v.})$$

$$(197\text{b}) \ldots\ldots c = \frac{1}{8k^2}(17\text{i.} - 54\text{ii.} + 64\text{iii.} - 34\text{iv.} + 7\text{v.})$$

$$(197\text{c}) \ldots\ldots d = \frac{1}{6k^3}(-3\text{i.} + 11\text{ii.} - 15\text{iii.} + 9\text{iv.} - 2\text{v.})$$

$$(197\text{d}) \ldots\ldots e = \frac{1}{24k^4}(\text{i.} - 4\text{ii.} + 6\text{iii.} - 4\text{iv.} + \text{v.})$$

If the *aggregate numbers* or *group-values* are used, instead of the *heights*, the denominators will be $1/k$, $1/k^2 \ldots\ldots 1/k^5$ instead of those above.

6. **Formulæ depending upon the leading differences in the groups or in group-heights.**—It is often convenient in practice to work with differences instead of the group-values or of heights. In the latter case the coefficients are similarly given by the following equations :—

The coefficients of equation (187) expressed in terms of successive leading differences of the group-heights are :—

$$(198) \ldots\ldots a = \frac{1}{1}\left(\text{i.} - \frac{1}{2}D_1 + \frac{1}{3}D_2 - \frac{1}{4}D_3 + \frac{1}{5}D_4\right)$$

$$(198\text{a}) \ldots\; b = \ldots\; \frac{1}{k}\left(D_1 - D_2 + \frac{11}{12}D_3 - \frac{5}{6}D_4\right)$$

$$(198\text{b}) \ldots\; c = \ldots\ldots\ldots\; \frac{1}{k^2}\left(\frac{1}{2}D_2 - \frac{3}{4}D_3 + \frac{7}{8}D_4\right)$$

$$(198\text{c}) \ldots\; d = \ldots\ldots\ldots\ldots\; \frac{1}{k^3}\left(+\frac{1}{6}D_3 - \frac{1}{3}D_4\right)$$

$$(198\text{d}) \ldots\; e = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\; \frac{1}{k^4}\left(\frac{1}{24}D_4\right)$$

In the above D_1, D_2, D_3, and D_4 are the *leading differences* of the *heights* only, viz., of i., ii.......v. As before, if the group values are subtracted, without first dividing by k, the denominators should be $1/k$, $1/k^2$......$1/k^5$, instead of those above given. Formulæ (198) to (198d) are correct for any number of groups up to five, the division lines on the right hand side shewing the results for two, three, four and five groups.

7. **Determination of differences for the construction of curves.** — When the equation of the curve is to hand, it is often required to find values of the ordinates corresponding to a series of values of the abscissa. This is most conveniently effected by obtaining the successive leading differences: from these the required values can be obtained. These are:—

$$(199)\ldots\ldots\ldots f(x) = a + bx + cx^2 + dx^3 + ex^4$$

$$(199a)\ldots\ldots D_1 f(0) = \ldots\; b \;+ c \;+ d \;+ e$$

$$(199b)\ldots\ldots D_2 f(0) = \ldots\ldots\ldots 2c \;+ 6d \;+ 14e$$

$$(199c)\ldots\ldots D_3 f(0) = \ldots\ldots\ldots\ldots\ldots 6d \;+ 36e$$

$$(199d)\ldots\ldots D_4 f(0) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots 24e$$

It may be remarked that when $k=1$ these difference values become

$$(200)\ \ldots\ldots D_1 f(0) = D_1\text{i.} - \frac{1}{2} D_2\text{i.} + \frac{1}{3} D_3\text{i.} - \frac{1}{4} D_4\text{i.}$$

$$(200a)\ldots\ldots D_2 f(0) = D_2\text{i.} - \frac{1}{2} D_3\text{i.} + \frac{1}{3} D_4\text{i.}$$

$$(200b)\ldots\ldots D_3 f(0) = D_3\text{i.} - \frac{1}{2} D_4\text{i.}$$

$$(200c)\ldots\ldots D_4 f(0) = D_4\text{i.}$$

in which the symbol D_1i., D_2i., etc., denotes the *leading differences* derived from the series from i., ii., iii., etc.

8. **Cases where position of curve on axis of ordinates has a fixed value.** —In the equation (187) it may happen that the curve is required to pass through the intersection O of the axes OX, OY; or at a fixed distance therefrom on the Y-axis. In this instance the solutions given are invalid, inasmuch as a is initially given, not determined from the group-values. The most convenient procedure is to subtract this value a from the heights i., ii., iii., etc., of the ordinates, or the value ka from the

group-values (or areas) I., II., III., etc. This procedure gives new values, viz., $y' = y - a$, and the solution required is then of the successive integrals (group values) divided by k.

$$(201)\ldots\ldots \frac{1}{k}\int y'dx = \frac{1}{k}\int(bx + cx^2 + \text{etc.})\, dx$$

that is, of $\frac{1}{2}bx + \frac{1}{3}cx^2 + \text{etc.}$

It is obvious that in this instance n groups will require an equation of the nth degree, instead of, as before, of the $(n - 1)$th, the imposed condition of a fixed value for a involving this limitation.

The following formulæ give the value of the constants in terms of the heights.

For two groups, curve passing through origin.

$$(202)\ldots\ldots b = \frac{1}{2k}(7\text{i} - \text{ii});\quad c = \frac{1}{4k^2}(-9\text{i} + 3\text{ii})$$

For three groups, curve passing through origin.

$$(203)\ldots\ldots b = \frac{1}{18k}(85\text{i} - 23\text{ii} + 4\text{iii})$$

$$(203\text{a})\ldots\ldots c = \frac{1}{2k^2}(-10\text{i} + 5\text{ii} - \text{iii})$$

$$(203\text{b})\ldots\ldots d = \frac{1}{9k^3}(11\text{i} - 7\text{ii} + 2\text{iii})$$

For four groups, curve passing through origin,

$$(204)\ldots\ldots b = \frac{1}{72k}(415\text{i} - 161\text{ii} + 55\text{iii} - 9\text{iv})$$

$$(204\text{a})\ldots\ldots c = \frac{1}{96k^2}(-755\text{i} + 493\text{ii} - 191\text{iii} + 33\text{iv})$$

$$(204\text{b})\ldots\ldots d = \frac{1}{36k^3}(119\text{i} - 97\text{ii} + 47\text{iii} - 9\text{iv})$$

$$(204\text{c})\ldots\ldots e = \frac{1}{288k^4}(-125\text{i} + 115\text{ii} - 65\text{iii} + 15\text{iv})$$

For five groups, curve passing through origin,

$$(205)\ldots b = \frac{1}{1800k}(12019\text{i} - 5981\text{ii} + 3019\text{iii} - 981\text{iv} + 144\text{v})$$

$$(205\text{a})\ldots c = \frac{1}{32k^2}(-343\text{i} + 273\text{ii} - 155\text{iii} + 53\text{iv} - 8\text{v})$$

$$(205\text{b})\ldots d = \frac{1}{360k^3}(2149\text{i} - 2111\text{ii} + 1429\text{iii} - 531\text{iv} + 84\text{v})$$

$$(205\text{c})\ldots e = \frac{1}{96k^4}(-133\text{i} + 147\text{ii} - 113\text{iii} + 47\text{iv} - 8\text{v})$$

$$(205\text{d})\ldots f = \frac{1}{1200k^5}(137\text{i} - 163\text{ii} + 137\text{iii} - 63\text{iv} + 12\text{v}.)$$

The constants in the terms of the leading differences of the heights are :—[1]

For two groups, curve passing through origin

$$(206) \ldots\ldots b = \frac{1}{k}\left(3\mathrm{i} - \frac{1}{2}D_1\mathrm{i}\right)$$

$$(206a) \ldots c = \frac{1}{k^2}\left(-\frac{3}{2}\mathrm{i} + \frac{3}{4}D_1\mathrm{i}\right)$$

For three groups, curve passing through origin,

$$(207) \ldots\ldots b = \frac{1}{k}\left(\frac{11}{3}\mathrm{i} - \frac{5}{6}D_1\mathrm{i} + \frac{2}{9}D_2\mathrm{i}\right)$$

$$(207a) \ldots c = \frac{1}{k^2}\left(-3\mathrm{i} + \frac{3}{2}D_1\mathrm{i} - \frac{1}{2}D_2\mathrm{i}\right)$$

$$(207b) \ldots d = \frac{1}{k^3}\left(\frac{2}{3}\mathrm{i} - \frac{1}{3}D_1\mathrm{i} + \frac{2}{9}D_2\mathrm{i}\right)$$

For four groups, curve passing through origin,

$$(208) \ldots\ldots b = \frac{1}{k}\left(\frac{25}{6}\mathrm{i} - \frac{13}{12}D_1\mathrm{i} + \frac{7}{18}D_2\mathrm{i} - \frac{1}{8}D_3\mathrm{i}\right)$$

$$(208a) \ldots c = \frac{1}{k^2}\left(-\frac{35}{8}\mathrm{i} + \frac{35}{16}D_1\mathrm{i} - \frac{23}{24}D_2\mathrm{i} + \frac{11}{32}D_3\mathrm{i}\right)$$

$$(208b) \ldots d = \frac{1}{k^3}\left(\frac{5}{3}\mathrm{i} - \frac{5}{6}D_1\mathrm{i} + \frac{5}{9}D_2\mathrm{i} - \frac{1}{4}D_3\mathrm{i}\right)$$

$$(208c) \ldots e = \frac{1}{k^4}\left(-\frac{5}{24}\mathrm{i} + \frac{5}{48}D_1\mathrm{i} - \frac{5}{72}D_2\mathrm{i} + \frac{5}{96}D_3\mathrm{i}\right)$$

For five groups, curve passing through origin,

$$(209) \ldots b = \frac{1}{k}\left(\frac{137}{30}\mathrm{i} - \frac{77}{60}D_1\mathrm{i} + \frac{47}{90}D_2\mathrm{i} - \frac{9}{40}D_3\mathrm{i} + \frac{2}{25}D_4\mathrm{i}\right)$$

$$(209a) \ldots c = \frac{1}{k^2}\left(-\frac{45}{8}\mathrm{i} + \frac{45}{16}D_1\mathrm{i} - \frac{11}{8}D_2\mathrm{i} + \frac{21}{32}D_3\mathrm{i} - \frac{1}{4}D_4\mathrm{i}\right)$$

$$(209b) \ldots d = \frac{1}{k^3}\left(\frac{17}{6}\mathrm{i} - \frac{17}{12}D_1\mathrm{i} + \frac{17}{18}D_2\mathrm{i} - \frac{13}{24}D_3\mathrm{i} + \frac{7}{30}D_4\mathrm{i}\right)$$

$$(209c) \ldots e = \frac{1}{k^4}\left(-\frac{5}{8}\mathrm{i} + \frac{5}{16}D_1\mathrm{i} - \frac{5}{24}D_2\mathrm{i} + \frac{5}{32}D_3\mathrm{i} - \frac{1}{12}D_4\mathrm{i}\right)$$

$$(209d) \ldots f = \frac{1}{k^5}\left(\frac{1}{20}\mathrm{i} - \frac{1}{40}D_1\mathrm{i} + \frac{1}{60}D_2\mathrm{i} - \frac{1}{80}D_3\mathrm{i} + \frac{1}{100}D_4\mathrm{i}\right)$$

[1] i denotes the height of the first group-result; D_1i = ii − i; D_2i = iii − 2ii − i; D_3i = iv − 3iii + 3iii − i; etc.; that is, they are the *leading* differences.

9. **Determination of group-values when constants are known.**—When the equation is in the form (187), p, q, r, etc., being 1, 2, 3, etc., the most ready way to compute a series of values of groups 0 to k, k to $2k$, $2k$ to $3k$, etc., is to form the leading differences, and from these the successive values of the groups can be readily formed. The following formulæ give the required result :—[1]

$$(210)\ldots\quad \mathrm{I.} = ak + \frac{1}{2}bk^2 + \frac{1}{3}ck^3 + \frac{1}{4}dk^4 + \frac{1}{5}ek^5 + \frac{1}{6}fk^6$$

$$(210a)\ \ldots D_1\mathrm{I.} = \ldots\ldots bk^2 + 2ck^3 + 3\tfrac{1}{2}dk^4 + 6ek^5 + 10\tfrac{1}{3}fk^6$$

$$(210b)\ \ldots D_2\mathrm{I.} = \ldots\ldots\ldots\ldots 2ck^3 + 9dk^4 + 30ek^5 + 90fk^6$$

$$(210c)\ \ldots D_3\mathrm{I.} = \ldots\ldots\ldots\ldots\ldots\ldots 6dk^4 + 48ek^5 + 260fk^6$$

$$(210d)\ \ldots D_4\mathrm{I.} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots 24ek^5 + 300fk^6$$

$$(210e)\ \ldots D_5\mathrm{I.} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots 120fk^6$$

When the equation is of a less degree than the fifth, zeros can be substituted for the coefficients ; thus for a fourth degree, $f=0$; for a third degree $f=0$ and $e=0$; and so on ; and the formulæ still hold good.

10. **Curve of group-totals for equal intervals of the variable expressed as an integral function of the central value of the interval.**—If we have a series of group-totals for equal intervals of the abscissa, as, for example, for 0 to k, k to $2k$, etc., and if those values divided by the common interval are represented by the ordinates at $\frac{1}{2}k$, $1\frac{1}{2}k$, $2\frac{1}{2}k$, etc., to a curve the equation of which is an integral function of the type of formula (187), then, whatever be the value of x in this equation, the ordinate for the point x will give very approximately the group-total for $x - \frac{1}{2}k$ to $x+\frac{1}{2}k$. That is to say, denoting the ordinate to the curve representing the groups $x \pm \frac{1}{2}k$ by Y, and that to the curve representing the original function by y, if

$$(211)\ldots\ldots\ Y = F\left(x + \frac{1}{2}k\right) = y\int_x^{x+k} dx = \int_x^{x+k} f(x)\,dx$$

for the values $x=0$, 1, 2, etc., then it follows that very approximately

$$(212)\ldots\ldots\ F\left(x + \frac{1}{2}k + q\right) = \int_{x+q}^{x+k+q} f(x)\,dx$$

provided that the forms of F and f are the same, that is, that they are both integral functions of a single variable. This result is important, and may be established by the following consideration.

If we compute $F(x) = \int f(x)\,dx$ so that the two are in agreement for $x=\frac{1}{2}k$, $1\frac{1}{2}k$ and $2\frac{1}{2}k$, in the first function, with the limits 0 to k, k to $2k$, and $2k$ to $3k$ in the second, then it is easy to establish that if the original

[1] D_1I., D_2I., etc., denote the series of *leading* differences, viz. (II. – I.); (III. – 2II. + I.); (IV. – 3III. + 3II. – I.); etc.

equation be $a+bx+cx^2$, and if the equation for the group-total, divided by the common interval, be $A+Bx+Cx^2$, when x is the value of the abscissa for the middle of the interval, then

$$(213)\ldots\ldots A = a + \frac{1}{12}ck^2;\quad B = b;\quad C = c.$$

If we extend the solution to the third power of x, that is, extend the limits to $3\frac{1}{2}k$ and $3k$ to $4k$ respectively, we have

$$(214)\ldots\ldots A = a + \frac{1}{12}ck^2;\quad B = b + \frac{1}{4}dk^2;\quad C = c;\quad D = d.$$

If we further extend the solution to the fourth power of x, and the limits to $4\frac{1}{2}k$ and $4k$ to $5k$ respectively, we obtain

$$(215)\ldots A=a+\frac{1}{12}ck^2+\frac{1}{80}ek^4;\ B=b+\frac{1}{4}dk^2;\ C=c+\frac{1}{2}ek;\ D=d;\ E=e.$$

If the fifth power of the variable be included, that is, the limits be $5\frac{1}{2}k$ and $5k$ to $6k$ respectively, then

$$(216)\ldots A=a+\frac{1}{12}ck^2+\frac{1}{80}ek^4;\ B=b+\frac{1}{4}dk^2+\frac{1}{16}fk^4;\ C=c+\frac{1}{2}ek^2;$$
$$D = d + \frac{5}{6}fk^2;\quad E = e;\quad F = f.$$

It will be observed that up to the second power of the variable, the effect is that A differs from a only by a constant, consequently the function F gives rigorously the correct result, viz., that given by integrating the function f. For powers higher than the second, the result is true only for $k=\frac{1}{2}, 1\frac{1}{2}$, etc., in F, and for any other values is more or less in error. This error cannot, in general, however, attain appreciable magnitude, because it is repeatedly reduced to zero at intervals of k, viz., at the values of the abscissa, $\frac{1}{2}k$, $1\frac{1}{2}k$, etc.

In practical statistical examples the coefficients b, c, d, e, f, etc., are generally in diminishing order of magnitude, and we see from the equations (213) to (216) that the corresponding numerical factors also rapidly diminish; hence the difference between the rigorous value $\int f(x)dx$ and the approximate value $F(x)$ must generally be very small, and, by the formulæ given, can be readily tested in any numerical examples.

11. **Average values of groups.**—An average value y_r of a group is the quantity

$$(217)\ldots\ldots y_r = \frac{1}{x_2 - x_1}\int_{x_1}^{x_2} y\,dx$$

in which y denotes the value of the ordinate, and x_1 to x_2, the range of the variable. Reverting to formulæ (187) and (188), and retaining the same meaning for the constants, the mean value of the range x to $x+k$ is

$$(218)\ldots y_r = A + \frac{1}{k}[B'\{(x+k)^{p+1} - x^{p+1}\} + C'\{(x+k)^{q+1} - x^{q+1}\} + \text{etc.}]$$

which takes a simpler form if p, q, r, etc., are 1, 2, 3, etc. Where x has a series of values 0, k, $2k$, etc., as in (189) the averages are given by omitting the factor k in the formulæ. More generally, that is, for any value of x and k we have

$$(219)\ldots\ldots y_r = a + b\left(x + \frac{k}{2}\right) + c\,(x^2 + xk + \frac{1}{3}k^2) +$$
$$d\,(x^3 + 1\tfrac{1}{2}x^2k + xk^2 + \frac{1}{4}k^3) +$$
$$e\,(x^4 + 2x^3k + 2x^2k^2 + xk^3 + \frac{1}{5}k^4)$$

For groups bounded by curves of the exponential type we may note that

$$(220)\ldots\ldots a^x = e^{x \log a} = e^{mx}$$

Thus, the rate of change at any point of the curve $y = ne^{mx}$ is

$$(221)\ldots\ldots \frac{dy}{dx} = d\,(ne^{mx})/dx = mne^{mx}$$

and the mean rate y_m is

$$(222)\ldots\ldots y_m = mne^{mx}.\frac{e^{mk}-1}{mk}$$

that is, this is the mean ordinate to the curve.

If the ordinates for the beginning, middle, and end of any range of values of the abscissa, that is, if the ordinates corresponding to the values x, $x+\frac{1}{2}k$, and $x+k$, are to hand, and the group-values are the integral of an equation of the type (199), then the value of y_r is

$$(223)..y_r = \frac{1}{6}(y_x + 4y_m + y_{x+k}) - \frac{1}{24}k^4\left\{\frac{1}{5}e + f(x+\frac{1}{2}k) + 3g\,(x^2 + xk + \frac{23}{84}k^2) + \text{etc.}\right\}$$

The negative term (in braces) is absolutely negative, x being positive, if e, f and g are positive, and it is usually so small as to be negligible. When $x=0$ and $k=1$, the value of (223) takes the very simple form

$$(224)\ldots\ldots y_r = \frac{1}{6}(y_0 + 4y_m + y_k) - \frac{1}{24}\left(\frac{1}{5}e + \frac{1}{2}f + \frac{23}{28}g\right)$$

y_m denotes, of course, the middle ordinate.

This result is important, because it shews that group-values can be calculated with considerable precision by the "prismoidal formula" if we have middle as well as terminal instantaneous values of each group.

VI.—SUMMATION AND INTEGRATION FOR STATISTICAL AGGREGATES.

1. **General.**—In effecting statistical summations, regard is to be had to two elements, viz.:—

(i.) Order of accuracy significant in the case in point ;

(ii.) Arithmetical consistency of results.

Curves drawn freehand among data, that represent either groups or instantaneous results, and which shew *visible* variations, can, for some purposes, be integrated with sufficient precision by careful graphing and the use of a planimeter.[1] When arithmetical smoothing has followed graphic, in order to enhance the accuracy, numerical calculations are virtually required as being of corresponding precision. As a rule group values (or the total area between any ordinates, the curve, and the axis of abscissæ) can, if the ordinates are relatively near each other, be computed by means of the prismoidal, Simpson's, Weddle's and similar rules. Finally, for work of the highest precision, actual integrations by the method of the infinitesimal calculus are required. In general, however, the precision then far transcends that of the data.

The extension of *implied* precision far beyond that of the data is seen in all actuarial tables : this matter is referred to later, since the year change in probability of life is a quantity of a much larger order than that to which results are expressed.

2. **Areal and volumetric summation formulæ.**—Statistics relating to population involve both *areal and volumetric summations*. The latter can, however, always be represented by an areal graph. If the curve represent instantaneous and not group-values[2] about a particular value of the variable, then the *areal* value can be computed without computing the equation of the curve and integrating it.

It has been shewn[3] that if an axis be equally divided, that is, if $x=0,\ k,\ 2k \ldots . nk$, and the curve passing through the terminals of the ordinates (y) from these points is assumed to be represented by an integral function of x, then suitable multipliers or weights may be determined, which, applied to the ordinates, will give the area. If there be an

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[1] Amsler's Integrator will cover a considerable area, and gives in the one operation (on four cylinders and discs) the values of following integrals, viz.:—

$$\int y dx\ ;\ \ \tfrac{1}{2}\int y^2 dx\ ;\ \ \tfrac{1}{3}\int y^3 dx\ ;\ \ \tfrac{1}{4}\int y^4 dx$$

that is, the *area*, the statical moment, the moment of inertia, and the cubic moment about the axis x. No mechanical integrator, however, can possibly approximate to the precision attainable by arithmetic.

[2] That is, represents the frequency y, for a given value x of the variable and not the group-mean for $x-\frac{1}{2}k$ to $x+\frac{1}{2}k$. See V., 10.—Curves representing group-totals, formulæ (211) to (216).

[3] See "Volumes of solids as related to transverse sections," by G. H. Knibbs, Journ. Roy. Soc. N.S.W., Vol. XXXIV., pp. 36-71, 1900. See Prop. (O), p. 70.

odd number of equidistant ordinates the curve may be of the same degree as the number of ordinates, viz., $(n+1)$; if the number of ordinates be *even*, the degree of the curve must be one less than that number (n).

It has been shewn also that if the curve bounding the area is of a *less* degree than that satisfied by the number of ordinates, then there is one-fold, two-fold, ... k-fold infinity of multipliers which will exactly give the area, according as the degree of the curve is 1, 2, ... k less than the number of ordinates.[1]

The formulæ can be readily constructed, and are exhibited in the table hereunder.[2] The significance of this table may be indicated as follows :—

When $n+1$ equidistant ordinates are given for a curve of the nth degree, there is only one system of weights that will give the integral correctly between the limits 0 and n. In the table this system is indicated in each case above by an asterisk (*). Further, when n is even, the unique series of weights, applicable to $n+1$ equidistant ordinates, is also applicable to a curve of the $(n+1)$th degree, but this is not true when n is odd.

When $n+2$ equidistant ordinates are given for a curve of the nth degree, any value whatever may be assigned to one of the weights (say w_0), and the corresponding values of the other weights may be expressed in terms of w_0. In this case there is evidently an infinite number of possible systems of weights, each of which will give the integral accurately for a curve of the nth degree. In the foregoing table the systems of this nature are indicated by a dagger (†), the coefficient (*i.e.*, 1) of the arbitrarily selected weight being shewn in heavy type. As an example, there may be taken the case in which seven equidistant ordinates of a fifth degree curve are given. Here the weightings shewn by the table are $w_0 = w_0$; $w_1 = 3.3 - 6w_0$; $w_2 = -4.2 + 15w_0$; $w_3 = 7.8 - 20w_0$; $w_4 = -4.2 + 15w_0$; $w_5 = 3.3 - 6w_0$; $w_6 = w_0$.

If w_0 be given the value 0.3 this series becomes $\frac{3}{10}\left(1, 5, 1, 6, 1, 5, 1\right)$, which will be recognised as Weddle's rule.

Similarly, when $n+3$ equidistant ordinates are given for a curve of nth degree, two weights may be arbitrarily selected and the remaining $n+1$ may be computed in terms thereof, thus admitting of a two-fold infinity of systems of weighting. In the foregoing table systems of this nature are indicated by a double dagger (‡).

Similarly, when $n+4$ ordinates are given for a curve of the nth degree there is a three-fold infinity of systems, when $n+5$ ordinates are given, a four-fold infinity, or, in general, when r ordinates are given for a curve of the nth degree there is an $(r-n-1)$-fold infinity of systems of weighting.

[1] Ibid, § 16, pp. 60-71. Examples of the development of k-fold infinity of multipliers are given on pp. 64-67.

[2] Prepared by Mr. C. H. Wickens, A.I.A.

TABLE VI.

Weights to be applied to a Series of Equidistant Ordinates, $y_0, y_1, \ldots y_n$, in order to determine the Value of the Integral $\int_0^{x_n} y\,dx$, that is, between the Limits covered by the Ordinates.

Series of Weights for Ordinates with same Suffix	Value of the Weights when the curve passing through their Terminals is a function of the— 2nd degree.	3rd degree.	4th degree.	5th degree.	6th degree.	7th degree.
3 Ordinates w_0 w_1 w_2	* $\left.\begin{matrix}1\\4\\1\end{matrix}\right\}\cdot\frac{1}{3}$	$\left.\begin{matrix}1\\4\\1\end{matrix}\right\}\frac{1}{3}$				
4 Ordinates w_0 w_1 w_2 w_3	† $\begin{matrix}\\ 9\\ 0\\ 3\end{matrix}\left.\begin{matrix}\\ \\ \\ \end{matrix}\right\}\frac{1}{4}\quad\begin{matrix}\mathbf{1}\\-3\\3\\-1\end{matrix}$	* $\left.\begin{matrix}1\\3\\3\\1\end{matrix}\right\}\cdot\frac{3}{8}$				
5 Ordinates w_0 w_1 w_2 w_3 w_4	‡ $\left.\begin{matrix}20\\-16\\8\end{matrix}\right\}\frac{1}{3}\quad\begin{matrix}\mathbf{1}&\\&\mathbf{1}\\-6&-3\\8&3\\-3&-1\end{matrix}$	† $\left.\begin{matrix}8\\-4\\8\end{matrix}\right\}\cdot\frac{1}{3}\quad\begin{matrix}\mathbf{1}\\-4\\6\\-4\\1\end{matrix}$	* $\left.\begin{matrix}7\\32\\12\\32\\7\end{matrix}\right\}\frac{2}{45}$	$\left.\begin{matrix}7\\32\\12\\32\\7\end{matrix}\right\}\times\frac{2}{45}$		
6 Ordinates w_0 w_1 w_2 w_3 w_4 w_5	$\left.\begin{matrix}175\\-200\\85\end{matrix}\right\}\times\frac{1}{12}\quad\begin{matrix}\mathbf{1}&&\\&\mathbf{1}&\\&&\mathbf{1}\\-10&-6&-3\\15&8&3\\-6&-3&-1\end{matrix}$	‡ $\left.\begin{matrix}225\\-325\\275\\-55\end{matrix}\right\}\times\frac{1}{24}\quad\begin{matrix}\mathbf{1}&\\&\mathbf{1}\\-10&-4\\20&6\\-15&-4\\4&1\end{matrix}$	† $\left.\begin{matrix}425\\-350\\600\\-50\\95\end{matrix}\right\}\times\frac{1}{144}\quad\begin{matrix}\mathbf{1}\\-5\\10\\-10\\5\\-1\end{matrix}$	* $\left.\begin{matrix}19\\75\\50\\50\\75\\19\end{matrix}\right\}\cdot\frac{5}{288}$		
7 Ordinates w_0 w_1 w_2 w_3 w_4 w_5 w_6	$\left.\begin{matrix}27\\-36\\15\end{matrix}\right\}1\quad\begin{matrix}\mathbf{1}&&&\\&\mathbf{1}&&\\&&\mathbf{1}&\\&&&\mathbf{1}\\-15&-10&-6&-3\\24&15&8&3\\-10&-6&-3&-1\end{matrix}$	$\left.\begin{matrix}24\\-45\\36\\-9\end{matrix}\right\}\times 1\quad\begin{matrix}\mathbf{1}&&\\&\mathbf{1}&\\&&\mathbf{1}\\-20&-10&-4\\45&20&6\\-36&-15&-4\\10&4&1\end{matrix}$	‡ $\left.\begin{matrix}123\\-252\\288\\-132\\33\end{matrix}\right\}\times\frac{1}{10}\quad\begin{matrix}\mathbf{1}&\\&\mathbf{1}\\-15&-5\\40&10\\-45&-10\\24&5\\-5&-1\end{matrix}$	† $\left.\begin{matrix}33\\-42\\78\\-42\\33\end{matrix}\right\}\frac{1}{10}\quad\begin{matrix}\mathbf{1}\\-6\\15\\-20\\15\\-6\\1\end{matrix}$	* $\left.\begin{matrix}41\\216\\27\\272\\27\\216\\41\end{matrix}\right\}\times\frac{1}{140}$	$\left.\begin{matrix}41\\216\\27\\272\\27\\216\\41\end{matrix}\right\}\times\frac{1}{140}$

The significance of the *, †, and ‡ is referred to in the text.

3. **The value of groups in terms of ordinates.**—It is often convenient to ascertain the value of groups between certain limits of a variable. If the ordinates be supposed to conform to the equation $a+bx$; or $a+bx+cx^2$, etc., etc., we can construct a series of equations which are rigorously true under the particular supposition, and may be regarded as approximations in the general case. By comparing the expression for the integral between assigned limits with the values of the ordinates, we deduce the following expressions for the *heights of the groups* in terms of the ordinates.

TABLE VII.

Values of Group Heights for Different Ranges of the Variable in Terms of the Ordinates to the Curve.

1st Approximation. Formulæ (225) to (228).

Ranges of Integral	0-$\frac{1}{2}$	$\frac{1}{2}$-1	1-$1\frac{1}{2}$	$1\frac{1}{2}$-2
Semi-group-heights	$\frac{1}{4}(3y_0+y_1)$;	$\frac{1}{4}(y_0+3y_1)$;	$\frac{1}{4}(-y_0+5y_1)$;	$\frac{1}{4}(-3y_0+7y_1)$

2nd Approximation. Formulæ (229) to (232).

Ranges of Integral	0-$\frac{1}{2}$	$\frac{1}{2}$-1
Semi-group-heights	$\frac{1}{12}(8y_0+5y_1-y_2)$;	$\frac{1}{12}(2y_0+11y_1-y_2)$
Ranges of Integral	1-$1\frac{1}{2}$	$1\frac{1}{2}$-2
Semi-group-heights	$\frac{1}{12}(-y_0+11y_1+2y_2)$;	$\frac{1}{12}(-y_0+5y_1+8y_2)$

3rd Approximation. Formulæ (233) to (236).

Ranges of Integral	0-$\frac{1}{2}$	$\frac{1}{2}$-1
Semi-group-heights	$\frac{1}{192}(119y_0+107y_1-43y_2+9y_3)$;	$\frac{1}{192}(25y_0+197y_1-37y_2+7y_3)$
Ranges of Integral	1-$1\frac{1}{2}$	$1\frac{1}{2}$-2
Semi-group-heights	$\frac{1}{192}(-9y_0+155y_1+53y_2-7y_3)$;	$\frac{1}{192}(-7y_0+53y_1+155y_2-9y_3)$

4th Approximation. Formulæ (237) to (240).

Ranges of Integral. Semi-group-heights.

$$(237)\ldots\ldots\ 0\text{–}\tfrac{1}{2} = \frac{1}{2880}(1694y_0+1969y_1-1191y_2+499y_3-91y_4)$$

$$(238)\ldots\ldots\ \tfrac{1}{2}\text{–}1 = \frac{1}{2880}(314y_0+3199y_1-921y_2+349y_3-61y_4)$$

$$(239)\ldots\ldots\ 1\text{–}1\tfrac{1}{2} = \frac{1}{2880}(-91y_0+2149y_1+1059y_2-281y_3+44y_4)$$

$$(240)\ldots\ldots\ 1\tfrac{1}{2}\text{–}2 = \frac{1}{2880}(-61y_0+619y_1+2589y_2-311y_3+44y_4)$$

1st Approximation. Formulæ (241) to (243).

Ranges of Integral	0–1	$\frac{1}{2}$–$1\frac{1}{2}$	1–2
Group-heights	$\frac{1}{2}(y_0+y_1)$	y_1	$\frac{1}{2}(-y_0+3y_1)$

2nd Approximation. Formulæ (244) to (246).

Ranges of Integral	0–1	$\frac{1}{2}$–$1\frac{1}{2}$	1–2
Group-heights	$\frac{1}{12}(5y_0+8y_1-y_2)$;	$\frac{1}{24}(y_0+22y_1+y_2)$;	$\frac{1}{12}(-y_0+8y_1+5y_2)$

3rd Approximation. Formulæ (247) to (249).

Ranges of Integral. Group-heights.

(247)........0–1 $= \frac{1}{24}(9y_0+19y_1-5y_2+y_3)$

(248)......$\frac{1}{2}$–$1\frac{1}{2}$ $= \frac{1}{24}(y_0+22y_1+y_2+0y_3)$

(249)........1–2 $= \frac{1}{24}(-y_0+13y_1+13y_2-y_3)$.

4th Approximation. Formulæ (250) to (252).

Ranges of Integral. Group-heights.

(250)...... 0–1 $= \frac{1}{720}(251y_0+646y_1-264y_2+106y_3-19y_4)$

(251)...... $\frac{1}{2}$–$1\frac{1}{2}$ $= \frac{1}{5760}(223y_0+5348y_1+138y_2+68y_3-17y_4)$

(252)......$1\frac{1}{2}$–2 $= \frac{1}{720}(-19y_0+346y_1+456y_2-74y_3+11y_4)$

In applying these formulæ the actual common-range of the interval on the axis of abscissæ is immaterial; that is, we may read throughout 0 to $\frac{1}{2}k$; $\frac{1}{2}k$ to k etc.; instead of 0 to $\frac{1}{2}$; $\frac{1}{2}$ to 1; etc.; the ordinates y_0, y_1, etc., being taken of course 0, k, $2k$, $3k$, etc. By these formulæ, therefore, we may halve groups.

It will be noticed that the coefficients are always symmetrically opposed for semi-groups standing in the same relation to the ordinates; for example, with two ordinates, 0 to $\frac{1}{2}$ is the same form as $\frac{1}{2}$ to 1; with three, 0 to $\frac{1}{2}$ agrees with $1\frac{1}{2}$ to 2, and $\frac{1}{2}$ to 1 with 1 to $1\frac{1}{2}$; with four ordinates, the only symmetrically opposed pair are 1 to $1\frac{1}{2}$ and $1\frac{1}{2}$ to 2. From this it is evident that, for the third and fourth approximations the formulæ for the remaining group-heights within the limits of the ordinates

used can be written down by inspection. Thus for the 3rd and 4th approximations the group-heights of the various semi-groups are as follow :—

3rd Approximation.

The ordinates for the semi-group			$2\frac{1}{2}$ to 3	are the inverse of those for		0 to $\frac{1}{2}$
,,	,,	,,	2 to $2\frac{1}{2}$	,,	,, ,,	$\frac{1}{2}$ to 1
,,	,,	,,	$1\frac{1}{2}$ to 2	,,	,, ,,	1 to $1\frac{1}{2}$

(as already given).

4th Approximation.

The ordinates for the semi-group			$3\frac{1}{2}$ to 4	are the inverse of those for		0 to $\frac{1}{2}$
,,	,,	,,	3 to $3\frac{1}{2}$	,,	,, ,,	$\frac{1}{2}$ to 1
,,	,,	,,	$2\frac{1}{2}$ to 3	,,	,, ,,	1 to $1\frac{1}{2}$
,,	,,	,,	2 to $2\frac{1}{2}$	,,	,, ,,	$1\frac{1}{2}$ to 2

4. **The value of group-subdivisions in terms of groups.**—It is often required to divide a group. Practically we may always halve a group and halve again if necessary. If we divide groups with a common interval (k) on the axis of abscissæ we may, with advantage, use the *group-height* (g) instead of the group number G; that is, we may use $g=G/k$. Then we obtain the following series of formulæ, which, like the last, are rigorously accurate if the groups are given by the integrals of the equation $a+bx$; $a+bx+cx^2$; etc.; etc. They may therefore be regarded, as in the previous instance, either as a series of approximations, or as rigorously accurate, according as they represent exactly or approximately the subdivisions of groups given by the integral equations referred to.

TABLE VIII.

Values of Group-heights for different half-ranges of the variable in terms of the heights of successive whole groups.

1st Approximation. Formulæ (253) to (256).

Ranges of integral	$0-\frac{1}{2}$	$\frac{1}{2}-1$	$1-1\frac{1}{2}$	$1\frac{1}{2}-2$
Semi-group-heights	$\frac{1}{4}(5g_1-g_2)$;	$\frac{1}{4}(3g_1+g_2)$;	$\frac{1}{4}(g_1+3g_2)$;	$\frac{1}{4}(-g_1+5g_2)$

2nd Approximation. Formulæ (257) to (260).

Ranges of integral	$0-\frac{1}{2}$	$\frac{1}{2}-1$
Semi-group-heights	$\frac{1}{8}(11g_1-4g_2+g_3)$;	$\frac{1}{8}(5g_1+4g_2-g_3)$
Ranges of integral	$1-1\frac{1}{2}$	$1\frac{1}{2}-2$
Semi-group-heights	$\frac{1}{8}(g_1+8g_2-g_3)$;	$\frac{1}{8}(-g_1+8g_2+g_3)$.

3rd Approximation. Formulæ (261) to (264).

Ranges of integral	$0-\frac{1}{2}$	$\frac{1}{2}$ to 1
Semi-group-heights	$\frac{1}{64}(93g_1 - 47g_2 + 23g_3 - 5g_4)$;	$\frac{1}{64}(35g_1 + 47g_2 - 23g_3 + 5g_4)$
Ranges of integral	$1-1\frac{1}{2}$	$1\frac{1}{2}-2$
Semi-group-heights	$\frac{1}{64}(5g_1 + 73g_2 - 17g_3 + 3g_4)$;	$\frac{1}{64}(-5g_1 + 55g_2 + 17g_3 - 3g_4)$

4th Approximation. Formulæ (265) to (268).

Ranges of integral. Semi-group-heights.

$$(265)\ldots\ldots\ 0-\tfrac{1}{2} = \frac{1}{128}(193g_1 - 122g_2 + 88g_3 - 38g_4 + 7g_5)$$

$$(266)\ldots\ldots\ \tfrac{1}{2}-1 = \frac{1}{128}(63g_1 + 122g_2 - 88g_3 + 38g_4 - 7g_5)$$

$$(267)\ldots\ldots\ 1-1\tfrac{1}{2} = \frac{1}{128}(7g_1 + 158g_2 - 52g_3 + 18g_4 - 3g_5)$$

$$(268)\ldots\ldots\ 1\tfrac{1}{2}-2 = \frac{1}{128}(-7g_1 + 98g_2 + 52g_3 - 18g_4 + 3g_5)$$

The opposite symmetry of the coefficients for semi-groups in symmetrically opposed positions, having regard to the total number of groups in question, is obvious, as in the case for ordinates. The same remarks apply, *mutatis mutandis*, as those made regarding the coefficients of the ordinates.

5. **Approximate computation of various moments.**—In connection with the application of the method of moments in statistical investigations of distribution (population and other) it is often necessary to compute moments from available data. This can also be done from the available ordinates in the following manner :—

It is obvious that the curved boundary of any group, covering a limited range of the variable, can be represented with considerable precision by a curve of the second degree : see V., § 11, formulæ (217) to (224). Let the *group-height* be denoted by g, that is, let g denote the group-area divided by k, that is, the group-range on the axis of abscissæ. If y' and y'' are the ordinates to the curve for $a-\frac{1}{2}k$ and $a+\frac{1}{2}k$ respectively, and y_a be the central ordinate, viz., at the distance a from the intersection of the axes, and if h be the distance of the mean of the terminals y' and y'' from the terminal of this central ordinate, that is, if $n=y_a-\frac{1}{2}(y'+y'')$, then the group-height is given by the equation

$$(269)\ldots\ldots g=\frac{1}{2}(y'+y'')+\frac{2}{3}h=\frac{1}{6}(y'+4y_a+y'')$$

and the equation to the curve is—

$$(270)\ldots\ldots y=y_a+\frac{y''-y'}{k}(x-a)+\frac{4h}{k^2}(x-a)^2=y_a+b\,(x-a)+c\,(x-a)^2$$

the origin being at the distance a from the ordinate y_a. This curve is regarded as valid only for the group to which it applies, and not for adjoining groups. From this last equation we can compute the successive moments, M_0 denoting the area, M_1 the statical moment, M_2 the moment of inertia, and M_3 the moment of the fourth order.

It is important to attend to the signs of b and c. If $y'' - y'$ is positive, that is, if the ordinate is increasing in the direction of $a + \frac{1}{2}k$, then b is plus; and c is plus if the curve is convex upward: that is, if h is positive.

Thus the several moments are:—

$$(271)\ldots\ldots M_0 = k\left(y_a + \frac{1}{12}ck^2\right) = k\left(y_a + \frac{1}{3}h\right)$$

$$(272)\ldots\ldots M_1 - aM_0 = \frac{1}{12}bk^3 = \frac{1}{12}k^2(y'' - y')$$

$$(273)\ldots\ldots M_2 - 2aM_1 + a^2M_0 = \frac{1}{60}k^3(5y_a + 3h)$$

$$(274)\ldots\ldots M_3 - 3aM_2 + 3a^2M_1 - a^3M_0 = \frac{1}{80}k^4(y'' - y')$$

and may be very readily computed from these formulæ, which are rigorously exact on the supposition made, and will be sensibly correct generally.

6. **Statistical integrations.**—Ordinarily, statistical data are subject to considerable error and uncertainty, and meticulous precision in regard thereto is, therefore, usually unmeaning. The approximations of statistical technique itself, should, however, aim at a somewhat higher order of accuracy than that characteristic of the data, in order that the error should not prejudicially accumulate through mere computational vitiation. The great majority of cases of integration occurring in ordinary statistical practice will be found to have been solved. Valuable tables of integrals are available.[1]

[1] (i.) Sammlung von Formeln der reinen und angewandten Mathematik. W. Láska, Braunschweig, 1888-1894, pp. 1-1071.

(ii.) Tafeln unbestimmter Integrale. G. Petit-Bois, Leipzig, 1906.

(iii.) Een Aanhangsel tot de Tafels van onbepaalde Integralen. D. Bierens de Haan.

(iv.) Exposé de la théorie des propriétés, des formules de transformation, et des méthodes d'évaluation des Intégrales définies, partie 1, pp. 1-82; partie 2, pp. 83-181; partie 3, pp. 183-698. Bierens de Haan, Amsterdam, 1860.

(v.) Nouvelles Tables d'intégrales définies. Bierens de Haan, 5 parties, pp. 1-733. Engels, Leide, 1867.

The integrals of curves of the type of (20), II., § 19, are sometimes required: that is,—

$$(275)\ldots\ldots \int a\,(bx)^{\pm m \pm nbx}\,dx = \frac{a}{b}\int y^{\pm m \pm ny}\,dy = A\int y^{m} e^{ny \log y}\,dy$$
$$= A\int e^{(\pm m \pm ny)\log y}\,dy$$

in which $A = a/b$; and $y = bx$. This last form may be expressed by an exponential series. Or

$$(276)\ldots \int y^{m+ny}\,dy = \int y^{m}\left\{1 + ny \log y + \frac{1}{2}(ny \log y)^2 + \ldots\right\} dy$$

which may be integrated term by term. Again

$$(277)\ldots \int x^{nx}dx = x\left\{1 - \frac{nx}{2^2} + \frac{n^2x^2}{3^3} - \frac{n^3x^3}{4^4} + \ldots\right\} + \frac{nx^2 \log x}{1\,!}\left\{\frac{1}{2} - \frac{nx}{3^2} + \frac{n^2x^2}{4^3} - \ldots\right\}$$
$$+ \frac{n^2x^3(\log x)^2}{2\,!}\left\{\frac{1}{3} - \frac{nx}{4^2} + \frac{n^2x^2}{5^3} + \ldots\right\} + \text{etc.}$$

Similarly, forms of the type of formula ($32a$), see II., § 23

$$(278)\ldots\ldots \int x^{\frac{1}{(\pm m \pm nx)^p}}\,dx = \int e^{\frac{\log x}{(\pm m \pm nx)^p}}\,dx$$

can, if m and n be regarded as positive, be put in the form

$$(279)\ldots \int\left\{1 + \frac{\log x}{(m+nx)^p} + \frac{(\log x)^2}{2!\,(m+nx)^{2p}} + \ldots + \frac{(\log x)^k}{k!\,(m+nx)^{kp}} + \ldots\right\}dx$$

which can be integrated term by term. The integrals, however, are tedious. For example:—

$$(280)\ldots \int \frac{\log x}{(m+nx)^p}\,dx = \frac{\log x}{(p-1)n\phi^{p-1}} + \frac{1}{(p-1)mn}\left\{\frac{1}{(p-2)\phi^{p-2}} + \right.$$
$$\left.\frac{1}{(p-3)\,m\phi^{p-3}} + \frac{1}{(p-4)m^2\phi^{p-4}} + \ldots + \frac{1}{2m^{p-4}\phi^2} + \frac{1}{m^{p-5}\phi}\right\}$$
$$+ \frac{1}{(p-1)m^{p-1}n}\log\frac{x}{\phi}$$

ϕ denoting $(m+nx)^p$.

If $p = 1$, and n is positive, this takes the simpler form—

$$(281)\ldots \int x^{\frac{1}{m+nx}} = \frac{1}{n}\log x \log(m+nx) - \frac{1}{2n}(\log nx)^2 \quad \frac{m}{n^2x}$$
$$- \frac{m^2}{2^2n^3x^2} + \frac{m^3}{3^2n^4x^3} + \text{etc.}$$

Owing to the very great elaboration of the terms of many of the integrals, practically it is preferable to compute a sufficient number of ordinates, and integrate by any suitable summation-formula (given hereinbefore).

7. **The Eulerian integrals or Beta and Gamma functions.**—The Beta and Gamma functions are of special importance in statistical integrations. They are :—

(282)......

$$\int_0^1 x^{l-1}(1-x)^{m-1}dx = \int_0^1 z^{m-1}(1-z)^{l-1} = \int_0^\infty \frac{y^{l-1}\,dy}{(1+y)^{l+m}} = \int_0^\infty \frac{y^{m-1}\,dy}{(1+y)^{l+m}}$$

that is, in the more brief notation—

$$(283)\ldots\ldots \mathrm{B}\,(l, m) = \mathrm{B}\,(m, l) = \frac{\Gamma(l)\,\Gamma(m)}{\Gamma(l+m)}$$

Further—

$$(284)\ldots\int_0^\infty e^{-x}\,x^{n-1}\,dx = \int_0^1 \left(\log\frac{1}{y}\right)^{n-1} dy = (n-1)\int_0^\infty e^{-x}\,x^{n-2}\,dx = \Gamma(n)$$

respectively, from which it is evident that :—[1]

$$(285)\ldots\ldots\Gamma(1) = 1 ;\quad \Gamma(n+1) = n\,! = n\,\Gamma(n)$$

Thus, in order to calculate $\Gamma(n)$ we have, if it be an integer, it is equal to $(n-1)\,!$, if not an integer, it can be readily found, since its logarithms have been tabulated for the range 1 to 2 to two places of decimals and to 9 places of figures.[2] Thus—

$$(286)\ldots\ldots n\,(n+1)\,(n+2)\ldots(n+k-1)\,\Gamma(n) = \Gamma(n+k)$$

which, logarithmically, is perfectly convenient to use.

By putting $kz = x$, in (284), it becomes obvious that

$$(287)\ldots\ldots\int_0^\infty e^{-kx}\,x^{n-1}\,dx = \frac{\Gamma(n)}{k^n}$$

$$(288)\ldots\ldots\frac{d}{dn}\Gamma(n) = \Gamma'(n) = \int_0^\infty e^{-x}\,x^{n-1}\,\log x\,dx.$$

Examples of the application of these formulæ have already been given : see IV., § 5, formulæ (150) to (166).

8. **Table of indefinite and definite integrals and limits.**—In an addendum small tables are given, for convenience, of indefinite integrals ; of definite integrals, for example, between limits such as zero and unity ; zero and infinity ; etc., and of limiting values. These embrace those which more frequently occur in statistical investigations.

[1] $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.

[2] Traité des Fonctions Elliptiques, Legendre. Paris 1825-8 (logarithms to 12 places). Sammlung von Formeln, W. Láska, pp. 290-1. Braunschweig (logarithms to 9 places). Biometrika, J. H. Duffell, Vol. VII., 1909-10, pp. 43-7 (logarithms to 7 places).

VII.—THE PLACE OF GRAPHICS AND SMOOTHING, IN THE ANALYSIS OF POPULATION-STATISTICS.

1. **General.**—Graphs of the data are necessary in any analysis of population-statistics purporting to aim at thoroughness. A graph indicates not only the general trend of the data, but also whether the individual items conform with great exactitude to that trend, or whether they deviate considerably therefrom. The criticism of deviations ordinarily depends upon whether *numbers* or *ratios* are being analysed. Where figures are of the nature of ratios, if, on the working-graphs the numbers be written, it is possible to see at a glance whether changes in any part of the graph of the crude data are significant or otherwise. Thus a ratio resulting from 30,000 divided by 10,000 would be materially changed so far as the numerical data are concerned by an alteration, say, of one-thousandth. To change the ratio say from 3 to 2.997 would mean an alteration of 30 in the numerator or of 10 in the denominator; whereas, if the original data were the numbers 3 and 1, an alteration of a single unit would greatly disturb the ratio.

In general, we are concerned with two kinds of alteration; one may be called the "*redistribution of the data without alteration of their aggregate*;" and the other may be called the "*alteration of data to coincide with what is deemed the most probable result,*" having regard to all the facts. It is, for example, sometimes desirable to keep the aggregate of the smoothed results identical with that of the data. In other cases this is less essential, and it may be said that probably much time is often wasted in making re-distributed data agree with the original as to the aggregate of units represented.

As to general method it may be noted that when the original facts have been plotted, a curve may be drawn freehand by anyone familiar with the characteristics of the various type-curves, and especially those of probability-curves. By means of sets of curves, French curves, and splines of various kinds,* the freehand curves may then be improved so as to be really smooth and conform to what might be called the probable indication of the data. When the numbers represented are large, limitations of scale may operate to limit the smoothness as deduced by scaled values, from the graphs, but a little simple *differencing* will suggest necessary adjustments, or the differences may be graphed. The adjustments having been made, the aggregate can be formed by adding together the scaled or properly differenced ordinates thus adjusted.

If this operation has been well done the total will be so nearly in agreement with the original data that a common *factor of correction* can be used throughout, that is, all the ordinates may be increased or diminished in the same ratio, and the finally deduced ordinates will then agree

* Splines of transparent celluloid are most convenient.

with the data, and at the same time form a smooth curve. If the data when plotted are visibly irregular, meticulous precision in adjustment is obviously but a waste of time. For this reason one of the great merits of the graphic method is that, not only can the analyst see at a glance the conformity or otherwise of the data to a particular type of curve, but he can also judge whether the data yield results of a high order of precision.

It has already been mentioned (see IV., § 1) that the initial and terminal characters of the curve and its mode (maximum and minimum) are important. It may be added, that if the curve is not drawn as unimodal in type, the reason for the adoption of a particular form must really depend on the character of the data, and may not be decided merely upon mathematical considerations.

2. **The theory of smoothing statistical data.**—It may often be known *a priori* that phenomena should exhibit a regular progression, and that data, when graphed, shewing as zig-zag lines, do not really represent the ideal fact, owing either to the paucity of the data, or to unavoidable error therein.

In a series of *group-values, i.e.*, totals or aggregates between a series of limits of a variable, it is important to bear in mind that—assuming the counts on which they depend to be correct—what is *known* is merely the series of aggregates themselves: the probable distribution yielding these aggregates has to be *conjectured*. When the totals or aggregates are themselves regarded as subject to error, then the distribution may be modified within the limits of probable uncertainty, some groups being diminished and others, particularly adjoining ones, increased.

There are four principal classes of data to which the process of curve-smoothing is applicable. These may be indicated as follows:—

(i.) Frequencies of a phenomenon at successive epochs or during successive periods of time; as, for example, population estimates at given dates and numbers of deaths occurring during successive years.

(ii.) Rates of occurrence of a phenomenon per unit of reference during successive periods; as, for example, birth-rates per thousand of population per annum for successive years.

(iii.) Frequencies in respect of successive values of characters capable of continuous variation; as, for example, the number of persons at each age recorded at a given census.

(iv.) Rates of occurrence of a phenomenon per unit of reference in respect of successive values of characters susceptible of continuous variation; as, for example, rates of mortality per unit per annum during a given decennium in respect of each age.

In all these cases the characteristic of continuous variation[1] is assumed to exist either actually or virtually. Where statistical results are *discontinuous* such a process is, strictly speaking, inapplicable; as, for

[1] See I., § 9.

example, in the tabulation of census population according to birthplace, occupation, or religion. In some cases, however, although the data are strictly speaking discontinuous, the principle may be applied partially; for example, in the case of a tabulation of dwellings according to number of rooms or according to number of inmates. In such cases the character possessed is *progressive without being continuous*; nevertheless, with proper qualifications, the smoothing principle may be applied even to these.

Another example, more nearly approaching but not attaining continuous variation, is the representation of dwellings according to rental value.

3. **Object of smoothing.**—From the foregoing it will be seen that the data to which the smoothing process is strictly applicable are those which may be regarded as functions of a continuous variable. But whether such functions are readily expressible by means of algebraic formulæ or not, is, of course, really immaterial. The essence of the matter is that in any instance the data are in the main such as admit of representation by means of a continuous line, or a continuous surface or solid in relation to continuous units of reference. When such representation has been made of the crude results of observation, it is ordinarily found that the line surface or solid exhibits evidences of marked irregularities as between adjacent points or series of points, their general trend, however, suggesting an underlying basis of orderly progression. This progression is, of course, affected by minor influences operating at individual points, and is more or less masked by the paucity of the data on which the representation has been based; thus, suggesting further that were it possible to obtain data of unlimited extent, these irregularities would become negligible. For this reason the object of the smoothing process may be said to be that of removing these apparently accidental irregularities, and of thus disclosing the basic or ideal uniformity which may be presumed to represent the facts in all their generality.

4. **Justification for smoothing process.**—The justifications for the smoothing process may thus be said to be :—

(*a*) That the irregularity does not represent the phenomenon in its generality, since much of the observed irregularity is known *a priori* to be due only to paucity of data ;

(*b*) or that it is known that the phenomenon subject to observation is really regular ;

(*c*) or, again, that the observed data suggest that regularity of trend will not efficiently represent them.

It has been objected that any system of smoothing is, strictly speaking, unwarrantable, since such a process virtually attempts to make the facts accord with more or less questionable preconceptions regarding them. To this view it may be rejoined that if the process were such as to produce results which, though smooth, differed systematically and materially in their distribution from the original observations, the objection would be valid. Where, however, due consideration is given to the

relative magnitudes of the original data, and the smoothed results accord therewith as closely as the data will allow when these exhibit a general trend, then the only preconception that can be regarded as operative is the justifiable one that ordinarily natural phenomena do not progress *per saltum*. In this connection it must be noted that where there is distinct evidence at any stage of a cataclysmic disturbance of results, the smoothing process for such points or periods will usually be invalid or not properly applicable. Examples of such cataclysmic disturbances of statistical data are war, famine, pestilence, earthquake, etc. Even in these cases, however, it appears admissible under certain circumstances to apply a smoothing process; as, for example, in cases where the disturbances referred to are of more or less frequent occurrence, and are not merely isolated instances.

One of the most cogent justifications for the smoothing process has its warrant in the fact that the recorded results of any statistical observations are necessarily approximative, and hence that the value of the function *recorded* for any given value of the variable is probably not usually more accurate than an estimate based on the recorded values in respect of preceding and succeeding values of the variable. This consideration suggests the idea of weighting successive observations to obtain most probable values, which idea forms the basis of one of the leading methods of adjustment. Again, where the results of the observations are to be employed as guides to future action, it is clear that these results should, as far as practicable, be freed from all fluctuations which may be considered merely accidental, and thus unlikely to be reproduced in future experience. This is of considerable importance in connection with the construction of mortality and sickness, superannuation, and similar tables to be used in the computation of rates of premium, and for the conduct of valuations.

5. **Mode of application of smoothing processes.**—It has already been indicated that one of the main objects of the smoothing process is the discovery of a smooth series which presumably underlies the irregular data furnished by a limited number of observations, and it has been implied that a process to be justifiable must, in addition to smoothness, be characterised also by what has been called "goodness of fit"; that is, within reasonable limits it must reproduce the *characteristic* features of the original data.

The methods of applying the smoothing process which have up to the present been employed, may conveniently be grouped in three classes, viz.:—(*a*) Graphic Methods; (*b*) Summation Methods; and (*c*) Methods of Functional Conformity.

These methods have been employed in connection with observations in many fields of research; as, for example, general statistics, actuarial science, physics and chemistry, astronomy, tidal theory, biology, etc. In the actuarial field, an extensive and systematic use of the process has been made, and a most detailed examination of the underlying principles has been carried out.

(*a*) **Graphic method.**—As its name indicates, this method is based on the attainment of the desired smoothness by means of a graphical representation and adjustment of the observed data. For example, the subject of observation being the infantile mortality experienced in a community during a given period, and the periods of observation being calendar years, a base line is taken and divided into equal parts, each of which represents a year. On these parts as bases a series of rectangles is constructed, the area of each rectangle being proportional to the rate of infantile mortality averaged for the corresponding year. The upper parts of these rectangles will present in the case supposed the appearance of flights of steps with uniform treads and unequal rises. The necessary smoothing may be effected by drawing a continuous free-hand curve through the upper portions of these rectangles in such a manner as to include *between certain limits* the same area *approximately* as is contained in the rectangles covering the same range.[1] The area enclosed by the part of the base line relating to any year, the ordinates drawn from the extremities of this part, and the portion of the curve between these ordinates will represent the smoothed result for the year under review. Whether, as in the example just given, the data should be represented by areas, or, as is sometimes more suitable, by ordinates, is a matter which is determined agreeably to the appropriate interpretation of the result to be attained. It may be noted that the method of representation by rectangular areas is specially applicable to cases where the data are functions not of single values of the variable, but of *ranges* of such values. For instance, in the above example, the rate of infantile mortality stated for any year is a function not of any one point of time in that year, but of the range of values representing the whole of the year. In most cases, however, the system of representation by means of ordinates would be equally valid, and sometimes more convenient.[2] Referring again to the above example, from a point on the base line representing the end of each year an ordinate could be drawn representing the rate of infantile mortality for that year, and a free-hand curve being drawn amongst the upper points of these ordinates, the ordinate to any point on the curve would represent the rate of infantile mortality for the year ending on the date corresponding to the foot of the ordinate. Similarly, the ordinate for smoothing might be drawn from the beginning or the middle of the line for each year, or, indeed, from any point uniformly selected in each, and a corresponding interpretation of any point taken on the curves drawn amongst the upper points of such ordinates would be applicable.

6. **On smoothing by differencing.**—A curve continually convex (or continually concave) upward might possibly be drawn with a single difference. We have, by the theory of differentiation—

$$(289). \; dy/dx = d\,(a+bx^p+cx^q+\text{etc.})/dx = pbx^{p-1}+qcx^{q-1}+\text{etc.};$$

[1] In practical examples it is rarely possible to make the curve such that the adjusted areas are continually identical with the rectangles on the same base.

[2] See, however, V., § 10, formulæ (212) to (216).

hence, if p, or q, etc., should happen to be *integers*, at some stage of differentiation, this particular term of the expression will be $x^0=1$, and hence that difference will vanish. Probably in no case are population-statistical results actually representable by integral values of p, q, etc., hence, strictly, there is no limit to the series of differences. These, however, ultimately become high negative powers of x, and consequently when x is large their value is small: they must ultimately become of negligible amount.

Again, statistical data often involve exponential forms, particularly those of the type ae^{-nx}, the differential of which is $-nae^{-nx}$, from which it is evident the successive differences are interminable. Since, however, $de^{-x}/dx=1/e^x$, the higher differences for large values of x become insensible. Hence, we shall always be justified in taking differences only to the stage where they are appreciable. Thus if at any stage of smoothing we make the second difference a constant, we are making the curve one which the equation $y=a+bx+cx^2$ will reproduce; if we go on then with a constant third difference, we add a stretch of a new curve, viz., $y'=a'+b'x+c'x^2+d'x^3$; and so on. Such methods are unobjectionable when the tangents to the curve at the point of junction may be regarded as sensibly identical.

7. **Effect of changing the magnitude of the differences.**—It is often useful to be able to recognise instantly the consequence of changing the magnitude of a difference. This can be indicated at once by a table.

Table IX.—Effect on the value of a function of a change of a unit in a leading difference.

Difference in which the change takes place.	Effect on the value of y where its suffix is—										
	0	1	2	3	4	5	6	7	8	9	10
1st difference ..	0	1	2	3	4	5	6	7	8	9	10
2nd difference ..	0	0	1	3	6	10	15	21	28	36	45
3rd difference ..	0	0	0	1	4	10	20	35	56	84	120
4th difference ..	0	0	0	0	1	5	15	35	70	126	210
5th difference ..	0	0	0	0	0	1	6	21	56	126	252

It will be recognised that these are the figures of Pascal's triangle taken diagonally, or the diagonal series in this are the figures of Pascal's triangle taken vertically. By means of such a table one can see at a glance the effect on any value of the function of changing a leading difference.

8. **Smoothing, by operations on factors.**—The smoothing of a succession of ordinates or of group-values may often advantageously be effected not by operating upon these numbers themselves, but upon their *ratios* to each other. This may be called *factorial smoothing*. Let A, B, C, D, etc., be the series of quantities to be smoothed. The ratios B/A, C/B, D/C, etc., are formed, and denoted by b, c, d, etc. These are graphed and smoothed by any process.[1] The smoothed values, denoted by b', c', d', etc., are then used to form a new series of quantities; thus $A=A$, $Ab'=B''$; $B''C'=C''$, etc. The sum of these is then made equal to the sum of the original series of quantities by a common factor k, thus—

$$(290)\ldots k = \frac{A+B+C+D+\text{ etc.}}{A+Ab+Abc+Abcd+\text{etc.}} = \frac{A\left\{1+b\left[1+c\left(1+d\,(1+\ldots)\right)\right]\right\}}{A\left\{1+b'\left[1+c'\left(1+d(1+\ldots)\right)\right]\right\}}$$

then the smoothed values A', B', etc., are $A'=kA$; $B'=kAb'$; $C'=kAb'c'$; $D'=kAb'c'd'$: etc.

Sometimes, on taking out the ratios, it becomes evident that they should have a common value, since they shew no systematic progression. In such a case, let m denote the mean value, then the denominator $A+Ab+Abc+$ etc. in (290) becomes $A + Am + Am^2 +$ etc.

Smoothing of this kind is serviceable for initial and terminal values.

9. **Logarithmic smoothing.**—In a similar manner quantities may sometimes be advantageously smoothed by smoothing their logarithms. In this connection we bear in mind that if a series of numbers are in geometrical progression their logarithms are in arithmetical progression. Let $\log A$, $\log B$, etc., be denoted by α, β, etc., which are graphed, and when smoothed denoted by α', β', etc. If the sum of A'', B'', etc., corresponding to the smoothed values, do not agree with that of the original values, k will be the factor of correction, and may be found as before, that is, by (290). This process may be called *logarithmic smoothing*, and like factorial smoothing, is often useful for initial and terminal values.

10. **On the difference between instantaneous and grouped results.**—When instantaneous results are smoothed the resulting smooth curve represents the equation which reproduces the values of y corresponding to given values of the abscissæ. When, however, group-results are smoothed by differencing, the resultant curve strictly represents the value of a group of the same base (supposed, of course, constant) with any central value throughout the range smoothed: see V., § 10. When, however, group results are few in number (that is, have relatively large bases) the graph must be drawn upon a different principle, viz., it must, as far as the probabilities of the case will admit, make the areas between bounded by the curve, the abscissæ, and the ordinates identical with the

[1] Arithmetically, *i.e.*, by difference, or mechanically, by splines, etc.

area of the group, or, in other words, the mean height of all the ordinates to the curve in any given range of the abscissa must be equal to the height of the group. That is, if h is the height of the group, then :—

$$(291)\ldots\ldots h = \frac{1}{x_2 - x_1}\int_{x_1}^{x_2} f(x)\,dx.$$

$f(x)$ denoting the smoothed curve drawn.

This method may be called "*the method of equivalent group-values,*" and it will, in general, either not depend on differencing at all, or depend thereon to a less extent than when the bases are relatively smooth and the groups numerous.

11. **Determination of the exact position and height of the mode.**—It is often desirable to ascertain with such precision as is possible the abscissa and height of the mode. Two approximate solutions are desirable, viz. :—(a) when the graph shews that *three groups* should be taken into consideration; and (b) when *four* groups. In the former case (a) the formulæ are extremely simple; in the latter (b) they are much less so. If more than four groups are to be taken into consideration it is better to determine the general equation of the curve and solve to obtain that value of x which makes $dy/dx=0$. As an approximate solution will be available from the graph, there is usually very little difficulty in obtaining an exact value of x. Then the corresponding value of y can be found from the equation: see V., §§ 3 to 7.

Case (a). In Fig. 35 let K denote the mean of the heights of the groups on either side of the maximum group and the height of this last, and let k be half the difference of the height of the groups on either side. Let also α denote the difference of the height of one group and the greatest group, and β similarly the difference of the height of the other group and the greatest group. Then

$$(292)\ldots\ldots K = \frac{1}{2}(\alpha+\beta);\ \text{and}\ k = \frac{1}{2}(\alpha - \beta).$$

Then a second degree curve, giving the same group values, gives the abscissa of the mode:—

$$(293)\ldots\ldots \mu = \frac{\alpha}{\alpha+\beta};\ \text{and}\ \mu' = \frac{\beta}{\alpha+\beta}$$

and the height λ, of the mode, above the maximum group is

$$(294)\ldots\ldots \lambda = \frac{1}{12}K + \frac{1}{4}\frac{k^2}{K}$$

If f, g, and h denote the heights of the rectangles we should have for the constants of the curve—

$$(295)\ldots\ldots a = \frac{1}{6}(11f + 2h - 7g)$$

$$(295\text{a})\ \ldots\ldots b = 3g - 2f - h$$

$$(295\text{b})\ \ldots\ldots c = \frac{1}{2}(h + f) - g$$

the base of the curve being considered unity.

In the case *(b)*, differences of height being as shewn in Fig. 36 the constants of the curve which must now contain dx^3 will be

$$(296)\ldots a = \frac{1}{12}(\gamma + \gamma'); \; b = \frac{1}{12}(15\beta - \delta'); \; c = -\frac{3}{12}(\gamma + \gamma'); \; d = -\frac{2}{12}(3\beta - \delta')$$

a being reckoned from the point K, half-way between A and B to the point L, that is, to the curve.

The value of the abscissa of the mode is given by

$$(297)\ldots x_m = -\frac{\gamma + \gamma'}{2(3\beta - \delta')}\left\{1 \pm \sqrt{\left(1 + \frac{2}{3}\cdot\frac{(15\beta - \delta')(3\beta - \delta')}{(\gamma + \gamma')^2}\right)}\right\}$$

The sign of the term under the radical can readily be determined in a practical example. The general expression for y_m is lengthy. In cases practically occurring we may compute it from x_m when that value is found: that is, it is

$$(298)\ldots y_m = \frac{1}{12}\left\{\gamma + \gamma'(1 - 3x_m^2) + (15\beta - \delta')x_m - 2(3\beta - \delta')x_m^3\right\}$$

the ordinate being reckoned from the line parallel to the axis of abscissæ and half-way between the points A and B in Fig. 36, *i.e.*, the line MJ in the figure.

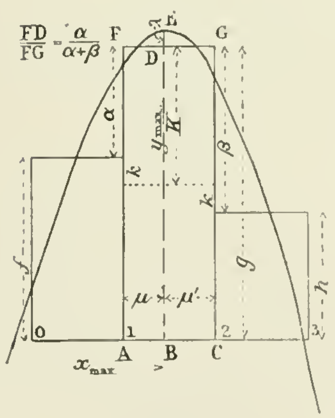

Fig. 35

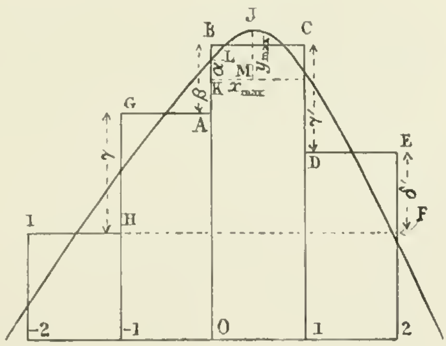

Fig 36.

The formulæ (293) and (294) and (297) and (298) are not quite satisfactory, and in general it is better to compute the coefficients of the equation which fits a considerable stretch of the curve, and find the position of the maximum by $dy/dx = 0$, if very great precision be required.

12. **The testing of smoothed or graphic results.**—When smoothed graphed results are obtained they will, in general, need, as already indicated, to be arithmetically tested. The fundamentals of arithmetical testing are the following :—

(i.) The sum of the graphed results should be sensibly (or exactly) equal to the sum of the original data ;

(ii.) The deviations, positive and negative, between the aggregate of the smoothed results and the data up to each given value of the argument should, consistently with the type of curve adopted, be a minimum ;

(iii.) The position and ordinate of the mode should be carefully fixed, and as well as the data will permit ;

(iv.) The position of the terminals should conform to the probabilities of the type of data so far as that can be determined.[1]

[1] In general, they cannot be determined mathematically. For example, the frequency of births of given ages, so far as mathematical relations are concerned, might be continued to start at the age 0, but in view of physiological considerations we should not be justified in starting at 0, but at, say, the age 11 ; similarly in regard to the terminal, which may be made to meet the axis of abscissæ for age 60 (or such later age as may be indicated as occurring, should satisfactory information be to hand).

VIII.—CONSPECTUS OF POPULATION-CHARACTERS.

1. **General.**—Thus far the consideration of the theory of population has been concerned only with its numerical aspect, and with the mathematical form of expressions under which it may be necessary to subsume the facts. These constitute an essential preliminary only. It remains now to consider in detail some of the various characters of importance.

Not only are population-statistics, in the narrower sense, significant both (i.) in themselves, and (ii.) in comparison, but so also are all facts that may properly be regarded as expressions of the various characteristics of a population. Following the nomenclature of biology, these may be called more briefly its *characters*. Such characters may relate to—

(*a*) *Vital phenomena*, that is, to birth, life and death, to reproduction in all its aspects, to disease and all the modes of its incidence;

(*b*) *Anthropometry*, that is, may relate merely to the human form and its variations, or to its growth and decrepitude;

(*c*) *Anthropology*, that is, they may refer to man's general evolution, both physical and psychical;

(*d*) *Sociology*, that is, they may concern man in respect of his social life, an important element in which is his economic evolution, and they may concern also the reaction of this upon his numbers and the density of his aggregation.

(*e*) *Migration, aggregation, segregation, or wide dispersion, colonisation, etc.*, that is, the direction and velocity of movement of populations, the tendency to live in more or less dense groups (large cities or villages) or to spread over the earth, etc.

All these have significance in regard to the rate of development of the world's people. It is well to bear in mind, also, that population-characters may be in two forms, viz., either actual or potential.

The *importance* of the subject is seen in the impossibility of maintaining the present rate of increase for any great length of time (see II., § 34); and its range of subjects is best seen through a conspectus. Characters may be simple or complex, their manifestation may be instantaneous or durational; and the evidence of their nature direct or derivative. The greatness of the range of population-characters, and the number of significant relations subsisting among them is so vast that *no statistical presentation of them can be exhaustive.* Thus important questions are continually arising involving demands for new statistical compilation, for human affairs can be properly analysed only with the aid of a well-founded and technically satisfactory statistic. The simplest population-characters are expressible in regard to units, as, for example, the numbers in a population; the wealth possessed, etc. The complex

are those which involve multiple *fields* of comparison, for example, the number of one sex, who, being between given limits of age, and belonging to a given occupation, die of a particular disease.

That the number of comparisons possible is very great is obvious from the fact that n things considered in their mutual instantaneous relations, that is, n things considered each in relation to $1 \ldots\ldots n-1$ other things, are 2^n-1. The following table will shew the number possible up to $n=10$.

TABLE X.

No. of Elements in Combination	Elements of Original Statistical Data.									
1	1	2	3	4	5	6	7	8	9	10
2	..	1	3	6	10	15	21	28	36	45
3	..	..	1	4	10	20	35	56	84	120
4	..	..	..	1	5	15	35	70	126	210
5	..	..	..	..	1	6	21	56	126	252
6	..	..	..	..	..	1	7	28	84	210
7	..	..	..	..	..	..	1	8	36	120
8	..	..	..	..	..	..	..	1	9	45
9	..	..	..	..	..	..	..	..	1	10
10	..	..	..	..	..	..	..	..	..	1
Total possible combinations of elements	1	3	7	15	31	63	127	255	511	1023

The total possible for 12 is 4095, for 20 is 1,048,575.

There were, for example, 17 main questions to be answered in the Australian Census; thus there would be $2^{17}-1$ (viz., 131,071) possible tables by combinations of these results, and a considerable proportion of these would be of real significance.

2. **Characters directly given or derivative.**—Important characters are not always immediately yielded by the data: they are often to be ascertained only by analysis. Thus, as in the case of statistics generally, population statistics may be either

A. Direct, viz.:—		(or) B. Derivative, viz.:—	
A (i.) Instantaneous (numbers at a given moment). (Examples): No. of persons living: wealth possessed by them at a particular instant; etc., or	A (ii.) Durational (or number of occurrences during a unit of time). (Examples): Number of persons born, married, or deceased during a day, month, or year; etc.	B (i.) Instantaneous (numbers deduced representing a state of things for a given moment or epoch). (Examples): Masculinity at birth, or at a census; wealth possessed, per individual; expectation of life; etc., or	B (ii.) Durational (numbers deduced of occurrences during a unit of time). (Examples): Birth, marriage, or death-rates per day, month, or year; average wealth deduced from probate returns; etc.

The above indication of the nature of population statistic reveals the reason of its extent, which is much greater than is implied in the

number of mere combinations of different fields of statistic considered in their instantaneous relations alone.

3. **Characters in their instantaneous and progressive relations.**—The characters of a population are fully studied only when examined both in their instantaneous relations, and in the progression of these with time. Suppose, for example, that characters A and B both vary with time, and that such variation can be expressed by rational integral functions thereof; then the constant relation of the characters is given by

$$(299)\ldots\ldots\ \frac{B}{A} \equiv \frac{F_2(t)}{F_1(t)} \equiv \frac{a_2(1 + b_2 t + c_2 t^2 + \text{etc.})}{a_1(1 + b_1 t + c_1 t^2 + \text{etc.})}$$

$$= \frac{a_2}{a_1}\left[1 + (b_2 - b_1)t - \{b_1(b_2 - b_1) - (c_2 - c_1)\}t^2 + \text{etc.}\right]$$

approximately; or including the term in t^3, and writing

$$(300)\ldots\ldots\beta = b_2 - b_1;\quad \gamma = \frac{c_2 - c_1}{b_2 - b_1};\quad \delta = \frac{d_2 - d_1}{b_2 - b_1}$$

a result of greater precision is given by :—

$$(301)\ldots\ \frac{B}{A} = \frac{a_2}{a_1}\left[1 + \beta t\{1 - (b_1 - \gamma)t + [b_1(b_1 - \gamma) - c_1 + \delta]t^2 + \text{etc.}\}\right]$$

The successive coefficients, in nearly all practical examples, converge with sufficient rapidity to admit of the employment of the formula for even large values of t.[1]

Derived characters involve, as a rule, a greater complexity of change of relation with the lapse of time. Thus, for example, suppose a rate for *persons* is to be deduced from the rates for *males* and *females*, and suppose also that these do not change identically with the lapse of time (as, for example, a death or morbidity-rate for a particular disease). Suppose then that these rates for males and females are respectively :—

$$(302)\ldots\ldots m_t = m_0(1 + b_m t + c_m t^2 + \text{etc.});\ \text{and}$$

$$(303)\ldots\ldots f_t = f_0(1 + b_f t + c_f t^2 + \text{etc.})$$

Let us suppose that the ratio of the difference of the sexes to their sum, that is $(M - F)/(M + F) = \mu$; or $(F - M)/(F + M) = \phi$; and that the variation of this function with time is expressed by

$$(304)\ldots\ldots\mu_t = \mu_0(1 + \beta' t + \gamma' t^2 + \text{etc.});\ \text{and}\ \phi_t = -\mu_t;$$

then it will follow that the ratio p for persons will be

$$(305)\ldots p = \left(\frac{1}{2} + \frac{\mu}{2}\right)m + \left(\frac{1}{2} - \frac{\mu}{2}\right)f = \frac{1}{2}(m + f) + \frac{1}{2}\mu(m - f)$$

[1] Say, ordinarily at least to $t = 100$.

that is, it will be the mean weighted according to the relative numbers of males and females. The result may at once be written out from (302) and (303), and re-expressed is

$$\begin{aligned}(306)..p_t = \frac{1}{2}\Big\{&[m_0+f_0+\mu_0(m_0-f_0)]+[m_0\{b_m(1+\mu_0)+\mu_0\beta' \\ &+f_0\{b_0(1-\mu_0)-\mu_0\beta'\}]t+[m_0\{c_m(1+\mu_0)+\mu_0(b_m\beta'+\gamma')\} \\ &+f_0\{c_f(1-\mu_0)-\mu_0(b_f\beta'+\gamma')\}]t^2+\text{etc.}\Big\}\end{aligned}$$

From this it is obviously impossible to secure consistency among formulæ for persons, males, and females, where the variation with time of those for the two last is not identical, without complexity of expression. Moreover, when variations with time have to be considered, as well as many fields of comparison, not only do *general formulæ* become too involved to be of practical value, but also the number of relations necessary to exhaust the statistic becomes hopelessly large. For this reason it is often desirable to compute the coefficients for males, females and persons independently: if this be done with care *the involved inconsistency may be regarded as negligible.*

4. **Conspectus of the population-characters with which the ordinary census is concerned.**—In Section 1 of Chapter II. of the general Census Report, a classified statement and a brief review of the objects and uses of a census are given. These present, however, only one aspect of some of the leading characters of population. In the following conspectus a somewhat different and more extensive sketch of such of these characters as are capable of statistical measurement, and which constitute normal bases for comparisons, is furnished:—

A.—Numerical constitution of population at a given epoch in regard to

(i.) Sex, and (ii.) age;
(iii.) birthplace; and (iv.) length of residence in country of enumeration;
(v.) nationality; and (vi.) race;
(vii.) conjugal condition; (viii.) duration of marriage; and (ix.) size of family;
(x.) infirmity;
(xi.) degree of education; and (xii.) school attendance;
(xiii.) religion;
(xiv.) occupation—(*a*) designation; and (*b*) grade;
(xv.) dwellings—(*a*) material: and (*b*) number of rooms; (*c*) mode of occupancy; and (*d*) rental;
(xvi.) localisation.

In each case the statistical data initially represent the number of persons possessing the character or group of characters specified, as, for example, the number of persons having a family of a given size, the number of persons having a given duration of marriage.

In the case of dwellings the enumeration is twofold, and comprises, for example, the number of dwellings of a given material, as well as the number of persons living in dwellings of a given material.

B.—Relative constitution of population in respect of characters enumerated in A.

In this section are comprised *the ratios of the numbers possessing a given character or group of characters to the numbers possessing a wider range of such characters*, as, for example, the ratio of males under 21 years of age to the total population of all ages and of both sexes.

C.—Variations of population at different epochs.

This may involve merely variations in aggregate population, or may comprise variations in the numbers possessing any combination of the characters enumerated in A, or in the relative constitutions deduced under B.

D.—Mean population at a given period.

As in the case of C, this may involve merely the aggregate population or may comprise the mean population possessing any combination of the characters enumerated in A. The mean population for any unit of time represents the number of such units of human life lived by the population or section thereof under observation.

E.—Fluctuations of population during a given period.

These arise from :—(i.) Births (see F) ; (ii.) deaths (see G) ; (iii.) migration (see H).

F.—Births.

(*a*) The statistical data initially represent the number of births classed according to the following categories, taken either singly or in combination.

(i.) Whether live or still birth ; (ii.) sex of child ;
(iii.) whether born in wedlock or not ;
(iv.) age of father ; and (v.) age of mother ;
(vi.) birthplace of father ; and (vii.) birthplace of mother ;
(viii.) occupation of father ;
(ix.) duration of parents' marriage (see I.) ;
(x.) locality ; and (xi.) date of birth ;
(xii.) date of registration ; and (xiii.) position of child in family, *i.e.*, whether first, second, etc.
(xiv.) single or multiple birth.

(*b*) The derivative statistical results comprise, *inter alia*, particulars concerning the relations between

(i.) Live and still births ; and (ii.) nuptial and ex-nuptial births ;
(iii.) male and female births ;
(iv.) number of births and population from which derived.

These may involve merely the relation between total births and total population, or the relation between the number of births possessing any character or group of characters enumerated in F (*a*) and the appropriate subdivision of population from which derived. In the one case the result would be the crude birth-rate, or ratio of total births to total population, in the other it would comprise such results as, say, the nuptial birth-rate in a given area amongst fathers of a given age, birthplace, and occupation, who had been married for a given period. Similarly (i.), (ii.) and (iii.) may involve merely totals possessing the characters specified, or may relate to subdivisions possessing any character or group of characters enumerated in E: as, for example, the relation between live and still births amongst the nuptial male births of women of a given age and birthplace, who had been married for a given period.

G.—Deaths.

(*a*) The statistical data initially represent the number of deaths classed according to the following categories, taken either singly or in combination:—

(i.) Sex of deceased; (ii.) age; and (iii.) birthplace;
(iv.) cause of death, (*a*) primary, and (*b*) secondary:
(v.) occupation; (vi.) length of residence; and (vii.) locality;
(viii.) age at marriage and re-marriage;
(ix.) number of issue, according to sex, and whether living or dead;
(x.) date of registration.

(*b*) The derivative statistical results consist mainly of particulars concerning the relations between the number of deaths possessing any character or group of characters enumerated in G (*a*) and the appropriate subdivision of population from which derived, such, for example, as the death rate from a specified cause in a given locality amongst males of a given age, birthplace and occupation.

(*c*) As derivative results of the second degree may be classed such particulars as

(i.) Index of mortality; and (ii.) corrected death-rates;
(iii.) expectation of life; and (iv.) detailed mortality tables.

H.—Migration.

Complete statistical data would initially represent an enumeration of migrants classed according to the characters specified in A, with the exception of (xi.) length of residence; and (xv.) dwellings. Such detail is quite impracticable, and the main characters available in Australia are:—

(*a*) For traffic by sea:—

(i.) Sex; and (ii.) whether adult or child, or preferably exact age;
(iii.) port of embarkation; and (iv.) port of disembarkation;
(v.) nationality or race; and (vi.) date of migration.

(*b*) For land-traffic by rail :—

(i.) Sex ;

(ii.) state in which arrived ; and (iii.) from which departed ;

(iv.) date of migration.

(*c*) For land-traffic by road :—

Similar details as in (*b*).[1]

I.—Marriage.

(*a*) The statistical data initially represent the number of marriages granted in a given period classed according to the following categories taken either singly or in combination :—

(i.) Age of bridegroom ; and (ii.) of bride ;

(iii.) conjugal condition of bridegroom ; and (iv.) of bride ;

(v.) birthplace of bridegroom ; and (vi.) of bride ;

(vii.) occupation of bridegroom ; (viii.) locality ; and (ix.) date of registration ;

(x.) by whom celebrated ;

(xi.) ability of bridegroom to sign register ; and (xii.) of bride.

(*b*) The principal derivative statistical results are those concerning the relations between the number of persons married during a given period and possessing any character or group of characters enumerated in I (*a*) and the appropriate subdivision of the population from which derived, such, for example, as the marriage rate amongst bachelors of a given age, birthplace and occupation.

J.—Divorce.[2]

(*a*) Satisfactory statistical data would initially represent the number of divorces granted in a given period classed according to the following categories taken either singly or in combination :—

(i.) Age of husband ; and (ii.) of wife ;

(iii.) duration ; and (iv.) issue of marriage (*a*) males ; (*b*) females ;

(v.) locality ; and (vi.) birthplace of husband ; and (vii.) of wife ;

(viii.) occupation of husband ;

(ix.) sex of petitioner ; and (x.) cause of petition ;

(xi.) date of rule *nisi* ; (xii.) and of making rule absolute ;

(xiii.) by whom marriage was celebrated.

(*b*) The principal statistical results derivative from the foregoing would be relations between the numbers of persons divorced during a given period and possessing any character or group of characters enumerated in J (*a*), and the appropriate subdivision of the population from which derived, as, for example, the proportion of husbands of a given age, birthplace and occupation, who had been petitioners in granted divorce cases.

[1] In Australia this last information is not available.

[2] Complete statistics not available in Australia.

K.—Sickness and Accident.[1]

(*a*) Satisfactory statistical data initially represent the cases of disablement by sickness or accident occurring in a given period classed according to the following categories taken singly or in combination :—

(i.) Sex ; (ii.) age ; and (iii.) birthplace of person disabled ;
(iv.) cause of disablement ;
(v.) occupation ; and (vi.) locality ;
(vii.) date ; and (viii.) duration of disablement ;
(ix.) conjugal condition of person disabled ; and (x.) number of issue :
(xi.) whether or not disablement terminated by death.

(*b*) Derivative statistical results would consist mainly of relations between :—

(i.) cases and appropriate population ;
(ii.) cases of deaths.

(*c*) Derivative results of a second degree consist of sickness tables constructed from initial data.

5. **The range of the wider theory of population.**—The conspectus just given has obviously been limited to matters with which the census and ordinary vital statistics are more directly concerned. In a wide consideration of population, however, the characters of importance include a much larger range, embracing what has already been indicated in § 1, hereinbefore, viz., the anthropometric, anthropological, and sociological, including the economic. This has already been referred to : see I., § 6, iii. and iv., and II., §§ 13-18. Because of this fact, a complete theory of population must take account of (*a*) the reactions of eugenic facts and arrangements upon the numbers and mode of growth of the population of the entire world and of its constituent peoples, and (*b*) even of the reactions thereupon of all economic and social conditions, including those arising from mobility. This is seen when one contemplates the part played by modern facilities in transport and communication. Nor are the physical and psychical characters of the population less foreign to a complete theory. For the same reason there are aspects of subjects not directly enumerable as population facts, which have immediate touch therewith ; such, for example, as national, municipal and private wealth and their fluctuation, concentration and dispersion ; the productivity of such wealth, the economics of national and municipal revenues, expenditures, and administrations ; the productivity of private wealth, and, indeed, of wealth of all kinds ; the correlations between population-fluctuations and such financial characters as national liabilities ; the quantity and velocity of the circulation of currency ; the relations between primary and secondary production and population development ; the growth of institutions expressive of a deepening recognition of social solidarity in co-operative effort, and in the nationalisation of the greater public services, etc. And finally, it may be said

[1] Complete statistics are not available in Australia.

that all facts which throw any light whatever on the possibility of world-production of food supplies and the fluctuations of population with abundance or want belong to the wider theory of population, and demand appropriate mathematical investigation.

These wider facts are, of course, beyond the range of the narrow limits of ordinary official statistic, but no comprehensive view of the significance of a study of population is possible, which excludes the study of the reaction of material, psychical, or social conditions upon its growth and fluctuation.

6. **The creation of norms.**—The significance of statistical results is fully recognised only by comparisons with the similar results for other populations. Such comparisons are effected in the most general way by the creation of norms for each population-character. The principle which governs the constitution of a norm is that it shall represent the character selected on the widest possible basis. Thus, if statistical data existed for every population in the world, world-norms would be possible for every character statistically recorded. Western civilisation is fairly homogeneous and statistical data are available for many characters. Thus it should be practicable in the near future to create a series of norms for the greater part of the western world. These might be regarded as the *normal* or *usual* value of any character in question, with which the same character in any particular population may be compared. It is obvious in order to compare a series of populations the best basis is the average value of any character: furthermore, if a compared character is affected by the deviation of any other from the average the value of the norm and of the deviation therefrom furnish the best basis for necessary corrections.

The essential nature of a norm is perhaps best seen by regarding it as representing the characters of all the populations included, considered as a single population. Thus the deviation of the characters and any particular population about the secular changes therein of this great aggregate gives the most informative presentment of the position of the population in question, that can possibly be had: in short, it makes the scheme of comparison as broad as is possible.

7. **Homogeneity as regards populations.**—Two communities may be said to be homogeneous with regard to any series of characters, when those characters are identical. In comparisons between communities in regard to any one character, it is necessary, in order that the comparison should be a just one, that all other characters which have any influence thereupon should be identical; or, to put this more generally, the comparisons of any selected characters in a community are legitimate only when these communities are homogeneous with respect to all other characters which may have any influence on the comparison. For example, the birth-rates of two communities are immediately comparable if the relative numbers of married and single at each age are the same, because the birth-rate then (presumably) reveals the fertility under identical physiological conditions.

Since, however, different communities are more or less heterogeneous, appropriate schemes must be developed through which rigorous comparisons can be effected. Thus, for example, corrections may be applied in such a way that any character compared or contrasted will not be affected by difference of other characters.

The most convenient way of securing such a result is to adopt, as a basis for all comparisons, a popnlation so characterised as to represent all others to be compared as nearly as possible. Such a population may be called a "normal" or a "standard" population, and any character in regard to which it has been standardised may be called a "norm."

8. **Population norms.**—In order that any character of a number of populations or communities may be conveniently compared, it will be necessary that whatever population be adopted as basis, it shall represent *each* as nearly as possible. It is easy to see that, in regard to any character under review, such a basis must be a *weighted mean*, so that the character adopted as basic shall be the character of the population formed by aggregating all populations which may have to enter into comparison. Thus if P, Q, R, etc., be populations, and p, q, r, etc., be the values of some one character in each, then the best basis of comparison is :—

$$(307)\ldots\ldots \frac{Pp + Qq + Rr + \text{etc.}}{P + Q + R + \text{etc.}} = \frac{Ss}{S} = s$$

S being the sum of $P + Q + R +$ etc., and s the *norm*.

It is immediately obvious that, in general, the secular changes of norms will be less marked than the secular changes in respect of the same character of the individual populations from which the norm is determined. For this reason it will be necessary for the progress of exact statistic to establish a *series of norms* for all elements the comparison of which are important. That is, we must adopt a standard or normal population of definite characters, or, in other words, create a series of population norms to serve as a basis for comparisons. The scheme then of comparison is to apply the ascertained attributes of each existing population to the standard population. This process will reveal what would have been manifested had each population been similarly constituted to the standard population.

9. **Variations of norms.**—Inasmuch as, in the present development of statistics, norms have not been created, except perhaps as regards the constitution of population of each sex according to age, it will suffice to indicate the outlines of a general method of studying the variation of norms. Since necessarily they can vary only slowly, a decennial determination will be probably always sufficient, and when a number of decennial changes are to hand, the investigation of their variation will become possible. Whether such variation will reveal any sign of periodicity or not it is at present impossible to say. It is not unlikely that periodic elements of variation will be fonnd superimposed upon slow secular changes. This, however, must be left for the future to determine, and the appropriate method of analysis will depend upon the character of the data.

10. **Norm representing constitution of population according to age.**—A norm for males and one for females of European race is of importance for properly comparing death, marriage, birth and other rates. The use of such a norm was proposed by Dr. Ogle at the meeting of the "Institut International de Statistique," in Vienna, 1891, and the *index of mortality* at present used is based upon such a norm, though not a properly constituted one. The aggregation of the populations of a considerable group of countries between which also there is migration, removes the specialising influence of this latter element, and secures the general advantages of large numbers. The following results were obtained from combining the populations of England and Wales, Scotland, Ireland, the United States, the German Empire, Norway, Sweden, Italy, Canada, Australia, and Newfoundland generally for the censuses of 1900 or 1901.[1] The numbers are given in each age-group, and above a given age :—

TABLE XI.
Population Norms for 1900.

Age.	European (1900). Numbers in Age-Group in total of 10,000.			India (1901). Numbers in Age-Group in total of 10,000.			Age.	European (1900). Numbers at and above age indicated.		
	Males.	Females.	Persons.	Males.	Females.	Persons.		Males.	Females.	Persons.
0	270	263	266	266	276	271	0	10,000	10,000	10,000
1–4	971	953	962	988	1,063	1,025	1	9,730	9,737	9,734
5–9	1,139	1,119	1,129	1,394	1,382	1,388	5	8,759	8,784	8,772
10–14	1,057	1,038	1,047	1,264	1,081	1,174	10	7,620	7,665	7,643
15–19	975	980	977	866	835	851	15	6,563	6,627	6,596
20–24	915	931	923	787	892	838	20	5,588	5,647	5,619
25–29	808	813	810	879	894	887	25	4,673	4,716	4,696
30–34	715	705	710	848	851	850	30	3,865	3,903	3,886
35–39	640	621	632	609	557	583	35	3,150	3,198	3,176
40–44	563	550	557	648	652	650	40	2,510	2,574	2,544
45–49	470	463	467	370	339	355	45	1,947	2,024	1,987
50–54	413	417	415	437	452	445	50	1,477	1,561	1,520
55–59	331	344	337	177	169	173	55	1,064	1,144	1,105
60–64	272	290	281	254	303	278	60	733	800	768
65–69	197	212	205	66	79	72	65	461	510	487
70–74	136	150	143	76	91	84	70	264	298	282
75–79	79	88	84	27	32	29	75	128	148	139
80–84	36	43	39	30	35	33	80	49	60	55
85–89	10	13	12	5	6	5	85	13	17	16
90–94	3	3	3	6	7	6	90	3	4	4
95–	1	1	1	3	4	3	95	1	1	1
Total ..	10,000	10,000	10,000	10,000	10,000	10,000		10,000	10,000	10,000

[1] See "The determination and uses of population norms representing the constitution of populations according to age and sex, and also according to age only." By G. H. Knibbs, and C. H. Wickens, Trans. 15th, Int. Congr. Hygiene and Demography, Washington. Vol. VI., pp. 352-378.

11. **Mean age of population.**—The mean age, x_m, of a population is given by the formula

$$(308)\ldots\ldots x_m = \frac{\int_0^\omega x l_x dx}{\int_0^\omega l_x dx} = \frac{1}{2} + \frac{\Sigma_0^\omega x L_x}{\Sigma_0^\omega L_x}, \text{ approximately.}$$

l_x denoting the relative frequency at the age x, ω the greatest age attained or considered, and L_x the number of age x last birthday, it being assumed that this number may, on the average, be regarded as of age $x + \frac{1}{2}$. Omitting the $\frac{1}{2}$, this last expression really gives the correct *mean age last birthday*. The *mean age next birthday*, x_n, of a population under the age n is

$$(309)\ldots\ldots x_n = \frac{nL_{n-1} + (n-1)\,L_{n-2} + \ldots\ldots + L_0}{L_{n-1} + L_{n-2} + \ldots\ldots\ldots + L_0}$$

From this formula it is evident that, with a table giving the number at and above each integral age, all that is requisite to obtain the mean age next birthday is to divide the total population into the sum of the numbers from the youngest to the oldest ages. Deducting $\frac{1}{2}$ gives the usual approximation to the mean exact age, while a deduction of unity gives the mean age last birthday.

The mean age in years of the normal or standard population is, for 1901 :—

Males.	*Females.*	*Persons.*
26.934 years.	27.341. years.	27.148. years.

This mean age is, of course, *not* what is known actuarially as the *expectation of life at age* 0, but is the average age of all persons living at a given moment, or, in other words, it is the average past lifetime of the population at a given moment. On the other hand, the expectation of life at age 0 is the average future lifetime of all persons born. In the case of a *stationary population*, however, with rates of mortality varying with age, but remaining constant for each age through a great length of time, the average past lifetime of the population at a given moment is equal to its average future lifetime, that is, the average age of the population is equal to the average "expectation of life" of the population as a whole.[1] Thus for the population of Europe in 1901 persons had lived on

[1] The expectation of life e_x^0 of the $l_x\,dx$ persons of the exact age x is the future lifetime T_x of these, divided by their number, that is—

$$e_x^0 = \int_x^\omega l_x\,dx \,/\, l_x = T_x \,/\, l_x$$

and consequently the total future lifetime of these $l_x\,dx$ persons is

$$e_x^0\, l_x\, dx = l_x\, dx\, T_x \,/\, l_x = T_x\, dx$$

Hence the total future lifetime of the whole existing population between 0 and ω is

$$\int_0^\omega e_x^0\, l_x\, dx = \int_0^\omega T_x\, dx$$

and as a whole existing population is $\int_0^\omega T_x\,dx$, the average future lifetime or expectation of life of the whole existing population is $\int_0^\omega T_x\,dx \,/\, \int_0^\omega l_x\,dx$, which may be shewn to be equivalent to $\int_0^\omega x l_x\,dx / \int_0^\omega l_x\,dx$, or the mean age.

the average about 27 years. The expectation of life changes with the lapse of time, and is appreciably lengthening. Thus the secular change of the norm will be the weighted average of the changes of the constituent populations.

12. **Population norm as a function of age.**—The number of persons, Y, at and above the age x may be closely represented by

$$(310) \ldots \quad Y = k\alpha^{a^x}\beta^{b^x} = 52674\,(0.99961)^{1.10808^x}\,(0.18998)^{1.01495}$$

which is a development of the Gompertz-Makeham type of formula. The constants indicated fit very closely the values of the norm given in preceding table.[1] This matter will be dealt with more fully hereinafter.

[1] For solution, vide op. cit. pp. 364-7.

IX.—POPULATION IN THE AGGREGATE, AND ITS DISTRIBUTION ACCORDING TO SEX AND AGE.

1. **A Census and its results.**—A well-conducted Census furnishes results which are substantially correct so far as the aggregate number of persons and the aggregate number of each sex is concerned. That is, if p, m and f denote the *errors* of the numbers of persons, males and females respectively, and P, M and F their respective aggregates, then p/P, m/M and f/F are all extremely small quantities, which can have no important bearing upon the general theory, or upon any deductions flowing from it. Unfortunately this is not true regarding the numbers of either sex between given age-limits.

In Chapter X of the Census Report, it has been shewn that for Australia the Census results bear intrinsic evidence of great improvement in regard to accuracy of statement respecting age ; see §§ 3 and 5. The nature of this is shewn in the tables given of numbers and percentages for the ages 28, 29 . . . . 32, and 48, 49 . . . . 52. The excessive statements, for example, for the ages 30 and 50, became markedly less. The results were as follow :—

Census	Age.	1891.	1901.	1911.	1911(adjusted).
Percentage of age-quinquennium including two years on either side ..	30	23.35	22.98	20.90	19.96
	50	29.06	25.77	21.75	20.16

A glance at Figs. 37 and 38 hereinafter will shew that the curves of numbers according to age for ages 30 and 50 do not depart very much from a straight line. For the former age the curves are concave upward ; for the latter, convex upward. Hence at 30 the mean should be somewhat less, and at 50 somewhat more, than 20 per cent. The ratio determined from the smoothed results are shewn in the final column. We shall consider the question of smoothing the results later.

For each it is seen that the numbers for the ages in question were excessive, enormously so for 50 years of age, in the 1891 Census. The error, however, was diminished for the Census of 1911, probably largely in

consequence of a special attempt to ensure the population appreciating the necessity for accuracy.[1] It may be said, however, that statements of age leave much to be desired.

2. **Causes of misstatement of age.**—Many people are so indifferent as regards their age that they are really unaware what it is, and for this reason tend to assign round numbers (viz., ages ending with the figure 0 or the figure 5), as roughly expressing about their ages. In the case of persons approaching 21 years of age, what may be called "matrimonial reasons" exist for an overstatement, and this may continue to operate for a year or two. In the case of females the tendency to overstate the age is, on the whole, negative for a considerable period of life.[2] For the older ages, however, there is probably a distinct tendency in the opposite direction.[3]

3. **Theory of error of statement of age.**—Assuming both a tendency to express in round numbers ending in 0 and 5, an age not accurately known, and also particularly in the case of females some tendency to understate age, except for ages above, say, 60, we ought in general to find the following characters in the crude results of a Census, viz. :—

(i.) In smoothing the crude results so as to conform to the general trend, the results for ages ending in 0 have to be considerably reduced ; while those ending in 5 have to be reduced a somewhat smaller amount.

(ii.) The amounts of the corrections for ages above and below the round numbers on the whole shew some asymmetry, though at the same time, owing to the masking effect operating in ages so close as $x + 0$ and $x + 5$, this character is not definite.

(iii.) The curves for males and females exhibit systematic differences of form due to systematic misstatement.

Figs. 37 and 38 shew the graphs of the numbers for each year from 0 to 100, for the Australian Commonwealth. It will be seen from these that, for a population profoundly affected by migration, no systematic difference of form actually exists of sufficient magnitude to unmistakably indicate systematic misstatement of age. The marked tendency to give ages ending with the figure 0 is, however, very evident, so also that to give ages ending with the figure 5 is also fairly clear.

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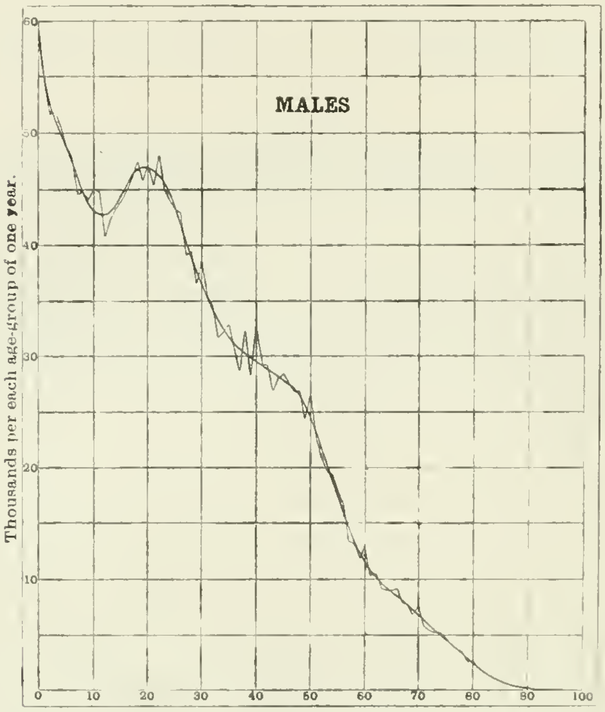
[1] Where the official administration of a community is sufficiently systematic to require every one to keep a *card of identification*, it is easy to get correct answers to this and similar questions. The public appreciation of the importance of correct answers is regrettably deficient.

[2] For matrimonial and economic reasons, and even reasons not entirely dissociated with personal vanity ; the two latter reasons also operate in the case of males, but to an appreciably lesser extent.

[3] Certain investigations shew that vanity concerning longevity is not wholly absent in either sex.

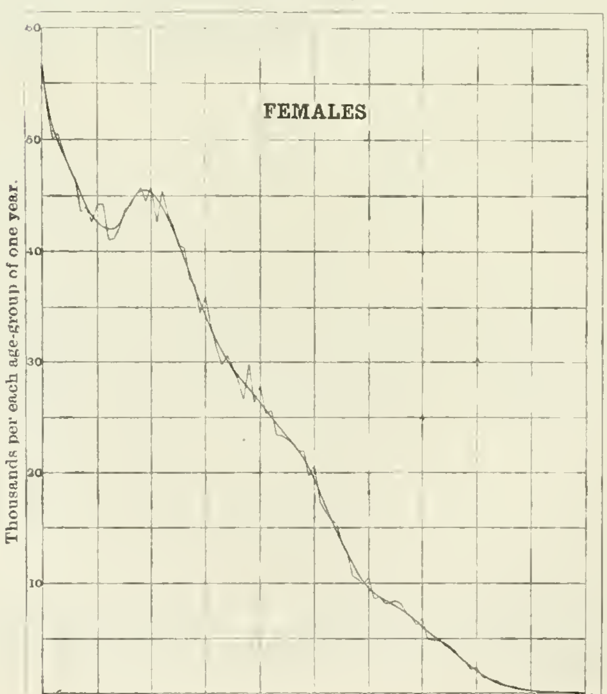
AUSTRALIA, 1911.



Commencing points of age-groups of one year at age indicated.

Fig. 37.

AUSTRALIA, 1911.



Commencing points of age-groups of one year at age indicated.

Fig. 38.

The curves in Figs. 37 and 38 are interpreted in the following way, viz. :—The ordinate or vertical distance to the curve at any point represents in thousands the number of males (or females) in the age-group of one year, commencing at the age in question. The zig-zag line denotes the results furnished immediately by the Census, and the curve the smoothed (and more probably correct) results.

4. **Characteristics of accidental misstatements, and their fluctuations.**—The Censuses of the various States of Australia never having been combined, it was desirable to compile the three preceding Censuses, viz., those for 1881, 1891, 1901, in order to deal thoroughly with that of 1911. The results were not in age-groups for single years for 1881, but were for the later Censuses. In doing this it was found on inspecting the graphs for 1891, 1901 and 1911, of the numbers enumerated for each age, that in the statements of age there were tendencies to concentrate on certain ages, and to avoid certain others. In order to definitely examine these tendencies a tabulation was made of the data in respect of the unit figure in the year of age stated in Australia at the Censuses of 1891, 1901 and 1911. To enable an estimate to be made of the degree of error involved in these statements of age, the smoothed results were similarly tabulated according to the unit figure in the year of age, and the ratio of the former set of results to the latter was obtained for each sex and each unit figure. The results should, of course, be unity if the errors balanced, or had no tendency in any direction.

The ratios so obtained are as follows :—

Table XII.—Ratio of Number Recorded to Adjusted Number, Censuses 1891, 1901, 1911, Australia.

Year of Census	Unit Figure in Age Last Birthday—									
	0	1	2	3	4	5	6	7	8	9
					MALES.					
1891 ..	1.1388	.9167	1.0088	.9545	.9969	1.0366	1.0207	.9513	1.0055	.9532
1901 ..	1.1044	.9369	1.0072	.9677	.9809	1.0343	1.0134	.9636	1.0144	.9667
1911 ..	1.0485	.9956	.9944	.9787	.9990	1.0085	1.0097	.9691	1.0191	.9695
					FEMALES.					
1891 ..	1.1251	.9288	.9978	.9848	.9943	1.0077	1.0117	.9640	1.0125	.9558
1901 ..	1.0926	.9270	1.0039	.9861	.9979	1.0106	1.0128	.9708	1.0165	.9738
1911 ..	1.0367	.9895	.9935	.9895	1.0056	1.0050	1.0066	.9770	1.0148	.9760

The outstanding indications furnished by this table are for both sexes

(i.) A marked tendency to concentrate on ages ending in 0.

(ii.) A less marked but persistent tendency to concentrate on ages ending in 5, 6 and 8.

(iii.) A marked tendency to avoid ages ending in 1, 3, 7 and 9.

(iv.) A tendency to state ages ending in 2 and 4 with fair accuracy, concentrations and avoidances being in evidence, but relatively small in respect of these ages.

The table also furnishes an indication of the *increasing accuracy of statement of age* at successive Censuses, the excess at ages ending in 0 having fallen from 13.88 per cent. in 1891, to 4.85 per cent. in 1911, in the case of males, and from 12.51 per cent. in 1891, to 3.67 per cent. in 1911, in the case of females.

Another interesting feature of the results is the evidence furnished that *inaccuracy of statement is more marked amongst males than amongst females.* Thus, for the Census of 1891 the mean deviation from unity (irrespective of sign) of the above ratios was .0438 for males, as against .0332 for females. The corresponding figures in 1901 were .0358 for males, as against .0281 for females, and in 1911 they were .0181 for males, as against .0143 for females.

ENGLAND AND WALES, 1911.

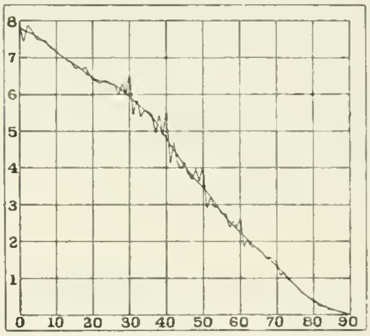


Fig. 39.

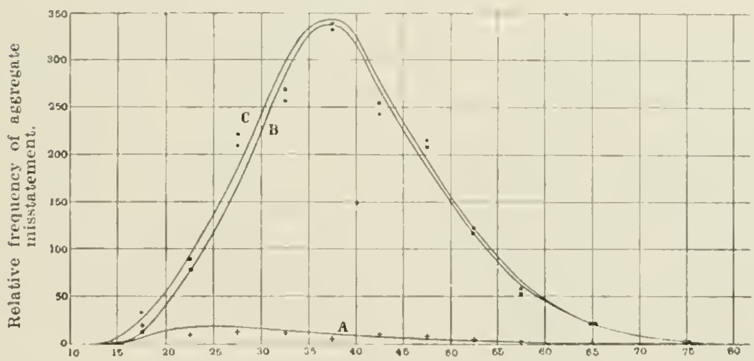
Another remarkable feature, worthy of attention, in the population-graphs for Australia, as compared with those of England and Wales, is the similarity of the features for ages 37, 38, 39, and 40, viz., in the graphs for "males" and for "females" of the former, with that for "persons" for the latter country. There is also some similarity at ages 50 and 60, due to excessive numbers for the ages ending in 0; see Fig. 39.

5. **Characteristics of systematic misstatement.**—It having been ascertained that in some cases the ages given in the Census cards were not correct, notwithstanding the explicit directions, persons who made misstatements were invited to send in corrections. Out of over 7000 received, 1660, containing definite information as to the age given and the amount of misstatement of age in the case of females, were tabulated in age-groups, and according to the number of years the age had been misstated. Of these, one-half (830) were for the State of Victoria, and the balance of 830 for the State of New South Wales. The tabulated results were as follows :—

Table XIII.—Analysis of 1660 Cases of Misstatements of Age at Census of 1911, Australia.

Correction in Years.		Correct Age.												No. per 1000.	
		Under 20	21 to 25	26 to 30	31 to 35	36 to 40	41 to 45	46 to 50	51 to 55	56 to 60	61 to 70	Over 70	Total.	% Crude.	Smoothed.
Over-statements	Over 5 ..	2	—	—	1	1	2	1	—	—	—	—	7	79	19
	5 ..	1	—	1	—	—	—	—	2	—	—	—	4	45	64
	4 ..	3	1	—	1	—	1	1	—	—	—	—	7	79	96
	3 ..	5	3	2	1	1	2	1	2	—	—	—	17	191	146
	2 ..	4	—	1	4	1	1	1	1	1	—	—	14	157	226
	1 ..	5	6	8	5	3	5	4	1	2	1	—	40	449	449
Total ..		20	10	12	12	6	11	8	6	3	1	—	89	1000	1000
Smoothed ..		5	17	18	14	11	8	6	4	3	2	1	89	-53.6	
Understatements	1 ..	10	55	80	56	72	49	41	22	9	6	1	401	255	255
	2 ..	2	21	62	62	87	48	54	22	5	9	—	372	237	193 + 37*
	3 ..	1	4	36	48	45	37	27	17	11	5	—	231	147	145
	4 ..	—	—	18	26	49	23	19	11	9	9	1	165	105	107
	5 ..	—		8	30	23	26	23	9	3	4	—	126	80	79
	6 ..	—	—	5	13	21	20	13	11	6	3	3	95	60	58
	7 ..	—	—	1	7	10	8	9	4	1	1	1	42	27	41
	8 ..	—	—	1	6	7	8	8	3	4	—	—	37	24	27
	9 ..	—	—	—	2	4	4	1	5	2	2	—	20	13	18
	10 ..	—	—	—	6	13	16	9	10	3	6	—	63	40	11 + 25*
	11–15 ..	—	—	—	—	2	2	5	3	1	1	—	14	9	3
	Over 15 ..	—	—	—	1	—	2	—	—	1	—	1	5	3	1
Total ..		13	80	211	257	333	243	209	117	55	46	7	1571	1000	1000
Smoothed ..		13	77	168	281	337	268	189	120	64	44	7	1571	=946.4	
Grand Total ..		33	90	223	269	339	254	217	123	58	47	7	1660		
Smoothed ..		18	94	186	298	348	276	195	124	67	46	8	1660	= 1000	

* The abnormality is about 37 in the one case, and 25 in the other. The 193 and 11 would be the normal values in a total of 1000 — 37 — 25 = 938.



True Age of Females.
Relative frequency of Overstatements (A) and Understatements (B) of age with females according to true age.

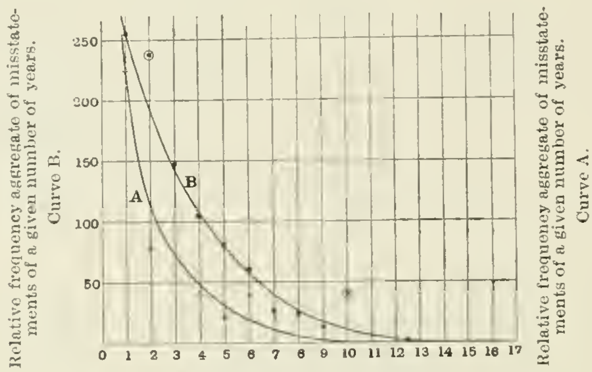
Fig. 40.

In the above table, the results of which are shown in Figs. 40, 41 and 42, the "smoothed" figures for the *aggregate number of overstatements according to age* probably very closely represent the tendency in general : the results, however, for under 20 years of age appear to be unduly large. The smoothed results for the *aggregate of understatements according to age* indicate the probable tendency in general. The smoothed result for the *total number of misstatements* (over and under) *according to age* are merely the sum of the preceding. The crosses, squares and circles

represent the age-group aggregates for overstatements, understatements and total misstatements, respectively. These results are shewn respectively by curves A, B and C in Fig. 40.

The smoothed results of the *aggregate number of overstatements according to the amount of overstatement* (see the vertical column at the right hand side of the table) probably represent the distribution, but the aggregate 89 is so small that it can be regarded only as a rough indication. The graph of this is curve A of Fig. 41.

AUSTRALIAN CENSUS, 1911.



Misstatement of age in years.
Curve A denotes overstatement; curve B denotes understatement.
Fig. 41.

The smoothed result of the *number of understatements according to the amount of understatement.* is probably represented by the final column in the table. In this, however, the abnormality of understatements of 2 and 10 years is very striking. The graph is curve B of Fig. 41, and the abnormal position for 2 and 10 year understatements is shewn by the small squares with circles surrounding. This abnormality is probably on the whole real; that is to say, misstatements of 2 and 10 years had a real predominance over the number which might have been expected according to a probable law of frequency based upon misstatements of other amounts (say, a frequency varying inversely as some power of the magnitude of the misstatement).[1] At the same time it is also possible that in part it represents defects in the *allegation* as the amount of misstatement.

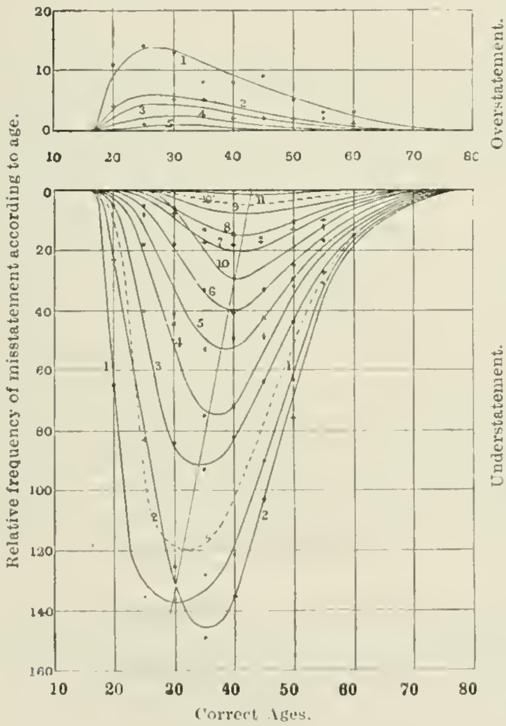
6. **Distribution of misstatement according to amount and age of persons.**—By forming a series of 10-year groups from Table XIII., with the central ages 20, 25, 30, etc. (completed years), and plotting these as ordinates, some idea is obtained of the form of the function representing the *relative* frequency of misstatement according to both age and magnitude of misstatement. Curves are then drawn among these positions, the results shewn on Fig. 42 being thus obtained. The families of curves are obviously fairly regular, and are skew. The positions of the ordinate-terminals, obtained as described, are shewn in the following way. The

[1] In a Census the frequency is for integral amounts of misstatement only.

character of the mark denoting the terminal of the ordinate for a misstatement of 1 year is a dot ; for 2 years a vertical cross ; 3, a square ; 4, a slanting cross ; 5, a circle and vertical line ; 6, a lozenge ; 7, a circle and horizontal line ; and 8, a slanting cross. After the age 55 the results are rather irregular.

The broken lines for understatements of 2 years and 10 years shew what may be regarded as the "normal" positions. That is, had there been no peculiar predominance in the adoption of ages differing by these amounts from the true age, the frequency curve would have been found in about the position of these broken lines. They are numbered with light-faced figures.

The frequency of misstatement according to age, as indicated in Table XIII. and Fig. 42, refers to the number actually existing in the age-groups, for which Table 18 of Part I. of the Australian Census may be consulted (pp. 32-33). To ascertain the frequency for equal numbers of females a correction is necessary, viz., division of each result by the number in the age-group to which it refers.



The figures on the curves denote the amount of misstatements in years.

Fig. 42.

Although over 7000 acknowledgments of misstatements of age were received, mostly from women, the proportion these bore to the aggregate number of misstatements was not ascertainable, and after a study of other errors revealed by the zig-zag character of the enumerated age-groups, it was decided to regard the characteristic misstatement as sensibly negligible. The absolute scale of the frequency is not known, since the total number of misstatements could not be inferred. Nevertheless its form is important as throwing light upon the relative frequency of misstatements of different amounts by women of different ages. The result may be summed up as follows :—

The analysis of acknowledged misstatements shewn in the table gives the following indications (of course for females only) :—

(i.) Understatement of age constitutes 94.64 per cent., and overstatement 5.36 per cent. of the aggregate cases of misstatement.

(ii.) Excepting in the case of understatements of 2 years and 10 years, which are evidently abnormal, the frequency of misstatement diminishes with the number of years misstated, at first very rapidly and later more slowly.

(iii.) The greatest frequency of understatement of all amounts corresponds to the age of about $37\frac{1}{2}$ years.

(iv.) The age corresponding to the greatest frequency of understatement of a given number of years increases with the amount of understatement approximately in the ratio of about $1\frac{1}{4}$ years for every year of understatement, except in the case of 2 and 10 years.

(v.) The frequency of understatements of 2 years is about 1.2 times that which would accord with the general tendency to understatement ; and the maximum is for the age of about 35 years.

(vi.) The frequency of understatement of 10 years is about 3.3 times that which would accord with the general tendency to understatement ; and its maximum is for the age of about 30 years.

While these indications, being based upon only 1660 investigated cases, have limited validity, they are probably substantially correct. An insufficient number of returns were received from males to draw any deductions as to the frequency of misstatement according to age and amount of misstatement.

For *correction* purposes misstatements regarding age are best tabulated according to the *age declared* ; on the other hand, for the expression of the measurement of misstatement they are better tabulated according to the *true age*. Since probably by far the greater number of persons give their age correctly, it is probably desirable to regard the curves for overstatement and understatement as discontinuous at the value zero.

7. **The smoothing of enumerated populations in age-groups.**—The generalities of smoothing have been partially dealt with in VII., hereinbefore ; see particularly §§ 1–9. Figs. 37 and 38 shew the graphs of the enumeration in age-groups of the Australian Census of 1911 ; obviously these are not the true results. It is obvious that the " smoothed" curve must be of higher accuracy than the zig-zag results, since there are strong reasons for believing that the numbers are sufficiently large to give a " smooth curve." The following principles may be taken as a guide in smoothing :—

(i.) Any smoothed curve so drawn as to equalise the zig-zag results (doubtless) better represents the facts than the original data.

(ii.) The drawing of the smoothed curve can be assisted by arithmetical and algebraic devices.

(iii.) The adoption of a particular position for the smoothed curve must be governed not only by mathematical considerations, but by the probabilities of each particular case.

(iv.) If arithmetic or algebraic methods are employed, they should be such as do not involve systematic error.

(v.) The accumulations of error at all ages should be as small as possible, and therefore should frequently change in sign, and the grand total should be approximately (or exactly) the enumerated total.[1]

The method of smoothing by drawing a curve fulfilling the conditions indicated is known as the *graphic method*. Before considering it further, we shall examine the essential character of smoothing by grouping, and the limitations of smoothing by grouping methods. First, we consider the error introduced by mere means of aggregates.

8. **The error of linear grouping.**—If a series of points lie on a curve say, *convex upwards*, their mean, weighted or otherwise, will obviously lie *below* the curve, that is, x'_m, y'_m, denoting the mean of the co-ordinates, and w the weight assigned to any point, the point having these co-ordinates, viz. :—

$$(311)\ldots\ldots x'_m = \frac{\Sigma\,(wx)}{\Sigma w}\,;\quad y'_m = \frac{\Sigma\,(wy)}{\Sigma w}$$

will, in the case supposed, be below the curve. If the original points lie on a straight line, the point will, of course, be on that line. Graphically, the point may be determined for equal weights thus :—

Let P, Q, R, S, etc., be any points : the point midway between P and Q is the mean of P, Q ; the point one-third of the distance of this mean from R, towards R, is the mean of P,Q,R ; and, similarly, that one-fourth of the distance of this last toward S, is the mean of P,Q,R,S ; and, in general, the mean of n points is $1/n$th of the distance of the mean of $(n-1)$ points towards the nth point.

It follows from this that when n values are taken of any quantities, which, being graphed, are found to lie, not upon a straight, but upon a curved line, then the mean of the independent variable (or argument) does not correspond to the mean of the dependent variable (or value of the function) unless the points representing them are all symmetrically situated about the middle point. Thus, if we have the numbers in a population at, say, ages 50 to 55, the mean does not correspond to the age 52. We proceed to consider the magnitude of the systematic error involved.

[1] Exact correspondence is neither essential nor extremely desirable, but as it is easy to secure, there is no reason why it should not be insisted upon. A simple way of securing it is to multiply each group-result by a correcting factor, viz., in VII., § 7, herein.

If we suppose the results to be representable by the equation $y = A + Bx + Cx^2 +$ etc., and take points on either side of the middle so that the correct value of y is A, we readily derive the following expressions shewing the errors of ternary, quinary, and larger groupings:—

$$(312)\ldots\ \frac{1}{3}\Sigma y = A + \tfrac{2}{3}\,Ck^2 + \tfrac{2}{3}\,Ek^4 + \text{etc. (ternary).}$$

$$(313)\ldots\ \frac{1}{5}\Sigma y = A + 2\,Ck^2 + 6\tfrac{4}{5}\,Ek^4 + \text{etc. (quinary).}$$

$$(314)\ldots\ \frac{1}{7}\Sigma y = A + 4\,Ck^2 + 28\,Ek^4 + \text{etc. (septenary)}$$

$$(315)\ldots\ \frac{1}{9}\Sigma y = A + 6\tfrac{2}{3}\,Ck^2 + 78\tfrac{2}{3}\,Ek^4 + \text{etc. (nonary).}$$

$$(316)\ldots\ \frac{1}{11}\Sigma y = A + 10Ck^2 + 178\,Ek^4 + \text{etc. (undecenary).}$$

If the number of terms in the groups be denoted by n, the law of increase in the numerical coefficients, γ say, of C and ϵ of E is as shewn hereunder:—

$$(317)\ldots\ldots\ \gamma C = \frac{1}{12}(n^2 - 1)\,C.$$

$$(318)\ldots\ \epsilon E = \left\{-\frac{1}{30}(n-1) + \frac{1}{30}(n-1)^2 + \frac{1}{20}(n-1)^3 + \frac{1}{80}(n-1)^4\right\}E$$

The latter may be put in the more concise form in (319) hereunder. Hence the error of a simple mean is shewn in the most general form by the following expression, viz. :—

$$(319)\ldots\ \frac{1}{n}\Sigma y = A + \frac{1}{12}(n^2-1)Ck^2 + \frac{1}{240}\left\{(n^2-1)(3n^2-7)\right\}Ek^4 + \text{etc.}$$

The values of Ck^2, Ek^4, etc., can be very readily expressed in terms of the ordinates to say the roughly smoothed curve. Thus, using accents to denote ordinates symmetrically situated on either side of the middle (unaccented) ordinate, we have—

$$(320)\ldots\ldots\ \frac{1}{3}(\eta_{,} - 2\eta + \eta') = Ck^2 + Ek^4 + \text{etc.}$$

$$(321)\ldots\ldots\ \frac{1}{5}(\eta_{,,} + \eta_{,} - 4\eta + \eta' + \eta'') = 2Ck^2 + 6\,\frac{4}{5}\,Ek^4 + \text{etc.}$$

We may therefore from the above equations obtain the value of y, free from the systematic error due to curvature. Thus

$$(322)\ldots\ldots\ y_0 = \frac{1}{3}\left\{\Sigma y - (\eta_{,} - 2\eta + \eta')\right\}$$

and from (313) and (321)

$$(323)\ldots\ldots\ y_0 = \frac{1}{5}\left\{\Sigma y - (\eta_{,,} + \eta_{,} - 4\eta + \eta' + \eta'')\right\}$$

for ternary and quinary groupings respectively. These correction-terms in the inner brackets are, as a rule, very small.[1]

[1] To reduce the arithmetical work any one number may be taken from each of the values of η.

The repeated application of any system of grouping leads to more highly smoothed results, but is unobjectionable only if freed from systematic error. It, however, even then, *never* wholly removes the vitiating influence of a value which is seriously defective or excessive.

It is easy to build up from the preceding formulæ a system of coefficients by means of which the repeated groupings can be performed in one operation. Thus, each ordinate being assumed to have equal weight, we have for repetitions of ternary groupings—

Table XIV.—Coefficients for Repeated Grouping.

No. of Repetitions.	Factor.	Resulting Grouping.	Weights to be Applied to Co-ordinates.										
0	$\frac{1}{3}$	Ternary					1	1	1				
1	$\frac{1}{9}$	Quinary				1	2	3	2	1			
2	$\frac{1}{27}$	Septenary			1	3	6	7	6	3	1		
3	$\frac{1}{81}$	Nonary		1	4	10	16	19	16	10	4	1	
4	$\frac{1}{243}$	Undecenary	1	5	15	30	45	51	45	30	15	5	1

The scheme of deriving these is evident.[1] In the same way it is necessary to build up also the scheme of corrections from (314), (315), etc.

9. **Graphic process of eliminating systematic error.**—A simple approximate method of graphically eliminating the systematic error indicated in the preceding section is based on the fact that the distance k between the mean of a series of n ordinates on a parabolic curve and the vertex of the curve is given in Table XV hereunder.

Table XV.—Position of Mean of n Points.

Number (n) of points on curve ..	$n =$	3	4	5	6	7	9	11
Proportional distance of mean of the ordinates from centre of chord towards vertex of curve	$k =$	$\frac{1}{3}h$	$\frac{4}{9}h$	$\frac{1}{2}h$	$\frac{8}{15}h$	$\frac{5}{9}h$	$\frac{7}{12}h$	$\frac{3}{5}h$
	$k =$	$.33h$	$.44h$	$.50h$	$.53h$	$.55h$	$.58h$	$.60h$

the height h being the distance from the middle of the chord to the vertex. Thus, if a series of means of n ordinates are plotted, and a curve be drawn through them, this series can be taken to give an approximate guide to

[1] Thus, $1 \;.\; 2 \;.\; 3 \;.\; 2 \;.\; 1 = \begin{array}{ccccc} & 1 & . & 1 & . & 1 \\ 1 & . & 1 & . & 1 & \end{array}$; $\begin{array}{c}1 \;.\; 1 \;.\; 1\\ \end{array}$.

the shape of the true curve. A section of double the stretch being then taken[1] the interval between the chord and curve along the ordinate is assumed to be four times the similar distance for the central ordinate of the original stretch. Hence in this case the points defined by the means should be moved the following amounts, viz., those in Table XVI.

Table XVI.—Distance of Vertex from Mean of n Points.

Number of ordinates for which a mean is taken	$n =$	3	4	5	6	7	9	11
Proportion of vertex-distance of the *double stretch* to be taken as a correction	$k =$	$\frac{1}{6}H$	$\frac{5}{36}H$	$\frac{1}{8}H$	$\frac{7}{60}H$	$\frac{4}{36}H$	$\frac{5}{48}H$	$\frac{1}{10}H$
	$k =$	$.167H$	$.139H$	$.125H$	$.117H$	$.111H$	$.104H$	$.100H$

H denoting the height of the vertex above the chord double stretch. This correction will eliminate the greater part of the systematic error, but not the whole, inasmuch as the curve has been flattened by taking the series of means: hence the corrections having been applied to the mean points a new curve may be drawn, and the process repeated if necessary. A smooth curve is then drawn among the points ultimately defined.

This process, however, yields results which, after all, are but little better than a direct attempt to draw a smooth curve among the points given by the ordinate-terminals; it is tedious, and its probability is but little greater than that obtained by directly drawing the smoothed curve and correcting it by arithmetical (or algebraic) methods ("hand polishing"). To avoid its tedium of drawing and hand-polishing, what are called *summation methods* have been used. In these a weighted mean is obtained, the weight factors having *opposite* signs in order to eliminate the systematic error indicated in formulæ (312) to (316).

10. **Summation methods.**—Summation methods in so far as they are rigorous, eliminate the systematic error involved in weighted means where the weights have no change of sign. Rigorously devised algorithms, applied to a series of ordinates strictly conforming to a curve of the nth degree, will reconstitute the given ordinates, whereas mere means of a series of ordinates will not only *not* do so, but will increase the error with every repetition of the grouping. The taking of the means of a series of ordinates is therefore valid only where the general trend is either linear, or so nearly linear as to make the corrections referred to negligible. Suppose, then, we have a series of ordinates, the terminals of which O, P, Q, R . . . . Z, are to be smoothed. Evidently we can draw an

[1] That is, if $n + 1$ be the number of ordinates, a curve defined by $2n + 1$ ordinates is taken; thus, if 3 points are originally taken, the curve of double stretch will be that defined by five points.

integral curve of the nth degree through any $n+1$ such points. Geometrically, the summation smoothing process is the following:—Draw a curve of the nth degree through the points O, O$+i$; O$+2i$; ...... O$+ni$: a similar curve through the points P,...... P$+ni$: a third through the points Q,...... Q$+ni$; and so on.[1]

This will give a series of curves of the nth degree, usually close to one another, and sometimes intersecting. The mean position of their intersections on the ordinates (or ordinates produced) is the smoothed curve required. The flexibility, or *fitting power*, of the curve depends, other things being equal (a) on the degree of the curve ; and (b) on the nearness of the points O, O$+i$, etc.; and consequently of P, P$+i$, etc., to each other.

It may readily be demonstrated, graphically or otherwise, that *as the value of i is increased, minor fluctuations are more and more obliterated.* The whole range being limited, the larger the value of n the more points on the curve are fitted by one stretch : hence the smaller i will be ; and the fitting power will consequently be increased.

Since the mean position of the intersection of the curves and the ordinates defines the position of their terminals at the smoothed curve ; and since each point O, P, etc., is the start of one of the component curves, any abnormality in its position (*i.e.*, deviation from the general trend) is reflected in the mean result ; that is, *it produces a deviation of a smaller amount in the direction of the abnormal point.*

The defect of all summation methods is seen, from their geometrical representation, to be the following :—

(i.) The degree of obliteration of minor fluctuations is quite arbitrary and depends upon the character of the summation-system.

(ii.) The result is vitiated by all abnormalities: the method, in fact, does not lead to real smoothing, but to the reduction of the magnitude of the oscillations of the curve.

This may be shewn analytically in the following way. We observe first that if there are $q+1$ points in the total range of q intervals of any component curve taken, then in a *complete*[2] series there will be $q+1$ intersection-points on the ordinates. The mean of these is to be taken. The first complete term arranged according to the powers of the common distance (k) between the ordinates, and the second term will be respectively :—

$$(324) \ldots\ldots y_q = \frac{1}{q+1}\{(a_q + a_{q-1} + \ldots a_0) + (b_{q-1} + 2b_{q-2} + 3b_{q-3} + \ldots)k$$
$$+ (c_{q-2} + 2^2 c_{q-3} + 3^2{}_{q-4} + \ldots)k^2 + (d_{q-3} + 2^3 d_{q-4} + 3^3 d_{q-5} + \ldots)k^3 + \ldots\}$$

[1] Where k is the common interval on the axis of abscissæ between ordinates, the common interval i between the points will always be an integral multiple of k greater than 1 ; that is $i = 2k$, or $3k$, or $4k$, etc.

[2] It is, of course, not *essential* that the series should be what has been called here *complete*, and in Woolhouse's method it was not complete. A complete series may be defined as one where, $q+1$ being the number of points including the terminal ones ranged over by any curve, the initial point of the $(q+1)$th curve is on the same ordinate as the final point of the initial range, viz., the zero (or first) curve.

$$(325)\ldots\ldots\ldots y_{q+1} = \frac{1}{q+1}\{(a_{q+1}+\ldots a_0)+(b_q+\ldots)k+(c_{q-1}+\ldots)k^2+(d_{q-2}+\ldots\ldots)k^3+\ldots\ldots\}$$

Thus the coefficients of the powers of x are changing every term, and consequently the equation of the smoothed curve of, say, $s+1$ points will be of the degree s, that is, it has no relationship whatever to the degree of the originating equations of the nth degree passed through the points 0; $0+i$; $0+2i$, etc.

It is thus seen that results of a "smoothing" by "summation" methods are in principle *toto cœlo* different from those obtained by methods which ensure conformity to some function adopted for considerations of the nature of the case.[1]

Numerous papers on the summation method have appeared from time to time in the Journal of the Institute of Actuaries by various investigators, of whom the principal are the following :—J. A. Higham, W. S. B. Woolhouse, G. F. Hardy, J. Spencer, T. G. Ackland, G. J. Lidstone, G. King, R. Todhunter. Some of these have contributed several papers on the subject. A specially valuable one, on "The rationale of formula for graduation by summation," by G. J. Lidstone, appeared in the Journal of the Institute of Actuaries, Vol. XLI., pp. 348 *et seq.*, and XLII., pp. 106 *et seq.* An important paper on the subject by Dr. J. Karup will also be found in the Transactions of the Second Actuarial Congress, p. 31 *et seq.*

The subject of graduation of summation has also quite recently been re-examined by Mr. C. H. Wickens,[2] and formulæ based on ranges of *three* determined points (0, $0+i$, and $0+2i$) and *four* determined points (*i.e.*, including also $0+3i$) are discussed for the developments of quinary formulæ and formulæ other than quinary, the adjective denoting the number of spaces into which i is divided. That is, if $i=rk$ then the formula derived is an r-ary formula. It is shewn that there are great advantages in making the series complete, and that in taking the mean it is advantageous to allow only half-weight to the terminal points of intersection on any ordinate.[3]

The following weights (Table XVII.) have been deduced by Mr. Wickens for the different ordinates about the middle ordinates, th

[1] Prof. Karl Pearson's scheme, adopted by many biometricians, is to resolve the data under a suitable type-form derived from a generalised theory of probability, certain criteria being used to decide which form should be preferred. A single Pearsonian curve, however, will not apply to population-enumerations, although the population-curve may be *empirically* considered to be a combination either of Pearsonian or of other curves.

[2] An extension of the principle underlying Woolhouse's method of graduation, read 30th October, 1911, Trans. Act. Soc., N.S.W., Session 1912, pp. 243-7.

[3] There are many physical analogies for this process. For example, if a physical property be measured at equidistant points along a line including the terminals the mean value is $(a + 2b + 2c + \ldots + 2y + z)/2N$, where N is the number of spaces into which the points divide the line.

marked (3) and (4) being deduced from curves passed through 3 points and 4 points respectively. The similarity is obvious. Other formulæ may be obtained from the paper in question.

Table XVII.—Summation-formula-coefficients to be applied to a Series of r Ordinates Deduced on the Basis of (3), and on a Basis of (4) Determined Points.

Ordinates $q+1$ (x/k)	Binary $i=2k$		Ternary $i=3k$		Quarternary $i=4k$		Quinary $i=5k$		Sonary $i=6k$	
	(3)	(4)	(3)	(4)	(3)	(4)	(3)	(4)	(3)	(4)
−12	..	..	..	..	..	..	..	..	0	0
−11	..	..	..	..	..	..	..	..	−5	$-3\frac{5}{9}$
−10	..	..	..	..	..	..	0	0	−8	$-7\frac{1}{9}$
−9	..	..	..	..	..	..	−2	$-1\frac{3}{5}$	−9	−9
−8	..	..	..	..	0	0	−3	$-2\frac{4}{5}$	−8	$-8\frac{5}{9}$
−7	..	..	..	..	−3	$-2\frac{1}{2}$	−3	$-3\frac{1}{5}$	−5	$-6\frac{1}{9}$
−6	..	..	0	0	−4	−4	−2	$-2\frac{2}{5}$	0	0
−5	..	..	−1	$-\frac{5}{9}$	−3	$-3\frac{1}{2}$	0	0	29	$25\frac{2}{3}$
−4	0	0	−1	$-1\frac{1}{9}$	0	0	12	$10\frac{4}{5}$	56	$53\frac{1}{3}$
−3	−1	−1	0	0	19	$17\frac{1}{2}$	23	$22\frac{2}{5}$	81	81
−2	0	0	7	$6\frac{2}{3}$	36	36	33	$33\frac{3}{5}$	104	$106\frac{2}{3}$
−1	9	9	13	$13\frac{1}{3}$	51	$52\frac{1}{2}$	42	$43\frac{1}{5}$	125	$128\frac{1}{3}$
0	16	16	18	18	64	64	50	50	144	144
1	9	9	13	$13\frac{1}{3}$	51	$52\frac{1}{2}$	42	$43\frac{1}{5}$	125	$128\frac{1}{3}$
2	0	0	7	$6\frac{2}{3}$	36	36	33	$33\frac{3}{5}$	104	$106\frac{2}{3}$
3	−1	−1	0	0	19	$17\frac{1}{2}$	23	$22\frac{2}{5}$	81	81
4	0	0	−1	$-1\frac{1}{9}$	0	0	12	$10\frac{4}{5}$	56	$53\frac{1}{3}$
5	..	..	−1	$-\frac{5}{9}$	−3	$-3\frac{1}{2}$	0	0	29	$25\frac{2}{3}$
6	..	..	0	0	−4	−4	−2	$-2\frac{2}{5}$	0	0
7	..	..	..	..	−3	$-2\frac{1}{2}$	−3	$-3\frac{1}{5}$	−5	$-6\frac{1}{9}$
8	..	..	..	..	0	0	−3	$-2\frac{4}{5}$	−8	$-8\frac{5}{9}$
9	..	..	..	..	..	..	−2	$-1\frac{3}{5}$	−9	−9
10	..	..	..	..	..	..	0	0	−8	$-7\frac{1}{9}$
11	..	..	..	..	..	..	..	..	−5	$-3\frac{5}{9}$
12	..	..	..	..	..	..	..	..	0	0
Sum of Coefficients	32	32	54	54	256	256	250	250	864	864

For the mode of obtaining the values given by these formulæ by processes of *summation*, reference should be made to the paper, in which also the smoothing coefficient is given as follows :—

Table XVIII.—Smoothing Coefficients.

Interval s	No. of Terms or Ordinates $4s-1$	Series (3) $\sqrt{(7s^2+1)/4s^3}$	Series (4) $\sqrt{7(s^2-1)(s^2+5)+36s/6s^4}$
2	7	.1683	.1683
3	11	.0741	.0615
4	15	.0415	.0316
5	19	.0265	.0193
6	23	.0184	.0130
7	27	.0135	.0094
8	31	.0103	.0071
9	35	.0082	.0056
10	39	.0066	.0045

The smallness of the smoothing coefficient is a measure of the efficiency in smoothing.[1]

11. **Advantages of graphic smoothing over summation and other methods.**—This graphing of the group-results of an enumeration (numbers according to years of age in the instance immediately under review) yields a succession of rectangles, or, if we prefer, points denoting their heights. Smoothing in such a case consists essentially in transferring numbers of those who alleged they were a given age to some other nearly identical age, the reason for this transfer being that it is judged *a priori* (and justly so) that the irregular distribution indicated by the data does not accord with the real facts. To do this there is no better way than to draw among the tops of the rectangles (or the points representing them) a smoothed curve following every variation of their general trend, which, *in the judgment of the analyst*,[2] is regarded as probably conforming to the facts. This can be done, and the result scaled and smoothed arithmetically, that is, by differencing. The aggregates as by enumeration and by the smoothed curve can be formed, and the accumulated differences examined to see that they are kept within probable limits; that is, are alternately positive and negative, and are never great (see VII., §12). The initial curve can then be amended whenever improvement seems possible; thus in its final form the grand total can be made identical with the enumeration, and the difference between the enumerated and smoothed aggregates up to any value of the variable (age) can be made the least possible *for the form of curve deemed to be best on examining the graph of the enumerated results*.[3]

The logic of this process has been admirably expressed by Whewell, and before him again by Sir John Herschel, in the following passages:—

> "This curve once drawn must represent . . . . the law . . . . much better than the individual raw observations can possibly . . . . do. . . . . The series of lines joining the consecutive points . . . cannot possibly represent reality. . . . . . If, however, we thus take the whole mass of the facts . . . . by making the curve which expresses the supposed observations regular and smooth . . . . we are put in possession . . . . of something more true than any (one) fact by itself."—Sir J. Herschel, Trans. Astr. Soc., Vol. V., pp. 1–4.

[1] See G. F. Hardy, Journ. Inst. Act., Vol. xxxii., p. 376.

[2] Any attempt to dispense with the element of judgment is really illusive. The adoption, for example, of a summation method will yield appreciably different results according to the range taken. Thus a real undulation in a population curve may be virtually obliterated by the process.

[3] There is a tendency to forget that technical processes are but instruments in the hands of the user, and formulæ employed confer no validity to the elements depending upon judgment.

> "The peculiar efficacy of the Method of Curves depends upon this . . . . that order and regularity are more clearly recognised when thus exhibited to the eye as a picture (and) not only enables us to obtain laws of Nature from *good* observations, but . . . . from observations which are very *imperfect*. . . . . We draw our main regular curve not *through* the points given by . . . . observations, but *among* them."—Whewell[1], Novum Organon Renovatum, Bk. III., Chap. vii., p. 204, 3rd Edit., 1858.

Finally, it may be remarked that by adopting the graphic method of smoothing, minor and unmeaning fluctuations are avoided. The invalidity of merely mechanically applying various summation formulæ has been shewn by G. J. Lidstone; he has indicated how, by the summation method, unmeaning fluctuations are introduced into what may be known *a priori* to be a straight line.[2]

12. **Graphs of Australian population distributed according to age and sex for various Censuses.**—Adopting the principles indicated, the graphs of the enumerated population of Australia for the Census of 1911 distributed according to age, shewed that, both for females and for males, the adoption of any function to which the results should be conformed was out of the question. It was evident also that a "summation method" was quite unsuitable. In the results for 1911 there was a sharp increase in the numbers for ages 13 to 18; then a zig-zag result up to age 22 before a decided decrease appeared. It was thus evident that results must be examined, and the smoothing based upon considerations as to the possibility of misstatement. The data therefore were simply graphically smoothed by drawing first a freehand curve among them, the changes of direction of this curve being made a minimum, so far as that was possible, while following all fluctuations deemed to represent the actual facts. This curve was then carefully drawn with the aid of splines, French curves, etc., the ordinates scaled off and adjusted arithmetically.[3] The result of this smoothing is shewn on Figs. 37 and 38

As has been shewn in § 10 and formulæ (324) and (325) hereinbefore, this is obvious from either geometrical or analytical considerations. For that reason the graphic process has been preferred to summation processes, which latter are regarded as theoretically invalid for the reasons indicated.[4]

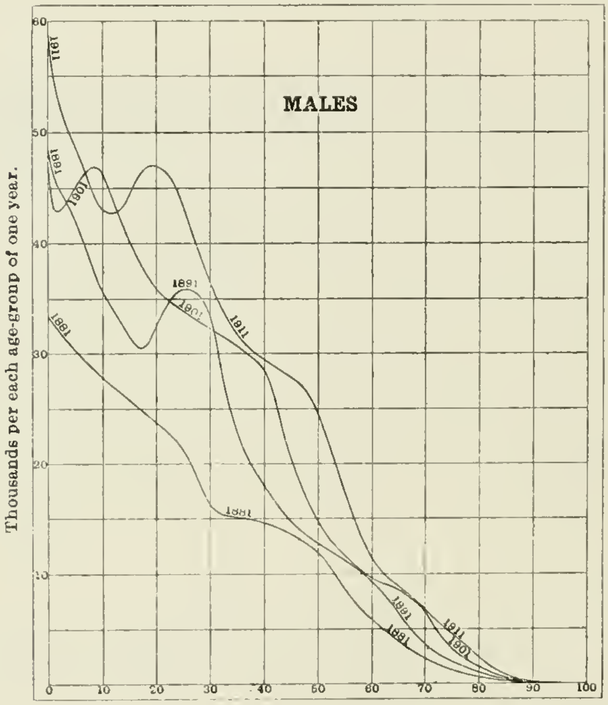
[1] See also T. B. Sprague, Journ. Inst. Act., Vol. XXX., pp. 161 3, 1892; James Sorley, Journ. Inst. Act., Vol. XXII., pp. 309-340, in particular 3: The Graphical Method, pp. 321-8; T. B. Sprague's works on "The Graphic Method, etc.," Journ. Inst. Act., Vol. XLI., p. 182.

[2] On the rationale of the Formulæ for graduation by summation. Journ. Inst. Act., Vol. XLI., 1907. p. 360, and diagrams A, B and C.

[3] Identical methods were also applied to the data of the earlier Censuses.

[4] In the summation methods, as we have seen, fluctuations are introduced into curves in order to conform to a convenient algorithm, rationally deduced. But a little reflection will convince any mathematician that the minute oscillations in the directions of the tangents, involved in the process, would be better eliminated, when that can conveniently be done: and in any case, in the presence of large departures of individual results from the smoothed curve, these small fluctuations have neither real significance nor validity.

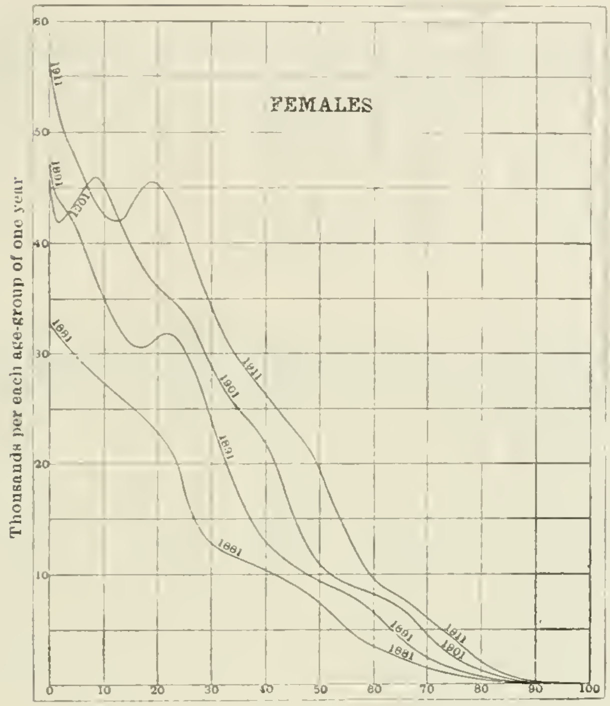
Graphs shewing the distribution according to sex and age have been prepared for the Australian Census of 1881, that of 1891 and 1901, as well as that of 1911. The results for 1881 were deduced from quinquennial groups; those for the latter Censuses from year-groups; and they are shewn on Figs. 43 and 44. It will be seen that intervals of ten years cause considerable differences in the forms of the curves; these differences are due of course to migration and to fluctuations in the birth and mortality rates.



Commencing points of age-groups of one year at age indicated.

Fig. 43.

The curves in Figs. 43 and 44 are interpreted in the following way, viz.:—The ordinate or vertical distance to the curve at any point represents in thousands the number of males (or females) in the age-group of one year commencing at the age in question.



Commencing points of age-groups of one year at age indicated.

Fig. 44.

13. **Growth of population when rate is identical for all ages.** A population P_0 increasing at the instantaneous rate ρ per unit of time becomes, if that rate be constant, as we have seen, $P_t = P_0\, e^{\rho t}$ see II., §§ 1-10, formulæ (1) to (14). Hence, if the numbers between the ages x and $x + dx$ for the epoch $t = 0$, are represented by $P_0 f(x) dx$, in which case

(326)...... $\int_0^\omega f(x)\, dx = 1$

and *the rate of increase be the same for all ages*, then the numbers between the ages x and $x + dx$ at any later date t, must be

(327)...... $P_t\, dx = P_0\, e^{\rho t} f(x)\, dx$.

the aggregate being $P_0\, e^{\rho t}$; that is to say $f(x)$ *remains constant*. Hence, if the age-groups be divided by the total population, the results will be identical, *i.e.*, the *relative* numbers will be seen to remain the same and *their graphs will be identical*. If, however, the aggregate numbers, denoted by $F(x)$, are graphed, *the graphs will not be identical*. For we have in the latter case

(328)...... $F_t(x)\, dx = e^{\rho t} F_0(x)\, dx$;

and by hypothesis ρ is not a function of x; hence

(329)...... $\dfrac{d F_t(x)}{dx} = e^{\rho t} \cdot \dfrac{d F_0(x)}{dx}$; or $\tan\theta_t = e^{\rho t} \tan\theta_0$

that is, the *slopes* of the tangents to the graph of the population are increased in the proportion $1 : e^{\rho t}$. In the absence of all information of "migration" and "natural increase" (increase by excess of births over deaths) the rate of increase of the preceding period must be assumed to continue not only for the population as a whole, but also for each age; which is expressed by

$$(330)\ldots\ldots P_t = P_0\, e^{\rho t} \int_{x_1}^{x_2} f(x)\, dx.$$

and (327) hereinbefore.

14. **Growth of population where migration element is known.** If the ages and numbers of migrants be known, as well as the ages and numbers of the dying, then it is possible to determine the numbers in each age-group by remembering that survivors after t years have increased their age by t years.

Except for very small communities, this method of estimating populations according to age (and sex), is, however, perhaps impracticable. We shall, however, later consider it. Here it may be noted that the estimation may be most conveniently treated in single year age-groups, *i.e.*, not by infinitesimal methods. The value of the method is that it would enable all rates to be finally made up intercensally, whereas, after a Census has rendered the intercensal adjustments possible, they have always to be corrected.

15. **Growth of population when rate of increase varies from age to age.** Changes in the birth-rate (= rate of immigration at age 0), in the death-rate for various ages (= rate of emigration at age x), in the rapidity of migration and age of migrants (= rate of immigration or of emigration at age x) causes a change to take place in the form both of $f(x)$ and $F(x)$ referred to in the previous section. The graphs of $f(x)$, *i.e.*, of *relative* numbers, at different epochs all give an area of unity between the limits 0 and ω (= end of the longest life); hence the curves for different epochs necessarily intersect; those of $F(x)$, *i.e.*, of *absolute* numbers, give the areas P_t, and may or may not intersect. We consider the consequence of those variations which change the form of $f(x)$; see Figs. 43 and 44.

Where we have to interpolate to obtain intercensal populations, or to extrapolate to predict a population, we may assume that the tangents to the curve $f_0(x)$ change uniformly with time; that is, they become those of $f_T(x)$ by a linear change with time, T denoting the intercensal period. Thus

$$(331)\ldots \frac{df_t(x)}{dx} = \frac{df_0(x)}{dx} + \sigma\frac{t}{T}\,;\ \text{where}\ \sigma = \frac{df_T(x)}{dx} - \frac{df_0(x)}{dx}$$

that is, σ is the total change in the tangent in the intercensal period T. Hence, given the total population at the time t, we can effect its distribution according to age by determining merely $f_t(x)$ on the supposition indicated.

This supposition (i) is of a more general character than that of supposing that the number at any age changes linearly: supposition (ii.). Graphically, the difference between the two is that, according to supposition (i.), the intercept on any ordinate between the graphs of $f_0(x)$ and $f_T(x)$, divided in the ratio t/T, gives the position of $f_t(x)$, while according to supposition (ii.) it is the intercept between $F_0(x)$ and $F_T(x)$ which is uniformly divided. The advantage of supposition (i.) is that only the *form* of $f_t(x)$ is fixed; the graph of $F_t(x)$ can then be made to agree with any intercensal estimate of population.[1]

16. **The prediction of future population and its distribution.**—The graphs of population of various countries for the years 1790 to 1910, Fig. 3 hereinbefore, discloses no general law. All shew what may be called oscillatory development. The graph of the population of Australia from 1788 to 1914 (see Official Year Book No. 8 of the Commonwealth of Australia, p. 127) shews also this feature in a fairly well marked degree, and those of the individual States exhibit more striking oscillations. Hence accurate predictions even of total population of any precision are not possible. Figs. 43 and 44 shew that accurate predictions for age-groups are not only *not* possible, but may be even more misleading than the assumption of an unchanged distribution according to sex and age. It may be noted, however, that there is a general similarity, though there is by no means identity, in the forms of the graphs for males and females. The great fluctuation in the masculinity of the population according to age is also evident from a comparison of the results shewn on Figs. 43 and 44. This, however, will be discussed later.

[1] See Census Report, Vol. I., Chap. IX., post-censal adjustment of population estimates for the intercensal period 1901–11.

X.—THE MASCULINITY OF POPULATION.

1. **General.**—The ratio between males and females in any population has been called its *masculinity*, and the fluctuations of such a ratio are obviously important. The following ratios of the aggregate number of males to the aggregate number of females in various populations will give an idea of how closely the number approximates to unity.

Table XIX.—Masculinity of Various Populations (about Year 1900).

Norway ..	1891	.932	Ireland ..	1901	.974	Australia ..	1901	1.101
Sweden ..	1895	.944	Italy ..	1901	.990	C. of G.Hope	1904	1.024
Scotland ..	1901	.946	United States	1900	1.044	India ..	1901	1.038
Eng. & Wales	1901	.954	Canada ..	1901	1.050	Ceylon ..	1901	1.140
Germany ..	1900	.969	Newfoundl'd	1901	1.053			

The results given hereinbefore, viz., in VIII., § 9, Table XI., shew that even when the total numbers for all ages for males is made equal to that for females, there are easily discerned differences between Eastern and Western populations.

In the following Table, viz., XX., the aggregate number of males in the different age-groups in the first eleven countries are divided by the aggregate number of females in the same age-groups, the results being shewn on line W; for the last three countries the similar quotients are shewn on line E.

Table XX.—Change of Masculinity with Age; Aggregate of Various Populations, about 1900.

Countries.	0	1–4	5–9	10–14	15–19	20–24	25–29	30–34	35–39	40–44	45–49
W ..	1.024	1.016	1.014	1.015	.992	.979	.991	1.005	1.021	1.020	1.012
E ..	1.003	.966	1.047	1.212	1.073	.919	1.022	1.037	1.135	1.035	1.131

Countries.	50–54	55–59	60–64	65–69	70–74	75–79	80–84	85–89	90–94	95–100	All Ages.
W ..	.988	.962	.934	.927	.906	.895	.847	.784	.674	.588	.9964
E ..	1.005	1.095	.870	.882	.873	.885	.873	.905	.880	.880	1.0390

The figures in the table shew the relatively large range of "masculinity" for different age-groups, and indicate the desirableness of the determination of a norm for purposes of comparison. We proceed to consider this aspect of the question.

2. **Norms of masculinity and femininity.**—The variations with the lapse of time, of the norm of distribution according to age for the male population of any community, and the same norm for the female population of the same community will not, in general, be identical. The progressive changes, which may have both periodic and aperiodic elements, are best studied by observing the fluctuation of the *masculinity* or of the *femininity* of the population. These characters as ordinarily defined are the number of males to one female (or in practice usually to 100 females), and the number of females to one male, respectively. Thus if m = the number of males, f the number of females, and $p = m + f$ the number of persons of any age, the masculinity μ_1 and femininity ϕ_1 for that age will be expressed by the formulæ :—

$$(332)\ldots\ldots\mu_1 = \frac{m}{f}\,;\ \phi_1 = \frac{f}{m}$$

with suffixes to denote the age. When these quantities and their variations are known, the changes taking place in the relative numbers of the sexes are determined as soon as the variations in the norm for the entire population (persons) are ascertained ; see VIII., §§ 8 to 10. The curve shewing the variations of the norms for both sexes at each age from epoch to epoch is not an essential, for their fluctuation is determinable from the fluctuation of the norm for persons, and the fluctuation of either the masculinity or the femininity. For this purpose a somewhat different definition of masculinity is desirable ; this we shall now consider.

3. **Various definitions of masculinity and femininity.**—For many purposes definitions other than that mentioned above have advantages. Both of the functions referred to for ordinary populations approximate to unity. But other functions may be adopted which hover either about $\frac{1}{2}$ or about zero. For example, the ratio of males (or of females) to the whole population, is a quantity which ordinarily approximates to $\frac{1}{2}$; or yet again the ratio of the *difference* of the number of males and females to the total population is a number which ordinarily approximates to zero. Algebraically, the three methods and their interrelations are as follows :—

1st Method :—

$$(333)\ldots\text{Masculinity} = \mu_1 = \frac{m}{f}\,;\ \text{Femininity}\,\phi_1 = \frac{f}{m} = \frac{1}{\mu_1}$$

Possible range 0 to $+\infty$; ordinary value about 1.

2nd Method:=

$$(334)\ldots\text{Masculinity } = \mu_2 = \frac{m}{m+f} = \frac{\frac{m}{f}}{\frac{m}{f}+1} = \frac{\mu_1}{1+\mu_1};$$

$$\text{Femininity } = \phi_2 = \frac{f}{m+f} = \frac{\phi_1}{1+\phi_1} = \frac{1}{1+\mu_1}$$

Possible range 0 to + 1 ; ordinary value about $\frac{1}{2}$.

3rd Method:=

$$(335)\ldots\text{Masculinity } = \mu_3 = \frac{m-f}{m+f} = \frac{\frac{m}{f}-1}{\frac{m}{f}+1} = \frac{\mu_1-1}{\mu_1+1};$$

$$\text{Femininity } = \phi_3 = \frac{f-m}{f+m} = \frac{\frac{f}{m}-1}{\frac{f}{m}+1} = \frac{\phi_1-1}{\phi_1+1}$$

Possible range − 1 to + 1 ; ordinary value about zero.

The mutual relations subsisting among these several quantities are set out in the following table :—

Table XXI.—Relations subsisting between Masculinity and Femininity according to Various Definitions.

Function.	Expressed in terms of—					
	μ_1	μ_2	μ_3	ϕ_1	ϕ_2	ϕ_3
μ_1	μ_1	$\frac{\mu_2}{1-\mu_2}$	$\frac{1+\mu_3}{1-\mu_3}$	$\frac{1}{\phi_1}$	$\frac{1}{\phi_2}-1$	$\frac{1-\phi_3}{1+\phi_3}$
μ_2	$\frac{\mu_1}{1+\mu_1}$	μ_2	$\frac{1}{2}(1+\mu_3)$	$\frac{1}{1+\phi_1}$	$1-\phi_2$	$\frac{1}{2}(1-\phi_3)$
μ_3	$\frac{\mu_1-1}{\mu_1+1}$	$2\mu_2-1$	μ_3	$\frac{1-\phi_1}{1+\phi_1}$	$1-2\phi_2$	$-\phi_3$
ϕ_1	$\frac{1}{\mu_1}$	$\frac{1}{\mu_2}-1$	$\frac{1-\mu_3}{1+\mu_3}$	ϕ_1	$\frac{\phi_2}{1-\phi_2}$	$\frac{1+\phi_3}{1-\phi_3}$
ϕ_2	$\frac{1}{1+\mu_1}$	$1-\mu_2$	$\frac{1}{2}(1-\mu_3)$	$\frac{\phi_1}{1+\phi_1}$	ϕ_2	$\frac{1}{2}(1+\phi_3)$
ϕ_3	$\frac{1-\mu_1}{1+\mu_1}$	$1-2\mu_2$	$-\mu_3$	$\frac{\phi_1-1}{\phi_1+1}$	$2\phi_2-1$	ϕ_3

4. **Use of norms for persons and masculinity only.**—Instead of having *three* norms, viz., one each for males, females and persons, it will often suffice to have one for persons, and one for masculinity. Thus in the norm of population the masculinity, by method 3, viz. $(m-f)/(m+f)$ is as follows for Europe (i.) and for India (ii.).

Table XXII.—Change of Masculinity with Age.

AGE GROUP.	0	1 to 4	5 to 9	10 to 14	15 to 19	20 to 24	25 to 29	30 to 34	35 to 39	40 to 44	45 to 49
(i.)	+.013	+.009	+.008	+.009	−.003	−.009	−.003	−.007	+.013	+.012	+.008
(ii.)	−.018	−.037	+.004	+.078	+.018	−.063	−.008	−.002	+.015	.003	.044

AGE GROUP.	45 to 49	50 to 54	55 to 59	60 to 64	65 to 69	70 to 74	75 to 79	80 to 84	85 to 89	90 to 94	95 to 105
(i.)	+.008	−.005	−.019	−.032	−.037	−.049	−.051	−.089	−.130	.000	.000
(ii.)	+.044	.017	+.023	−.088	−.090	−.090	−.085	−.077	.090	−.077	−.143

5. **Relation between masculinity at birth and general masculinity of population.**—It has been suggested that some *tendency* exists which, while not very strongly expressing itself, is nevertheless sufficiently evident to equate the numbers of the sexes in the population of any country, or at least that the masculinity at birth is in some way affected by the masculinity of the population.[1] Masculinity here denotes merely the ratio of males to females. that is, M/F.

The population of Australia has enormously changed in its masculinity in a few decades, and consequently affords an opportunity of examining this supposition. The masculinity at birth is compared with that of the population for the years 1829–1913, the latter passing through a wide range of falling values. The results are shewn in the following table :—

Table XXIII.—Average Masculinity of Population and of Births, New South Wales, over Various Periods.

Period.	Average for Years.	Masculinity of Population.	Masculinity of Live Births.	Period.	Average for Years.	Masculinity of Population.	Masculinity of Live Births.
1829–34 ..	6	2.961	1.016	1840–49	10	1.625	1.034
1835–39 ..	5	2.436	1.031	41–50	10	1.560	1.035
40–44 ..	5	1.752	1.026	42–51	10	1.510	1.036
45–49 ..	5	1.498	1.038	43–52	10	1.412	1.036
50–54 ..	5	1.309	1.031	44–53	10	1.433	1.033
55–59 ..	5	1.281	1.033	45–54	10	1.404	1.035
1830–39 ..	10	2.680	1.026	46–55	10	1.375	1.032
31–40 ..	10	2.568	1.018	47–56	10	1.352	1.033
32–41 ..	10	2.443	1.021	48–57	10	1.325	1.029
33–42 ..	10	2.314	1.020	49–58	10	1.308	1.032
34–43 ..	10	2.205	1.029	50–59	10	1.295	1.032
35–44 ..	10	2.094	1.028	60–69	10	1.233	1.058
36–45 ..	10	1.979	1.028	70–79	10	1.196	1.045
37–46 ..	10	1.877	1.026	80–89	10	1.209	1.050
38–47 ..	10	1.784	1.027	90–99	10	1.147	1.054
39–48 ..	10	1.698	1.030	1900–13	13	1.186	1.058

[1] Dusing, Das Geschlechtverhältniss im Konigreich Preussen.

This table seems to shew that, on the whole, the masculinity of birth μ_b can be expressed approximately by such an equation, for example, as

$$(336)\ldots\mu_b = \frac{M_0}{F_0} = 1.06 - 0.0325\,(\mu_P - 1) + 0.0333\,(\mu_P - 1)^{\frac{5}{4}}\,;$$

μ_P denoting the total number of males divided by the total number of females in the population over the period considered. The tabulated mean values of the masculinity of the population, and the position of the curve which represents the formula, are shewn on Fig. 45. The result may, of course, not be *directly* due to the masculinity of the population: both may have varied through some condition itself varying with time. Fig. 46 shews such a variation. This, too, implies an opposite progression; that is, it indicates clearly that while the masculinity of the population was, on the whole, diminishing, that of the birth was, on the whole, increasing.

The results for Victoria point less decisively in the same direction. They are as follows :—

Table XXIV.—Masculinity in Victoria.

Period	1851–60	1861–70	1871–80	1881–90	1891–1900
Of Population ..	1.765	1.303	1.142	1.108	1.049
Of Births	1.046	1.047	1.044*	1.049	1.050

* In conflict with the general indication.

These shew that as the masculinity of the population was diminishing, that of birth was increasing, with the exception of the decennium 1871-1880.

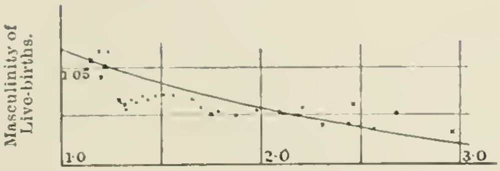
For the Commonwealth of Australia the results for the masculinity of the population at the *beginning* of a year compared with that of the births in the same year, set out in the order of the masculinities of the population, are :—

Table XXV.—Masculinity in Australia.

Masculinity ..	1909	1910	1911	1908	1907	1912	1913
Of Population	1.0764	1.0771	1.0787	1.0793	1.0824	1.0854	1.0885
Of Birth ..	1.0520	1.0638	1.0473	1.0493	1.0489	1.0454	1.0476

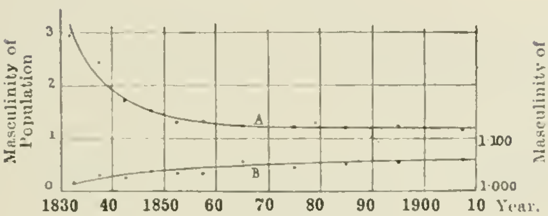
The *trends* are again in opposite directions, but not markedly.

Fig. 45.



The curve is that given by formula (386) above. The dots are individual results.

Masculinity of Population.



The curve A denotes masculinity of population, B masculinity of live-births. The dots are individual results.

Masculinity of Population and of Live-births, New South Wales, 1829-1913.

Fig. 46.

In the following table is set out the masculinity of the births, and in decreasing order of the population of a number of countries ; these give no definite indication :—

Table XXVI.—Masculinity of Various Countries, Arranged in Order of Masculinity of Population.

Country.	Year of Estimation	Masculinity of Population $M \div F$	Period for which Determined.	Masculinity of all Births.	Masculinity of Ex-nuptial Births only
Greece	1889	1.1037	1881–85 ..	1.118	1.059
Australia ..	1907	1.0793	1901–13 ..	1.051	1.042
Servia	1890	1.0548	1885–89 ..	1.047	1.035
Rumania ..	1889	1.0373	1886–90 ..	1.077	1.034
Italy	1881	1.0050	1887–91 ..	1.058	1.044
Belgium ..	1890	.9950	1887–91 ..	1.045	1.022
France	1891	.9930	1887–91 ..	1.046	1.029
Hungary ..	1890	.9852	1887–91 ..	1.050	1.029
Netherlands ..	1889	.9766	1887–91 ..	1.055	1.047
Ireland	1891	.9713	1887–91 ..	1.055	1.048
Finland	1890	.9690	1886–90 ..	1.050	1.052*
German Empire	1890	.9615	1886–90 ..	1.052	1.047
Spain	1887	.9615	1878–82 ..	1.083	1.079
Austria ..	1890	.9578	1887–91 ..	1.058	1.055
Denmark ..	1890	.9515	1885–89 ..	1.048	1.050*
Switzerland ..	1888	.9161	1887–91 ..	1.045	1.016
England & Wales	1891	.9399	1887–91 ..	1.036	1.041*
Sweden	1890	.9389	1887–91 ..	1.050	1.043
Scotland	1891	.9330	1887–91 ..	1.055	1.059*
Norway	1891	.9157	1887–91 ..	1.058	1.059*
Aver.(unweighted)	—	.9838	—	1.0568	1.0446

* The masculinity of ex-nuptial births is greater in these instances than that of all births ; in the other instances it is less.

6. **Masculinity of still and live nuptial and ex-nuptial births.**—J. N. and C. J. Lewis[1] studied the "variations of masculinity under different conditions" in 1906. Omitting seven of their quoted cases, in which the information is incomplete, they shew that still-births disclose a masculinity of 2 to 4 per centum greater than that for live-births. The unweighted averages of their cases with the omission mentioned (see p. 162), viz., 17, give for the masculinity of live-births (M/F), 1.0504, and for that of still-births 1.3032; that is, a masculinity 1.2407 greater than that of live-births. Results have been tabulated for *Western Australia* for the years 1897 to 1913 for live and still-births, and from 1908-1913 for ex-nuptial and nuptial still and live-births. These give the same general indication. The results are as follows:—

Table XXVII.—Masculinity-ratios for Nuptial, Ex-nuptial and Still-births, Western Australia,* 1897 to 1913.

	Masculinity.								
	1897–1902.			†1902–1907.			1908–1913.		
	M.	F.	M ÷ F	M.	F.	M ÷ F	M.	F.	M ÷ F.
Nuptial still-births	—	—	—	—	—	—	804	641	1.254
Ex-nuptial still-births	—	—	—	—	—	—	49	37	1.325
All still-births ..	507	373	1.359	672	528	1.273	853	678	1.258
Ex-nuptial live-births	759	687	1.1048	982	884	1.1109	1116	1037	1.0762
Nuptial live-births	15457	14658	1.0545	21226	20108	1.0556	23941	22882	1.0463
All live-births ..	16216	15345	1.0508	22208	20992	1.0579	25057	23919	1.0476
All births‡ ..	16723	15718	1.0639	22880	21520	1·0632	25910	24597	1.0534

* See Statistical Register, Western Australia, 1906; p. 12, 1914, Pt. I., p. 14.
† 1902 has been included twice in order to have 3 six-year periods.
‡ Including, that is, still-births.

The experience in Australia from 1901 to 1913 gave an unweighted average of the masculinities determined for each year, for all births, and for ex-nuptial births, the following results, viz.:—

		Average Masculinity.	Range of Masculinity.
Australia ..	All live-births	1.0508	1.0411 to 1.0638
,, ..	Ex-nuptial births ..	1.0417	1.0098 to 1.0621
Various Countries	All live-births	1.0568	1.036 to 1.118
(See Table XXV.)	Ex-nuptial births ..	1.0446	1.016 to 1.079

The unweighted average ratio of the "ex-nuptial" to all live-births was 5.954 per centum for Australia.

[1] See Journ. Inst. Act., Vol. xl., pp. 154-188, April, 1906.

It was stated by R. Mayo-Smith in his " Statistics and Sociology,"[1] that " among illegitimate" (*i.e.*, ex-nuptial) " children the excess of boys is less than among legitimate" (*i.e.*, nuptial). William Farr, however, pointed out in his " Vital Statistics,"[2] that he believed that " it is assumed in the French returns that foundling children are illegitimate," but that such an assumption is probably invalid, and he considered the matter to be in doubt. The Australian results, however, tend to confirm those for Europe given in Table XXVI.

7. **Coefficients of ex-nuptial and still-birth masculinity.**—It is a somewhat remarkable fact that ex-nuptial and still-births shew increased masculinity, and that among still-births the ex-nuptial shew a somewhat different masculinity to the nuptial. For the analysis of this the following notation will be convenient :—

Live male and female	births,	total	$= m + f = b$
,, ,,	,,	nuptial	$= m_0 + f_0 = b_0$
,, ,,	,,	ex-nuptial	$= m_1 + f_1 = b_1$
Still male ,,	,,	nuptial	$= m_2 + f_2 = b_2$
,, ,,	,,	ex-nuptial	$= m_3 + f_3 = b_3$
,, ,,	,,	total	$= m' + f' = b'$

If we call the ratio of the masculinity in the one case (say the ex-nuptial) to that in the other (say the nuptial) the *masculinity intensification-coefficient k*, its significance will vary according as we use μ_1, μ_2, μ_3; see Table XXI. It may easily be shewn that

$$(337)\ldots\ldots\text{For } \mu_1\ ;\ k_n = \frac{m_n}{m} \div \frac{f_n}{f}\ ;$$

$$(338)\ldots\ldots\text{For } \mu_2\ ;\ k_n = \frac{m_n}{m} \div \frac{b_n}{b}\ ;$$

$$(339)\ldots\ldots\text{For } \mu_3\ ;\ k_n = \frac{m_n - f_n}{m - f} \div \frac{b_n}{b}\ ;$$

that is, in regard to any character in the first case it is the relative number of males born *divided* by the relative number of females born ; in the second case it is the relative number of males born *divided* by the relative number of births ; in the third case it is the ratio of the differences of the males and females, divided by the relative number of births. The coefficient intended can be indicated by suffixes and accents ; thus the intensification-coefficient of ex-nuptial still-births on total still-births would be k'_3 ; of ex-nuptial on nuptial live-births, k_{10} ; and so on ; see the preceding scheme of notation in the beginning of this section.

[1] Macmillan, London, 1895, p. 77. [2] E. Stanford, London, 1885 p. 104.

The coefficients for Western Australia are as in the following table :—

Table XXVIII.—Masculinity Intensification-Coefficients, Western Australia, 1897–1913.

Ratio of Masculinity of	To the Masculinity of	1897–1902	1902–1907	1908–1913
All still-births ..	All live-births ..	1.293	1.203	1.201
Ex-nuptial still-births	Ex-nuptial live-births	—	—	1.057*
Ex-nuptial live-births	Nuptial live-births ..	1.049	1.052	1.029

* Depends upon limited numbers : see Table XXVII.

For Western Australia for 1897 to 1913 inclusive, the ratio of masculinity of all still-births, 1.287, on all live-births, 1.054, is 1.221. This agrees excellently with the result of a series of values for Europe shewn in Table XXVIII., the mean of which is 1.2397.

Table XXIX.—Ratio of Masculinity of Still-Births to that of Live-Births, in various Countries.

	Years.	Ratio.		Years.	Ratio.		Years	Ratio
Paris	8	1.157	Germany ..	5	1.220	Austria	5	1.249
Paris	10	1.179	**W. Australia**	17	1.221	Belgium	5	1.264
Livonia	10	1.205	Prussia ..	10	1.225	Switzerland ..	5	1.292
Montpellier ..	—	1.208	Hungary ..	5	1.238	Sweden + Finland	9	1.299
Alsace-Lorraine	10	1.208	Italy ..	5	1.239	Sweden	—	1.300
Netherlands ..	5	1.210	Amsterdam	12	1.241	France	5	1.360
Mean	..	1.195	Mean ..	..	1.231	Mean	..	1.294

8. **Masculinity of First-born.**—It has been supposed that masculinity has some relation to primogeniture. For the six years 1908 to 1913 inclusive, there were in Australia 111,545 births, of which 25,708 were first births. The number of males and females gave the following results, viz. :—

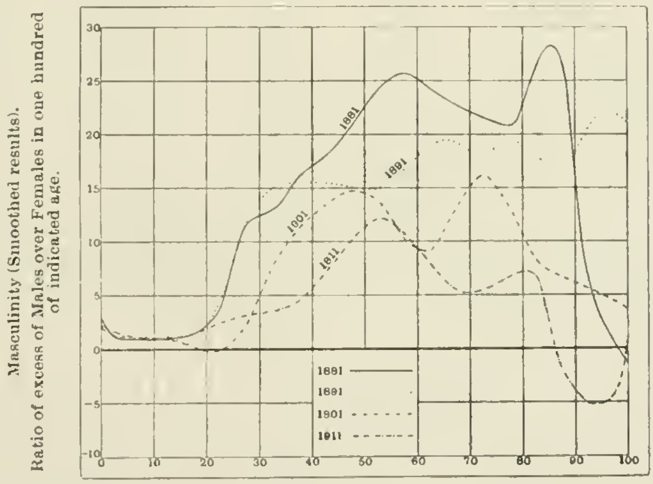
Period.	Masculinity of Australian		
	First-births.	Other births.	All births.
1908–1913 ..	1.05260	1.05001	1.05066

Tabulated according to ages between marriage and birth, the results were :=

Period	Masculinity of Australian First-births, the Interval after Marriage being—			
	Under 1 year	1 year	2–5 years	5–25 years
1908–13	1.0534	1.0514	1.0578	1.0091
Difference from Masculinity of all live-births for same period, viz., 1.0507	+.0027	+.0007	+.0071	−.0416

The numbers, however, are relatively small for the last group, in which there were only 3490 births. The difference between the different groups and the masculinity of all live-births for the whole period is not more remarkable than the difference between the masculinity of all live-births between one year and another. Bertillon's result from 1,140,860 births in Austria was 1.086 for first, and 1.054 for subsequent births; while Geissler's result for Saxony for 4,794,304 births was 1.054. Lewis for Scotland obtained from 85,964 births, for first births, 1.054; for subsequent births, 1.048; Streda for Alsace-Lorraine, from 47,198 births, for first births, 1.058; for subsequent births, 1.059.[1]

9. **Masculinity of populations according to age, and its secular fluctuation.**—In any country where migration has a large influence, and especially where also the migration is of a somewhat specialised character, the masculinity is likely to shew considerable changes. In the following Table, viz., XXX., are given the masculinities (μ_3) in age-groups, for four Censuses, viz., 1881 to 1911, the masculinities in this case being $(M-F)/(M+F)$. This character is strikingly different from that of England. The significance of the fluctuations of the masculinity are best seen in Fig. 47.



Variation of Masculinity of Australian population according to age

Fig. 47.

[1] See Journ. Inst. Act., vol. xl., 1906, p. 164.

Table XXX.—Masculinity* in Age-groups at Censuses 1881, 1891, 1901, 1911, Australian Commonwealth, and England, 1911. Computed from Smoothed Results.

Age-Group.	Australian Commonwealth.				England.
	1881.	1891.	1901.	1911.	1911.
0–4 ..	.01018	.01374	.01227	.01588	+.00463
5–9 ..	.00898	.00975	.01105	.01064	—.00060
10–14 ..	.00943	.01195	.00981	.00869	—.00126
15–19 ..	.01332	.00389	.00223	.01485	—.00804
20–24 ..	.03493	.04192	.00157	.02472	—.05366
25–29 ..	.12482	.11802	.02183	.03155	—.05440
30–34 ..	.12489	.15534	.07807	.03485	—.04359
35–39 ..	.15176	.14833	.11272	.04356	—.03459
40–44 ..	.17886	.16100	.13292	.07038	—.03693
45–49 ..	.20734	.14761	.14744	.10160	—.03811
50–54 ..	.24498	.15267	.13833	.12294	—.04132
55–59 ..	.25546	.16233	.10217	.10885	—.04883
60–64 ..	.23988	.19446	.08809	.07725	—.06437
65–69 ..	.22504	.19310	.13194	.05274	—.09299
70–74 ..	.22228	.17717	.16770	.05417	—.14419
75–79 ..	.20038	.19886	.13247	.06685	—.17745
80–84 ..	.26350	.17799	.07707	.07253	—.21752
85–99 ..	.28955	.12313	.06902	—.02107	—.27160
90–94 ..	.03175	.25424	.05306	—.05164	—.35311
95–99 ..	—.05263	.23967	.06215	—.04651	—.40237
100 ..	+.20000	.17647	—	+.05263	—.43750
Masculinity of total Population	.07983	.07362	.04824	.03840	—.03269

* (Males — Females) ÷ Persons.

An examination of these results shews that where there is a considerable migration element, predictions as to the future movement of the masculinity, by extrapolation, are somewhat uncertain both for any age-group and for all ages. Moreover, interpolations will lead to results which can be regarded only as fairly accurate.

10. **Theories of Masculinity.**—The results given shew that the masculinity of still-births is considerably higher than that of live-births, roughly in the proportion of about 1.15 to about 1.35 greater; and that masculinity at birth generally is about 1.05 or 1.06. These facts are remarkable, and have given rise to various attempted explanations. J. A. Thomson in his "Heredity"[1] says that, according to Blumenbach, Drelincourt in the 18th century brought together 262 groundless hypotheses as to the determination of sex, and that Blumenbach regarded

[1] Murray, London, 1908, p. 477.

Drelincourt's theory as being the 263rd. Blumenbach postulated a "Bildungstrieb" (formative impulse), but this was regarded as equally groundless. It has been suggested that war, cholera, epidemics, famine, etc., are followed by increase in the masculinity. These will have to form the subject of later investigations. At present it would seem that the first necessity is a sufficiently large accumulation of accurate statistic, as a basis for study. The one point which is clear is that death *in utero* (at least in the later stages) is marked by much greater masculinity than that which characterises live-births. This will be referred to later in dealing with infantile mortality.

That the effect of *war* is not apparently discernible in existing statistics, is evident from the following table, viz., Table XXXI, shewing the experience of France from 1865 to 1876. It will be seen that the war-years, 1870 and 1871, and subsequent years reveal no change in the masculinity.

Table XXXI.—Experience of France, 1865 to 1876.

Year.	Rates per 1000 of Mean Population.			Deaths of Children under 1 year of age per 1000 births.	Excess of Males over Females in each 1000 births.
	Marriage.	Birth.	Death.		
1865 ..	7.85	26.5	24.3	191	2.5
1866 ..	8.00	26.4	23.2	162	2.6
1867 ..	7.85	26.4	22.7	170	2.1
1868 ..	7.85	25.7	24.1	192	2.3
1869 ..	8.25	25.7	23.5	176	2.4
1870 ..	6.05	25.5	28.4	191	2.3
1871 ..	7.25	22.9	35.1	240	2.4
1872 ..	9.75	26.7	22.0	152	2.3
1873 ..	8.85	26.0	23.3	180	2.4
1874 ..	8.30	26.2	21.4	158	2.6
1875 ..	8.20	25.9	23.0	170	2.4
1876 ..	7.90	26.2	22.6	165	2.3

XI.—NATALITY.

1. **General.**—The phenomena of human reproduction, as affecting population, and the whole system of relations involved therein, may be subsumed under the term "natality." In one aspect they measure the reproductive effort of a population; in another they disclose the rate at which losses by death are made good; in a third they focus attention upon social phenomena of high importance (*e.g.*, nuptial and ex-nuptial natality); in yet another they bring to light the *mode* of the reproductive effort (*e.g.*, the varying of fecundity with age, the fluctuation of the frequency of multiple-birth, etc.) In this section we shall deal with the questions which relate more directly to birth-rate, and shall treat of those which relate more directly to nuptiality in section XII, and to fecundity in section XIII.

Birth-rates are not immediately comparable. The physical and social development of two communities being identical, their birth-rates become roughly comparable only when the relative numbers of married and of single women at each age are identical. In regard to the initial qualification, it may be pointed out that any of the races of Western Europe, for example, may be immediately compared on the basis of identical numbers at the same ages; but a population of the natives of India would not be comparable to one of Western Europe because of earlier physical development and earlier marriage. Comparisons of this special character, however, may sometimes be founded on principles indicated by the theory of "corresponding states" in physical investigations. This matter will be referred to later.

Populations similarly characterised in respect of features, material to any question at issue, may be called homogeneous in that respect. In order to compare the birth-rates of populations, otherwise homogeneous, but differently constituted in regard to age, it is necessary to take account at least of three things, viz., (i.) the numbers at each age; (ii.) the relative fecundity at each age; and (iii.) the relative numbers of married and single women. In other words, a convenient and strict comparison can be made satisfactorily only on the basis of what may be called a "standard" or "normal" female population. This normal population should represent the mean of the whole series of populations proposed to be compared (*i.e.*, the relative numbers of married and of single females at each age should be their ratio to the entire aggregate). Comparison is then effected by attributing to this population-norm the nuptial and ex-nuptial birth-rates actually existing in the populations to be compared with one another. Such a comparison is free from the effect of accidental differences in constitution as to age; thus the relative magnitude

of the birth-rates and populations compared are revealed. The principles of developing norms of this type have already been considered; see VIII., §§ 8 to 12.

We consider first the nature of a birth-rate.

2. **Crude birth-rates.**—While the total number born in any population during any period, divided by the average number of the population during the period, *i.e.*, the *crude birth-rate*, is one element of the rate at which the population is reconstituted, its nature and limitations are important from certain points of view. We propose to consider these. Since both births and population vary with time, we may regard their variations of rate as represented by the functions $f(t)$ and $F(t)$. Thus if B_m denote the number of births occurring in a unit period (say 1 year), and P_m be the mean population during that period, the average period-rate (annual rate in the case supposed), which may appropriately be referred to the middle of the period, is:—

$$(340)\ldots\ldots\ \beta_m = \frac{B_m}{P_m} = \frac{B}{P}\,\frac{\int_{-\frac{1}{2}}^{+\frac{1}{2}} f(t)\,dt}{\int_{-\frac{1}{2}}^{+\frac{1}{2}} F(t)\,dt}$$

the instantaneous value passing through the range of values which determine the form of the functions f and F. P is the population as at the middle of the year, and B the rate per annum at which births are occurring at that moment.

In general, no serious error will be introduced in the value of β if, instead of P_m, the population at the middle of the year is used, though more accurate results will be to hand if population-determinations at the end of each half-year, or each quarter, or better still each month, are used to ascertain the mean. The necessary formulæ would be respectively

$$(341)\ldots P_m = \tfrac{1}{2}(P_0 + P_1)\,;\ \text{or} = \tfrac{1}{6}(P_0 + 4P_{\frac{1}{2}} + P_1)\,;\ \text{or}$$

$$= \tfrac{1}{12}(P_0 + 4P_{\frac{1}{4}} + 2P_{\frac{1}{2}} + 4P_{\frac{3}{4}} + P_1):\ \text{or}$$

$$= \tfrac{1}{24}(P_0 + 2P_{\frac{1}{12}} + 2P_{\frac{2}{12}} + \ldots . 2P_{\frac{11}{12}} + P_1\,;\ \text{or}$$

$$= \tfrac{1}{40}\{(P_0 + P_{\frac{2}{12}} + P_{\frac{4}{12}} + P_{\frac{10}{12}} + P_1) + 2P_{\frac{6}{12}} + 5(P_{\frac{1}{12}} + P_{\frac{5}{12}} + P_{\frac{7}{12}} + P_{\frac{11}{12}}) + 6(P_{\frac{3}{12}} + P_{\frac{9}{12}})\}\,;$$[1]

or any of these indicated in VI., § 2, Table VI.

[1] The question of the formulae to be preferred was discussed for quarterly results in the Population and Vital Statistics Bulletin for Australia, No. 1, pp. 20, 21, and the coefficients adopted were 1, 4, 2, 4, 1, though previously 1, 2, 2, 2, 1 had been used. The use of formulae based upon integral functions supposes that the recorded population at the moment of record is substantially free from large deviations from the number represented by the functional change. If the functional change is small, and the "accidental" deviation is large, the use of the functional formulae does not yield the advantages expected, and has the disadvantage of multiplying the "accidental" deviation possibly by a very large or a very small factor (as the case may be); if the former, the result is not satisfactory.

Such formulae are, of course, more than abundantly accurate for all statistical purposes.

Birth-rate is influenced by—

(*a*) the sex and age constitution of the population;

(*b*) all forces restricting the fecundity of a population (*e.g.*, frequency of, and the age of, marriage; social tradition and habits; etc.);

(*c*) the frequency of multiple-births;

(*d*) infantile mortality (since mothers who lose their offspring are again exposed to the risk of maternity), etc.

These influencing factors will be considered either in this section, viz., XII., or in later sections.

3. **Influence of the births upon the birth-rate itself.**—Let it be supposed that the population of two communities be initially P and that in the same period B births occur in one and $2\ B$ in the other, of which in each case the proportion s survive; the numbers being thus sB and $2sB$ at the end of the period. If there were no migration, and no deaths, other than those arising from the births, the deduced birth-rates would be

$$(342)..\beta_1 = \frac{B}{P+\frac{1}{2}sB}\ \text{and}\ \beta_2 = \frac{2B}{P+sB}\ ;\ \text{but}\ 2\beta_1 = \frac{2B}{P+\frac{1}{2}sB}$$

a larger quantity. Hence the effect of an increase of a birth-rate, when a proportion of the births is incorporated in the population, is to somewhat diminish that ratio of births to population, which really represents the relative frequency of birth, unless at least the population is increasing in some manner which counteracts this. The preceding result is more obvious if put in the form—

$$(342a)..2\beta_1 = \frac{2B}{P+sB}\left\{1+\frac{1}{2}s\frac{B}{P}-\frac{1}{4}\left(s\frac{B}{P}\right)^2+\text{etc.}\right\}\ ;\ \beta_2 = \frac{2B}{P+sB}$$

More generally we have—

$$(342b)....\ \beta_1 : \beta_2 :: \frac{B_1}{P_1(1+\frac{1}{2}r_1)} : \frac{B_2}{P_2(1+\frac{1}{2}r_2)}$$

(r denoting the increase, supposed linear); shewing that the birth-rates and births are in the same ratio only if the *mean* populations are identical. Hence as measures of fecundity birth-rates need some slight correction, owing to their influence on the magnitude of the population. They are strictly comparable in this respect only when two populations are homogeneous, and differences of birth-rate themselves disturb the homogeneity and thus involve the application of some correction.[1]

[1]There is an analogous case in connection with the computation of interest earned on assurance and similar funds. Thus if I denote the interest earned in the course of a year, A and B the funds at the beginning and end of the year respectively, and i the *effective* rate of interest earned on the funds during the course of the year, then the value of i is approximately given by the following formula, now generally adopted in practice:—

$$i = I/\left\{\tfrac{1}{2}(A+B)-\tfrac{1}{2}I\right\}$$

4. **Influence of infantile mortality on birth-rate.**—Denoting the number of births by B, and of infantile deaths by M, and the number of women of child-bearing age by P, we shall have for the birth-rate β, attributed not to the whole population but to the P women, and for μ the rate of infantile mortality—

$$(343)\ldots\ldots\beta = B/P;\ \mu = M/B : \beta\mu = M/P.$$

Suppose μ to change to some other value $\mu' = M'/B'$: M' being the number of deaths and B' the number of births under the changed state of things, assumed to have become constant. Then, since mothers who lose their children are exposed to an increased risk of maternity, the ratio of which is only the proportion q (a proper fraction) of the full risk, we shall have for the number at risk as originally, viz., N, and also after a change in the prevailing rate of infantile mortality, N'.

$$(344)\ldots\ldots N = P - B + qM;\ \text{and}\ N' = P - B' + qM'.$$

If the reproductivity of these two groups is the same, then $B/N = B'/N'$; from which it follows that—

$$(345)\ldots\ldots \frac{P + qM}{B} - 1 = \frac{P + qM'}{B'} - 1$$

and consequently, discarding the unit from each side and writing in the values of the quantities as by (343) above, we have—

$$(346)\ldots\ldots \frac{1}{\beta} + q\mu = \frac{1}{\beta'} + q\mu';$$

that is—

$$(347)\ldots\ldots\beta' = \beta\ \{1 + q\beta'(\mu' - \mu)\}$$

It will be found that this change is sensibly a linear one, or any increment in the rates of mortality will cause a sensibly constant but small proportional increase in the birth-rate. If we call the birth-rate, freed from the influence of infantile mortality, the *normal birth-rate* β_0, then—

$$(348)\ldots\ldots\beta_0 = \beta\ (1 + k\mu).$$

in which k may be regarded as a constant for a particular community, and a particular epoch. The value of k was found on the average for Europe to be about $+ 0.033\ \mu$, or $\beta_0 = \beta + 0.033\ \mu$, the birth-rate β being expressed per 1000 of the population, and the infantile mortality rate μ expressed per 1000 births. An examination of the data for different countries gave the following results :—

TABLE XXXII.—Influence of the Rate of Infantile Mortality on the Crude Birth-rate for Various Countries, about Year 1900.

COUNTRY.	PERIOD.		Value of β_0 and k in $\beta = \beta_0 + k\mu$.†	
	Birth.	Mortality.	β_0	k
New Zealand	1881–1905	1882–1906	13.2	+ 0.191
Commonwealth	1887–1905	1888–1906	16.8	+ 0.118
Sweden	1881–1904	1882–1905	17.1	+ 0.100
Norway	1881–1905	1882–1906	20.5	0.100
Prussia	1881–1905	1882–1906	19.1	+ 0.085
Various Countries* ..	1901	1902	19.4	0.083
Netherlands	1881–1905	1882–1906	22.6	+ 0.063
France	1881–1905	1882–1906	12.7	0.061
Denmark	1881–1905	1882–1906	22.4	+ 0.060
Japan	1881–1904	1882–1905	22.3	+ 0.053
Ceylon	1881–1905	1882–1906	26.4	+ 0.042
Jamaica	1881–1905	1882–1906	34.3	+ 0.022
Switzerland	1881–1904	1882–1905	25.3	+ 0.018
Ireland	1881–1905	1882–1906	25.8	— 0.026
England and Wales ..	1881–1905	1882–1906	38.6	— 0.058
Scotland	1881–1905	1882–1906	38.9	— 0.068

* For one year only. † The birth-rate being expressed per 1000 of the population, and the infantile mortality per 1000 births.

The infantile mortality rate (μ) in the table is expressed by the number of infants dying per 1000 of infants born.

The crude birth-rate (β) is the number of births per 1000 of the total population.

It will be seen that the magnitudes of k, and therefore of q, have no general relation to the magnitude of the birth-rate; that is, a particular value of the risk-factor is characteristic of a particular country.

In an investigation made in 1908[1] it was shewn that the influence of infantile mortality was very irregular in its operation, and the following deductions were stated, viz. :—[2]

(i.) When either all mothers of deceased infants, or any constant proportion thereof, may be regarded as subject to equal risk of fecundity (*i.e.*, equally likely to bear children) then equal increases in the rate of infantile mortality tend to be followed by equal though relatively small increases in the birth-rate.

(ii.) The influence of infantile mortality on the birth-rate must always be very small. (The contrary proposition is not, of course, necessarily true).

This type of investigation aims rather at ascertaining the form of the function expressing the correction, so that the form being determined, the constants can then be ascertained from the data. It would appear that yearly irregularities of birth-rate are so great as compared with the influence of infantile mortality that the latter is virtually masked by the former. Probably in any rigorous investigation of a measure of the fecundity of a population the birth-rate should be corrected in some such way as has been indicated.

[1] By the writer. See Journ. Roy. Soc., N.S.W., Vol. xlii., pp. 238-250, particularly Fig. 1 on p. 243 therein.

[2] Loc. cit. pp. 241-2.

5. **World-relation between infantile mortality and birth-rate.**—In order to ascertain whether in a world-wide survey of infantile mortality and birth-rates any correlation manifested itself we may extend the purview of all countries where fairly accurate statistics are available, viz., the following :—

Australia, Austria, Belgium, Chili, Ceylon, Denmark, England and Wales, France, Ireland, Italy, Jamaica, Japan, Netherlands, New South Wales, New Zealand, Norway, Queensland, Russia, Scotland, South Australia, Spain, Sweden, Switzerland, Tasmania, Victoria, West Australia.

The populations are, of course, repeated with different rates, and are *equivalent* to 8776 millions,[1] the results forming groups of available results ; according to the magnitude of the infantile mortality we get the results shewn in Table XXXIII. hereunder, the ranges of infantile mortality being shown therein.[2]

In Fig. 48, graph A denotes the relative frequency of the given ranges of infantile mortality.[3] It will be observed that the graph is dimorphic, that is, that while the characteristic rate of infantile mortality is about .0150 (150 as usually expressed), there is also a second mode for the rate of about .0255. The corresponding crude birth-rates are about .029 and .048 respectively (or residual birth-rates, see hereinafter, about .025 and .035). It will be seen that there can be a very high rate of infantile mortality with low birth-rate, but it would appear, only for very limited populations.[4]

TABLE XXXIII.—Relations of Infantile Mortality and Birth-rate, various Countries, about Year 1900.

Population Represented (millions)	Ranges of Infantile Mortality for Individual Populations.	Mean of Infantile Mortalities.	Mean of Crude Birth-rates.	Infantile Mortality of Aggregate.	Crude Birth-rate of Aggregate.	12 Months Residual of Birth-rate.
344	.0688–.0959	.0821	.0291	.0911	.02692	.02447
479	.1018–.1232	.1120	.0291	.1119	.02889	.02566
2035	.1276–.1474	.1371	.0288	.1387	.02865	.02468
2172	.1519–.1724	.1618	.0291	.1598	.02904	.02440
1116	.1762–.1974	.1872	.0340	.1880	.03391	.02753
851	.2032–.2179	.2098	.0367	.2085	.03365	.02663
297	2213–.2372	.2286	.0380	.2279	.03808	.02940
696	.2406–.2559	.2490	.0480	.2491	.04757	.03572
668	.2601–.2771	.2688	.0479	.2710	.04763	.03472
189	.2800–.2920	.2870	.0446	.2845	.04885	.03495
105	.3040–.3290	.3133	.0385	.3075	.04549	.03150
147	.3325–.3490	.3406	.0366	.3392	.03701	.02446
91	.3660–.4120	.3890	.0372	.3800	.03681	.02282

[1] The method is, of course, not perfectly satisfactory ; for, as pointed out by the writer (on p. 245), loc. cit. the populations are not homogeneous, and doubtless if more moderate-sized districts could be analysed the material would give a clearer indication of the true nature of the relation.

[2] See also loc. cit., p. 246, and Fig. 2, p. 247, in the same paper.

[3] See page 150 hereinafter.

[4] Similar indications are given by the analysis before referred to. See loc. cit. p. 248, Fig. 3.

This more general result shews that propositions (i.) and (ii.) in the preceding section can be regarded as true only for individual populations and probably for very limited periods of time; the effects are readily masked by more potent influences.

In the table hereunder (XXXIV.), of results in the present century, the following countries have been included, viz., *in column* (i.) New Zealand, 1913; Norway, 1912; Australia, 1913; Sweden, 1911; France, 1912; Netherlands and Denmark, 1913; Switzerland, 1913; Ireland, England and Wales, and United Kingdom, 1913; Finland, 1912; Scotland and Ontario, 1913; Belgium, Italy and Prussia, 1912; Serbia, 1911; German Empire, 1912; Spain, 1907; Bulgaria and Japan, 1910; Jamaica, 1913; Austria and Hungary, 1912; Ceylon and Roumania, 1913; Russia (European), 1909; Chile, 1911; and *in column* (iv.) France and Belgium, 1912; Ireland, England and Wales, and Ontario, 1913; Sweden, 1911; United Kingdom, 1913; Switzerland, 1912; Scotland and Denmark, 1913; Norway, 1912; New Zealand, Netherlands, and Australia, 1913; German Empire, Prussia, Finland, Austria and Italy, 1912; Spain, 1907; Japan, 1910; Jamaica, 1913; Serbia, 1911; Hungary, 1912; Chile, 1911; Ceylon, 1913; Bulgaria, 1910; Roumania, 1913; Russia (European), 1909. The results are the weighted means (or what is the same thing, the values are for the population-aggregates) of the populations, combined in successive groups of ten, arranged (in ascending order) according to infantile mortality in the one case, and according to birth-rate in the other.

These results shew unequivocally that there is, *in general*, a relation between birth-rate and infantile mortality. The calculated results are as follows; β denoting birth-rate per unit of population, and μ denoting infantile mortality rate per birth:—

Determined from groupings in the order of infantile mortality:—

(349). . $\beta = 0.00956 + 0.1405\,\mu$; (which gives $\mu = 0.06804 + 7.117\,\beta$);

and determined from grouping in the order of birth-rate:—

(350). . . . $\mu = -0.03661 + 5.970\,\beta$; (which gives $\beta = 0.06132 + 0.1675\mu$).

The mean of these results is expressed with sufficient precision by—

(351). . $\beta = 0.00785\,(1 + 19.6\,\mu)$; $\mu = 0.0510\,(1 - 127\,\beta)$

β being the rate per unit of population, and μ per birth.

TABLE XXXIV.—General Relation between Infantile Mortality and Birth-rate, Aggregates of various Countries, 1907 to 1913.

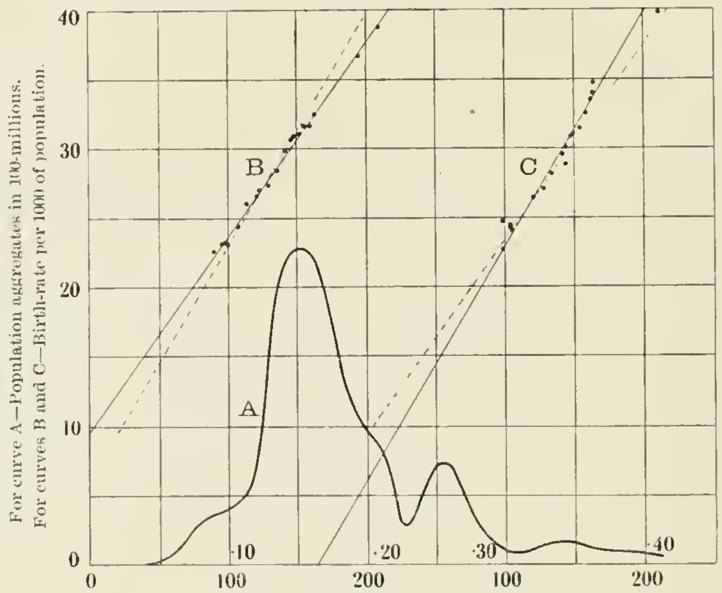
Infantile Mortality and Birth-rate.					Birth-rate and Infantile Mortality.				
Population in Millions.	Infantile Mortality.*	Birth-rate.†	Calculated.‡	Reduced Birth-rate.†	Population in Millions.	Birth-rate.†	Infantile Mortality.*	Calculated.§	Reduced Birth-rate.*
(i.)	(ii.)	(iii.)			(iv.)	(v.)	(vi.)		
107.6	90	22.6	22.2	20.6	154.2	22.7	99	99	20.5
152.5	96	23.0	23.1	20.8	116.9	24.0	105	107	21.5
153.3	96	23.1	23.1	20.9	110.4	24.1	104	107	21.6
153.2	97	23.0	23.2	20.8	112.2	24.4	103	109	21.9
150.4	99	23.0	23.5	20.7	80.1	24.8	99	111	22.3
118.3	107	24.3	24.6	21.7	143.5	26.4	121	121	23.2
147.2	113	26.1	25.1	23.2	179.1	27.1	128	125	23.6
185.4	121	26.7	26.6	23.5	136.2	28.2	134	142	24.4
184.5	122	26.9	26.7	23.6	161.3	28.8	144	135	24.7
246.3	129	27.3	27.7	23.8	191.6	29.6	142	140	25.4
228.6	135	28.4	28.5	24.6	208.0	30.0	144	132	25.7
186.9	142	29.8	29.5	25.6	256.1	30.8	148	147	26.2
234.2	146	30.7	30.1	26.2	255.9	30.8	148	147	26.2
230.4	147	30.8	30.2	26.3	252.7	31.0	150	148	26.4
256.6	151	31.0	30.8	26.3	269.0	31.4	154	151	26.6
270.1	155	31.6	31.3	26.7	206.3	32.5	159	157	27.3
239.3	159	31.6	31.9	26.6	169.5	33.6	163	164	28.1
205.6	163	32.5	32.5	27 2	170.6	33.9	164	166	28.3
319.2	194	36.7	36.8	29.6	149.1	34.8	163	171	29.1
256.5	208	38.9	38.8	30.8	230.6	39.8	211	201	31.4

* Per 1000 births. † Per 1000 population. ‡ By formula (349).
§ By formula (350).

From these the lines B and C respectively are plotted and the calculated values in columns (iii.) and (vi.) are computed. The dotted lines shew the positions of the other graph for the purpose of comparison, and the line which represents formula (351) is between the two.

That these results, though not identical, are very similar, is seen from the graphs B and C, shewing the two series of values. What they establish is that, on the whole, the birth-rate and infantile mortality increase together. Moreover, when the birth-rate is reduced to its effective value twelve months later (that is, for one year of age), it is much more uniform on the whole. Since, as shewn, the increase of risk of maternity is relatively small (348), it follows that, on the whole, tho social conditions which characterise a large birth-rate are those associated with a high rate of infantile mortality. This, of course, is not necessarily so, but expresses the general fact. In short, a high birth-rate is usually associated with a high rate of infantile mortality, but high infantile mortality will, *per se*, not appreciably affect the birth-rate. The importance of this result is obvious.

GENERAL RELATION BETWEEN INFANTILE MORTALITY AND BIRTH-RATE.



For curve A.—Infantile mortality rate per birth.
For curves B and C.—Infantile mortality rate per 1000 births.
Fig. 48.

6. **Residual birth-rates.**—Owing to the very high death-rate of infants, the crude birth-rate, taken alone, is not a satisfactory expression of the effective recuperative force of a population against the ravages of death. It is not practicable, however, to assign any particular age as specially appropriate for estimating the *virtual efficiency of birth-rate,* and as we have seen high birth-rates, however, are ordinarily associated with a high rate of infantile mortality.

For example, New Zealand and Australia had birth-rates in 1912 of 26.5 and 28.7 per thousand population, and *infantile death-rates* (*i.e.*, deaths under 12 months per 1000 born) of 51 and 72, while Ceylon and Chile, in 1911, had birth-rates of 37.9 and 38.5, and infantile death-rates of 218 and 332. This question will be referred to later.

Birth-rates corrected so as to represent the number living after a given period may be called *residual birth-rates,* and the quantity multiplied into a birth-rate to give its residual value may be called the *survival coefficient, or survival factor.* We shall consider these. Owing to the fact that of all the deaths which occur in 12 months, about 42 per cent. occur in the first month, the infantile mortality may be referred to the same calendar year as the births without sensible error, or we may correct

it as explained hereinafter. Let β be the birth-rate and γ the rate of infantile mortality, the first expressed per unit of the population, the latter per birth. Then the residual birth-rate β_r is[1]—

$$(352)\ldots\ldots\beta_r = \beta\ (1 - \gamma)$$

The quantity in brackets is the "survival-factor" and β_r is the "residual birth-rate." For a population in which the number of births was constant and the rate of mortality for the first twelve months was constant, the probability of persons of age 0 living to age 1, viz., ${}_0p_1$, would be the same as the survival factor, since under these conditions it would denote the ratio of those surviving one year, viz., l_1 to the number born, viz., l_0. Consequently, subject to this limitation—

$$(353)\ldots\ldots(1 - \gamma) = {}_0p_1 = l_1 \,/\, l_0.$$

For a population in which the number of births is *increasing*, say, at the rate rt, and the rate of infantile mortality *diminishing*,[2] say, at the rate $r't$, these quantities become functions of time and are affected by the interval of time between the year for which the births are recorded and the somewhat later year for which the infantile deaths *ought to be recorded*, in order to properly refer to the birth-group. As, however, the error arising is of a small order as compared with the accidental deviations from year to year, it is questionable whether a correction is worth applying. It may be mentioned that in Australia it was found by an investigation for the years 1909 and 1910, that all children who die in the first year of life live on the average 99.3 days, and children are registered on the average 38.2 days after birth.[3] The difference, 61.1 days, or say two months, is regarded as the difference between the years. Thus the infantile mortality in the following table was calculated on the births occurring one-sixth of a year earlier. Similarly the birth-rate given for the equivalent year to n, say β_e, is—

$$(354)\ldots\ldots\beta_e = \tfrac{1}{6}\,\beta_{n-1} + \tfrac{5}{6}\,\beta_n\,.$$

It may also be noted that an investigation of the question shewed that of the deaths in Australia under 1 year of age occurring in any calendar year, 0.72 to 0.74 per cent.—average about 0.73—arose from births which occurred within that calendar year, and 0.27 from those which occurred in the preceding year. This proportion is doubtless approximately true also for other countries.

[1] These rates are commonly expressed per 1000 of the population, and per 1000 born respectively, in which case the formulæ will be $\beta_r' = \beta'\,(1 - \frac{\gamma'}{1000})$; β' and γ being 1000 times greater than β and γ.

[2] Infantile mortality has for years past been steadily diminishing in many countries.

[3] This has ceased to be true because of the "maternity bonus."

This would suggest that the coefficients in the above equation (354), should be $\frac{1}{4}$ and $\frac{3}{4}$ instead of $\frac{1}{6}$ and $\frac{5}{6}$, but, only if the average lateness of the registration of births and deaths were the same, which, however, was not the case. The practical result of the difference is not great. It will appear from a rigorous investigation in the next two sections, that with the rate of infantile mortality as it stood during the years 1909 to 1913, the proper proportion is about 0.731, a proportion which will be modified only by the difference in the registration interval. This interval, owing to the payment of the maternity bonus, resulting in earlier registration of births, has now become smaller.

TABLE XXXV.—Residual Birth-rates, Australia, 1904-14.

Year.	Crude Birth-rate, for Calendar Year.*	Infantile Death-rate† Calendar Year.	Crude Birth-rate for Equivalent Year.	Survival Factor.	Residual Birth-rate
1903 ..	25.29				
1904 ..	26.41	81.77	26.073	.91823	23.94
1905 ..	26.23	81.76	26.260	.91824	24.11
1906 ..	26.57	83.26	26.497	.91674	24.29
1907 ..	26.76	81.06	26.728	.91894	24.61
1908 ..	26.59	77.78	26.618	.92222	24.54
1909 ..	26.69	71.58	26.673	.92842	24.76
1910 ..	26.73	74.81	26.723	.92519	24.72
1911 ..	27.21	68.49	27.297	.93151	25.43
1912 ..	28.65	71.74	28.410	.92826	26.37
1913 ..	28.25	72.71	28.317	.92729	26.26
1914 ..	28.05	71.47	28.083	.92853	26.08

* Per 1000 population. † Per 1000 births.

The final column is the efficient birth-rate, the end of the first year of life being taken as an appropriate point of time for determining the efficiency, since the larger death toll from infantile troubles may be regarded as then past.

7. **Determination of proportion of infantile deaths arising from births in the year of record, number of births constant.**—Births, and infantile and other deaths, are recorded as occurring during successive equal periods of time, usually calendar years, half-years, quarters, months, etc. ; and the deaths during such periods are distributed according to a series of age-limits, for adults usually whole years, 0–1, 1–2, etc. In the case of "infantile deaths" or deaths of children under one year of age, they are distributed according to age-limits of weeks, months, quarters, etc. Consequently the infantile deaths occurring in any year are drawn from the births [and immigrants] both in the year of record

and in the previous year. More generally deaths of persons between the ages x_1, and x_2 recorded in any period of time, say $-t_z$ to 0, are drawn [where there is no immigration] from those born [in the country] during the period $-(x_2+t_z)$ to $-(x_1+0)$.[1] In the same way deaths recorded in any period $-t_{2z}$ to $-t_z$ would be drawn from those born [either in the country or from migrants entering it] during the period $-(x_2+t_{2z})$ to $-(x_1+t_z)$.

If the frequency of births be denoted by $k' F_1(t)$, the number of survivors after any period of time, x, of persons born at the moment t, will, so long as the death rates at each age remain constant, also be this function multiplied by the probability of surviving to the age x. Thus if this probability be denoted by λ_x, or that of dying be denoted by δ_x, $= 1 - \lambda_x$, then the survivors of age x, say S_x, and those who have not attained that age, say D_x, will be—

$$(355)\ldots\ldots S_x = \lambda_x F_1(t)\,;\quad \text{and}\quad D_x = \delta_x F_1(t)$$

for we may make $k' = 1$ if ratios only are needed.[2]

With births increasing, the successive records of the dying of any given age will also shew a similar progressive increase, proportional to that of the births, the death-rates at each age being constant. Thus the aggregate of births between the times t_1 and t_2, will be—

$$(356)\ldots\ldots {}_{t_1}B_{t_2} = k\int_{t_1}^{t_2} F_1(t)\,dt.$$

which would give merely $B = K(t_2 - t_1)$ if the frequency of births were constant, K being the number per annum when t is expressed in years. If the frequency be not constant, but of the form indicated hereinafter, viz., that in equation (359), then it will be [see also II., §6, (10)]—

$$(357)\ldots {}_{t_1}B_{t_2} = K(t_2 - t_1)\left\{1 + \frac{1}{2}a(t_2 - t_1) + \frac{1}{3}b(t_2^2 + t_2 t_1 + t_1^2)\right.$$
$$\left. + \frac{1}{4}c(t_2 + t_1)(t_2^2 + t_1^2) + \text{etc.}\right\}$$

which, when t_1 is 0, takes the simpler form—

$$(358)\ldots\ldots {}_0B_t = Kt\left\{1 + \frac{1}{2}at + \frac{1}{3}bt^2 + \frac{1}{4}ct^3 + \text{etc.}\right\}$$

[1] The words in square brackets may be omitted, if proper care be taken in the practical computations in regard to the influence of migration.

[2] That is $\lambda_x = l_x / l_0$ in an "actuarial population"; or is l_x if l_0 be made unity. Similarly $\delta_x = (l_0 - l_x)/l_0$.

With respect to survivors to age x, it may be noted that, in the earliest stages of life, λ_x decreases with the greatest rapidity, hence of the deaths occurring in any year the greatest number is contributed by those of the smallest age ; at least in the case of all aged less than 10 or 11 years,[1] and therefore for ages less than 10 years the greater number is contributed by what may be called the *ordinary year of reference*, viz., in the case of infantile deaths, the year of observation [or year of record] itself.[2] We may obtain an exact measure of this if we have the values of λ_x or δ_x For perfect rigour we must put these quantities $= F_2(x, t)$ for at the present time the value λ_x is sensibly increasing every year, and thus δ_x decreasing every year, for nearly all ages. It will, however, simplify the solution, and lead to no sensible error, if we omit the t and assume that either quantity is simply a function of x, say $\lambda_x = F_2(x)$ and $\delta_x = F_3(x)$.

Before envisaging the pertinent questions in their practical form, rather than in their more general and theoretical form, it may be pointed out that both these probabilities may readily be expressed as the sum of a series of exponential terms in the form (360) hereunder. Moreover, as has already been shewn, see II., §§ 2 to 10, formula (2a), (6), (9) to (13), the birth-frequency [and if desired this may include the migration element], may be put in the form—

$$(359)\ldots\ldots b_t = F_1(t) = K\,(1 + at + bt^2 + \ldots\text{etc.})$$

in which a, b, c, etc., may of course be positive, negative, or zero. As above-stated, either—

$$(360)\ldots\lambda_x, \text{ or } \delta_x = F_2(x) \text{ or } F_3(x) = (k_1 e^{-n_1 x} + k_2 e^{-n_2 x} + \text{etc.})$$

in which, for values of x not greater than 10 or 11, n is numerically a diminishing quantity. On expanding the exponential terms we have—

$$(361)\ldots\ldots\lambda_x \text{ or } \delta_x = A\,(1 - \alpha x + \beta x^2 - \text{etc.})$$

in which it may be easily seen that—

[1] The instantaneous rate of mortality, or the so-called "*force* of mortality" in actuarial terminology, is, in Australia, a minimum between the ages 11 and 12 years of age for males, and between 10 and 11 for females.

[2] The "ordinary" rate of infantile mortality is the ratio of the number of infants dying under one year of age, in any year of observation, to the number of infants born in the same year. This ratio is usually multiplied by 1,000, to avoid decimals.

$$(362)\ldots\ldots A = \Sigma k;\ \alpha = \frac{\Sigma(kn)}{\Sigma k \,.\, 1\,!};\ \beta = \frac{\Sigma(kn^2)}{\Sigma k \,.\, 2\,!};\ \text{etc.}$$

Incidentally, it may be repeated that the function b_t may be made to embody all complications arising from migration, forasmuch as birth may be regarded mathematically, merely as a case of immigration at age 0, and the constant can be so determined as to represent birth and [net] immigration combined for a succession of ages.

8. **Equivalent year of birth in cases of infantile mortality.**—In order to avoid circumlocution and to simplify the statement of the problem, we shall assume the period of observation of infantile mortality to be successive calendar years; and the record of births to be also according to calendar years. The necessary variation of this statement for other equal periods is self-evident. From what has been indicated in the preceding section it is clear that the infantile deaths in any year can be referred to a birth-year, which precedes the calendar year by some period less than a half-year. We proceed to evaluate this interval, which obviously depends upon :—(a) the rate at which the cases come under initial observation, that is upon the frequency of birth [or of birth and migration]; and (b) upon the decrease in the rapidity of death in the first year of life. The birth-*rate* in the ordinary sense is, of course, immaterial. For so limited a period as one year, we can, for the purpose in view, assume that $1 + rt$ expresses the increase with time of the frequency of births. If we make the origin of the variable, *i.e.*, time, the *end* of the year of observation, we have to consider the deaths of persons between the age-limits 0 and 1, occurring during the period -1 to 0, drawn obviously from births [and immigration] during the period -2 to 0, inasmuch as survivors born at the time $-2 + t'$ will be within the age-limits up to the time $-1 + t'$, which also is in the year of observation. Consequently also deaths among these must be taken into account. Thus at the time $-t$, the ages of persons, the deaths among whom will be included in the category of infantile deaths, will be between 0 and $x = 1 + t$. *This connects the time-limits with the age-limits.* The range of relative frequency extends from 1 for $t = 0$, to $1 - 2r$ for $t = -2$, (coming under observation, however, only for $t = -1$, whenever infantile deaths are observed for the same year as births). The question for resolution then is :—

Given the form and constants of the function expressing the variation in the frequency of births, and the form and constants of the function expressing the probability of living to age x, where x is less than 1, what proportion of the infantile deaths in any calendar year is drawn from the year of observation, and what proportion is drawn from the year preceding that of observation. The nature of the problem is illustrated by Figs. 49 to 52, p. 157, which illustrate either the case of deaths, or

that of births and survivors. The deaths occurring in a small unit of time, Δt say, arising from births in the same unit, is represented by the height of the first parallelepiped HA. It embraces all persons of age 0 to age $0 + \Delta x$, the period of observation being 0 to $0 + \Delta t$, the number being the height HA, and Δt being equal to Δx, these quantities being thus dx and dt when indefinitely small. The number of deaths at the end of a period, say a year, from the survivors of those born at its beginning, is represented by the height BI, and after a second period, say two years, by the height CM, Fig. 49. The succession of deaths are thus represented by the parallelepipeds 1.1, 1.2, 1.3, etc. They are followed by deaths occurring among those born during the period Δt to $2\Delta t$, represented by 2.1, 2.2, etc., and so on, these, in an increasing population, being somewhat larger than the former series, since the births from which they are drawn are greater in number. The parallelepiped F E, or 12.1, represents the deaths in the last period, viz., $2 - \Delta t$ to 2; 7.1 represents those in the period 1 to $1 + \Delta t$, the deaths among the survivors in the successive elementary periods being 7.2, 7.3, etc. Thus, from Figs. 49 and 50, it is at once evident that A'_0, the deeper shaded figure BIKLDB, represents deaths from survivors from the previous period. The medium shaded figure, B_0, represents deaths in the period under consideration born in that period, since G to L is contemporaneous with G to F. The broken shaded figure A_0, or AHGKIBA, represents the deaths occurring in the preceding periods from births in that period; they are similar to B_0. The black shaded figure, A_1, or BILDCB, represents deaths at ages outside the limit, that is, at ages greater than one year (exactly). In short, A_0, $B_0, \ldots \ldots D_0$ represent deaths within the year of record of persons less than one year old; A'_0 to C''_0, represent deaths of survivors from the preceding year. Similarly in regard to the other figures, A_1 to C_1 represent deaths of persons of age 1 to 2, attaining that age in the year previous to the year of record; while A'_1 to B'_1 represent deaths of persons of age 1 to 2 who attain that age during the year of record. Similarly, *mutatis mutandis*, in regard to A_2, B_2, and A'_2, etc.

The figures of the type $A_0, B_0, \ldots A_1, B_1$, etc., are represented by the solid Fig. 52, those of the type $A_1, B_1, \ldots A_2, B_2$, etc., are represented by the solid Fig. 51.

If the origin for $x + 0$, $t + 0$, be the point A in Figs. 49, 50, 52 then the lengths of an element of volume in areas A_0, A'_0, and B_0, taken parallel to A G Q V, Fig. 52, are respectively $t - x = 1 - x$, x, and $2t - (t + x) = 1 - x$.

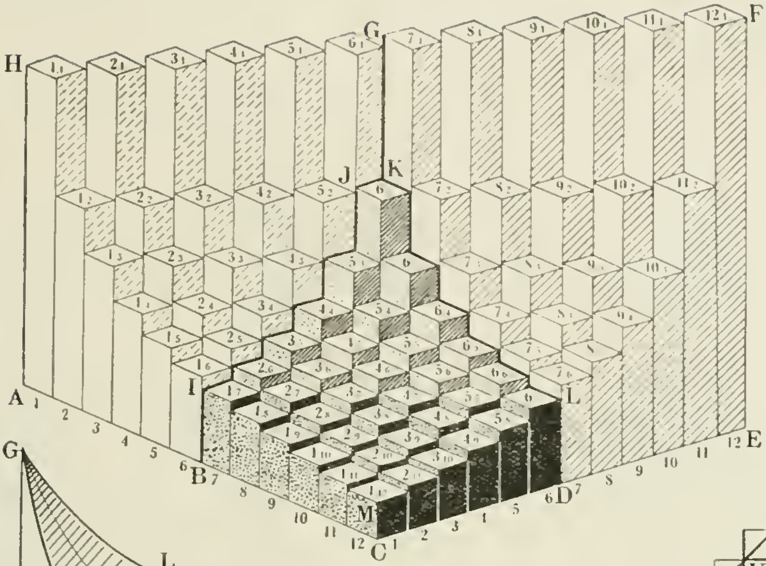


Fig. 49.

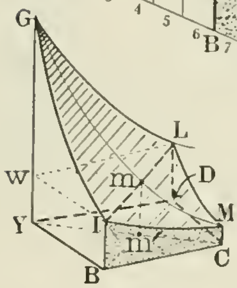


Fig. 51.

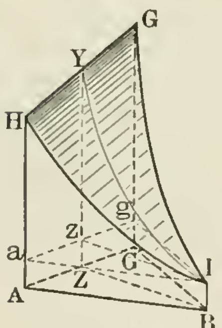


Fig. 52.

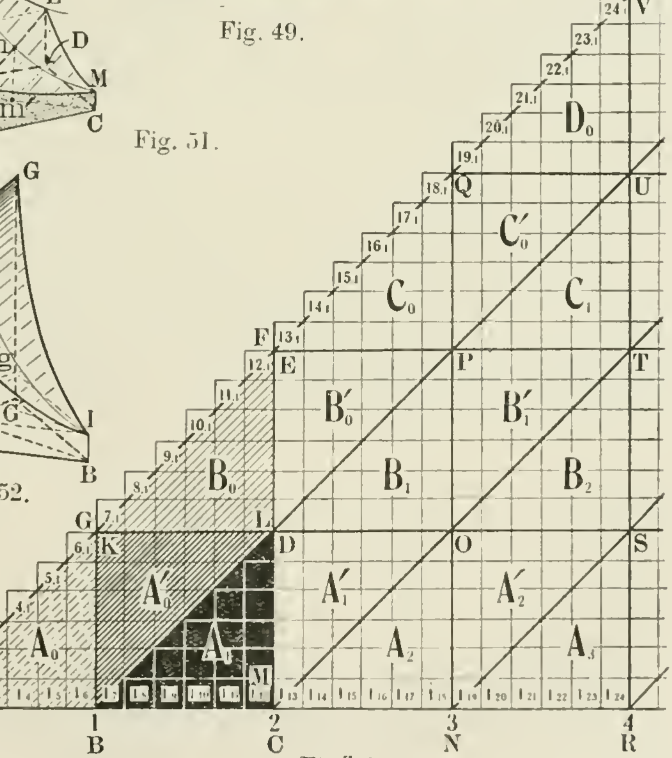


Fig. 50.

If the number of births in successive units of time be *constant*, it is obvious that $A_k = B_k = C_k$, etc. ; $A'_k = B'_k = C'_k$, etc. ; k being any suffix, and that we have also—

$$(363)\ldots\ldots A_0 = B_0 = \ldots\ldots \int_0^1 (1 - x)\, F_3(x)\, dx$$

$$(364)\ldots\ldots A'_0 = B'_0 = \ldots\ldots = \int_0^1 x\, F_3(x)\, dx$$

Consequently the ratio B_0 / A'_0 is found by dividing the value of (363) by that of (364).

9. **Proportion of infantile deaths arising from births in year of record, number of births increasing.**—If the number of births be *increasing*, and the increase be assumed to be at the rate of $e^{rt} = 1 + rt + \frac{1}{2} r^2 t^2 +$ etc. ; the quantities will increase, that is, with any common suffix $A < B < C$. Since the maximum value of r is about 0.03, the effect of the omission of the term t^2, where it is one year, is of the order of a two-thousandth, and may be ignored. Consequently, the increase in the number of births may, with abundant precision, be taken to vary as $1 + rt$. We may take the origin for t as at the point G in Fig. 50, hence an element of the volume of B_0, and of A'_0 will be respectively—

$$(1 - x)\, F_3(x)\, \delta x\, (1 + \frac{1}{2} r - rx) \text{ ; and } x\, F_3(x)\, \delta x\, (1 - rx) \text{ ;}$$

the element being taken parallel to H G F. The terms in brackets, containing r, represent the mean heights of the volume-elements taken along the lines Y I and G M respectively in Figs. 51 and 52. Hence, multiplying out, the relative values of the volumes representing B_0 and A'_0 are respectively as follows :—

$$(365)\ldots B_0 = \int_0^1 \left\{ (1 + \frac{1}{2} r)\, F_3(x) - (1 + \frac{3}{2} r)\, x\, F_3(x) + r\, x^2\, F_3(x) \right\} dx$$

$$(366)\ldots A'_0 = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \int_0^1 \left\{ x\, F_3(x) - r\, x^2\, F_3(x) \right\} dx$$

From an analysis of the deaths of infants during four years, viz., 1909, 1910, 1912 and 1913, the relative values of the functions to be integrated were found to be as in the following table, viz., XXXVI., in which the figures in the first column represent the number of deaths out of 1,000,000 births, occurring up to the time after birth indicated in the first column, that is up to age x.

TABLE XXXVI.—Values of above Integrals for Various Periods, Basis 1,000,000 Births, Australia, Years 1909-10; 1911-12.

Period, or Age x.	$\int_0^x F_3(x)\,dx$	$\int_0^x x\,F_3(x)\,dx$.		$\int_0^x x^2\,F_3(x)\,dx$.	
	Number.	Number.	Proportion. *	Number.	Proportion *
1 day	9,360	11	.0012	0	.0000
2 ,, ...	14,032	27	.0019	0	.0000
4 ,, ...	17,257	57	.0033	0	.0000
5 ,, ...	18,500	73	.0039	1	.0001
10 ,, ...	22,573	154	.0068	2	.0001
15 ,, ...	25,335	247	.0097	5	.0002
20 ,, ...	27,497	350	.0127	10	.0004
25 ,, ...	29,228	457	.0156	17	.0006
30 ,, ...	30,645	563	.0184	25	.0008
30.437 ,, or 1 mth.	39,757	564	.0184	26	.0008
60.874 ,, 2 ,,	37,231	1,362	.0366	128	.0034
91.311 ,, 3 ,, ..	42,796	2,517	.0588	370	.0086
121.747 ,, 4 ,, ..	47,735	3,953	.0828	790	.0165
152.181 ,, 5 ,, ..	52,165	5,612	.1076	1,413	.0271
182.621 ,, 6 ,, ..	56,128	7,424	.1323	2,246	.0400
213.058 ,, 7 ,, ..	59,688	9,349	.1566	3,289	.0551
243.495 ,, 8 ,, ..	62,920	11,367	.1807	4,551	.0723
273.932 ,, 9 ,, ..	65,871	13,455	.2043	6,030	.0915
304.368 ,, 10 ,, ..	68,563	15,585	.2273	7,716	.1125
334.805 ,, 11 ,, ..	71,045	17,755	.2499	9,615	.1353
365.242 ,, 12 ,, ..	73,366	19,973	.2722	11,736	.1600

* Proportion of number in second column.

From these results, the values given in the next table, viz., XXXVII., have been computed, agreeably to formulæ (365) and (366). The results shew that with quarterly records, over 94 per cent. of the recorded deaths are referable to births in that quarter: with half-yearly about 87 per cent., and with yearly records about 73 per cent., provided the intervals between occurrence and record are the same in both cases.

TABLE XXXVII.—Proportion of Deaths under One Year of Age, born during the Year in which the Death is Recorded. Australia, 1909, 1910, 1912, 1913.

Interval of Time Considered.	Rate of Increase of Births.				
	1.00 (constant).	1.01	1.02	1.03	1.04
15 days	.9903	.9903	.9904	.9905	.9906
1 month	.9816	.9817	.9818	.9820	.9821
2 ,, ...	.9634	.9636	.9639	.9641	.9644
3 ,, ...	.9412	.9415	.9418	.9422	.9425
4 ,, ...	.9172	.9176	.9181	.9185	.9190
5 ,, ...	.8924	.8930	.8936	.8942	.8949
6 ,, ...	.8677	.8685	.8693	.8701	.8709
7 ,, ...	.8434	.8444	.8454	.8465	.8475
8 ,, ...	.8193	.8204	.8216	.8227	.8239
9 ,, ...	.7957	7970	.7983	.7996	.8009
10 ,, ...	.7727	.7742	.7756	.7771	7786
11 ,, ...	.7501	7517	.7534	.7550	.7567
12 ,, ...	.7278	.7296	.7314	.7332	.7351

It will be seen from the above table that, for any ordinary increase of birth-rate the effect thereof upon the proportions given may be ignored for ordinary purposes; that is, with the infantile-death rates experienced in Australia, it may be assumed that 0.73 of the deaths recorded in any year occur in that year, whatever the increase of the birth-rate, and this will be approximately true generally.

10. **Secular fluctuation in birth-rates.**—The birth-rates since 1860 for Australia are as shewn in the following table:—

TABLE XXXVIII.—Crude Birth-rates, and Marriage-rates, Australia, 1860 to 1914.

Year of Decade.	Rates per 100,000 of the Population.											
	1860.		1870.		1880.		1890.		1900.		1910.	
	B	M	B	M	B	M	B	M	B	M	B	M
0 ..	4,256	842	3.866	712	3.525	717	3.498	764	2.733	724	2,673	837
1 ..	4,228	864	3.800	694	3.526	760	3,447	747	2.716	732	2.721	879
2 ..	4.327*	876	3,707	696	3,448	810	3.365	674	2.671	726	2,865	907
3 ..	4,166	837	3,744	740	3,482	836	3.279	619	2.529†	667	2,825	866
4 ..	4,291	861	3,679	720	3,560	828	3,083	608	2.641	702	2.805	880
5 ..	4,210	834	3,588	728	3,569	816	3,038	623	2,623	725		
6 ..	3,979	762	3,592	714	3,537	788	2,843	655	2,657	749		
7 ..	4,042	751	3,499	731	3.560	759	2,821	668	2,676	787		
8 ..	4,046	755	3,539	737	3,550	798	2,715	672	2,659	776		
9 ..	3,865	733	3,577	716	3,465	767	2,727	703	2,669	790		

* The highest value was in 1862. † The lowest value was in 1903.

These rates shew a fairly steady decrease till 1903, and then an inconspicuous rise. Their significance will be dealt with later, viz., in the part treating of "fecundity."

Secular as well as any other fluctuations in the birth-rate are of course influenced by the marriage-rate. This rate is also shewn in the above table. The figures in the columns denoted by B are birth-rates, and in those denoted by M are marriage-rates. Fig. 53 shews both results by dots, the general trends—ignoring small oscillations—being indicated by the broken lines among the dots. The scale of values for the marriage curve is ten times as much enlarged as that for the birth-rate curve: see p. 165.

The long-continued fall in the birth-rate, which has been characteristic of Australia, is characteristic also of the countries of the western world. The rates for as many years as are available for various countries are as shewn in the table hereunder. These are also graphed in Fig. 53. They afford unmistakable evidence of what may be called the Malthusian

drift of the world during the last 50 years, which drift, however, is on the whole contemporaneous with a conspicuous reduction of infantile mortality, so that the "residual birth-rates" would shew a much less marked effect. It is after all the residual rate which is of greater importance.

TABLE XXXIX.

Crude Birth-rates for Various Countries—1860-1914—per 10,000 of the Population.

Year.	Australia.	England and Wales.	Scotland.	Ireland.	France.	Prussia.	Italy	Switzerland.	Norway.	Sweden.	Denmark.	Netherlands.	Belgium.	Austria.	Hungary.	Mean.
1860..	426	343	356	..	262	386	..	..	..	348	..	319	306	379	..	381
1861..	423	346	349	..	269	377	..	..	..	326	318	354	308	372	..	344
1862..	433	350	346	..	265	372	..	..	..	334	310	332	301	379	..	242
1863..	417	353	350	..	269	395	..	..	..	330	311	364	318	403	..	352
1864..	429	354	356	240	266	397	379	..	..	336	303	357	315	403	..	345
1865..	421	354	355	257	265	393	385	..	..	328	314	361	314	378	..	344
1866..	398	352	354	262	264	393	390	..	..	331	322	354	327	379	421	350
1867..	404	354	351	260	264	371	367	..	..	308	305	354	321	366	388	340
1868..	405	358	353	268	257	369	354	..	..	275	312	349	325	379	424	341
1869..	387	348	343	267	257	379	372	..	..	282	295	343	316	393	426	339
1870..	387	352	346	277	255	383	369	298	..	288	305	361	323	396	417	330
1871..	380	350	345	281	229	338	370	291	202	304	302	354	310	389	430	331
1872..	371	356	349	278	267	397	379	300	297	300	303	360	323	391	410	339
1873..	374	354	348	271	260	396	363	299	299	308	308	362	325	399	422	339
1874 .	368	360	356	266	262	401	349	305	307	309	309	364	326	397	427	334
1875..	359	354	352	261	259	407	377	320	312	312	319	366	325	399	450	345
1876..	360	363	356	264	262	407	392	330	318	308	326	371	332	400	463	350
1877..	350	360	353	262	255	399	370	323	318	311	324	366	323	387	436	343
1878..	354	356	349	251	252	387	362	316	311	298	317	361	315	386	431	337
1879..	358	347	343	252	251	390	378	308	320	305	320	367	315	392	458	340
1880..	352	342	336	247	246	378	339	298	307	294	318	355	311	380	428	323
1881..	353	339	337	245	249	370	380	300	300	291	323	350	314	377	429	351
1882..	345	338	335	240	248	367	371	291	309	294	324	353	312	391	438	331
1883..	348	335	328	235	248	371	372	288	309	289	318	343	305	382	449	328
1884..	356	336	337	239	247	376	390	285	310	300	334	349	305	387	456	334
1885..	357	329	327	235	243	377	386	280	313	294	326	344	299	376	448	328
1886..	354	328	329	232	239	377	370	280	309	298	326	346	296	380	456	328
1887..	356	319	317	231	235	377	389	280	308	297	320	337	294	382	442	326
1888..	355	312	313	228	231	374	375	278	308	288	317	337	291	379	438	322
1889..	346	311	309	227	230	371	383	276	297	277	313	332	295	379	437	313
1890..	350	302	304	223	218	366	358	264	303	280	306	329	287	367	403	311
1891..	345	314	312	231	226	377	372	278	309	283	309	337	296	370	423	319
1892..	337	304	307	225	223	363	362	274	296	270	295	320	289	362	404	309
1893..	328	307	308	230	228	375	365	277	307	274	305	338	295	379	426	316
1894..	308	296	299	230	223	366	355	273	298	271	301	327	290	367	415	307
1895..	304	303	300	233	217	369	349	273	306	275	300	328	285	381	418	310
1896..	284	296	304	237	225	369	348	281	304	272	305	327	290	380	405	309
1897..	282	296	300	235	222	365	347	283	300	267	298	325	290	375	403	306
1898..	271	293	301	233	218	367	335	285	303	271	302	319	286	363	377	302
1899..	273	291	298	231	219	363	339	290	309	264	297	321	288	373	393	303
1900..	273	287	296	227	214	361	330	286	301	270	297	316	289	373	393	301
1901..	272	285	295	227	220	362	326	290	296	270	297	323	294	366	378	300
1902..	267	285	293	230	217	355	334	285	289	265	292	318	284	371	389	298
1903..	253	285	294	231	211	344	317	274	288	257	287	316	275	353	369	290
1904..	264	280	291	236	209	347	329	273	281	258	289	314	271	356	374	290
1905..	262	273	286	234	206	335	327	269	274	257	284	308	261	339	363	285
1906..	266	272	286	235	206	337	321	269	267	257	285	304	257	350	365	285
1907..	268	265	277	232	197	330	317	262	264	255	282	300	253	340	367	281
1908..	266	267	281	233	201	227	337	264	263	257	285	297	249	337	369	282
1909..	267	258	273	234	195	317	327	255	263	256	282	201	237	331	377	278
1910..	268	251	262	233	196	305	333	250	261	247	275	286	237	325	357	273
1911..	272	244	256	232	187	294	315	242	259	240	267	278	229	314	350	265
1912..	286	238	259	230	190	289	324	241	256	237	267	281	226	313	363	247
1013..	282	239	255	228	190	..	..	..	252	231	256	281	..	..	..	246
Mean	354	335	338	243	235	366	357	284	296	287	304	335	296	374	411	

This curve of birth-rate averages, convex upwards, discloses a continuous and accelerating decrease with time, *i.e.*, the Malthusian gradient shews an accelerating increase. This fact is significant, and is worthy of special notice. If the tendency to reduction of the birth-rate were one which bore a constant ratio to the rate itself we should have—

$$(367)\ldots\ldots d\beta/dt = -\kappa\beta\,;\ \text{or}\ d\beta/\beta = -\kappa\,dt\,;$$

β denoting the birth-rate, and κ a constant. In this case we should have by integrating—

$$(368)\ldots\ldots \log.\beta = -\kappa t + c\,;\ \text{or}\ \beta = Ce^{-\kappa t}$$

in which log. $C = c$, or $C = e^c$, a curve which of course is concave upwards, not convex. Again, a linear diminution of the birth-rate, viz., one of the form—

$$(369)\ldots\ldots \beta = \beta_0(1 - kt)$$

is an accelerating reduction of the relative increase of the population by births, consequently the convexity upwards of the curve implies a still more rapid reduction than a linear one.

11. **The Malthusian law.**—The question naturally arises whether the birth-rate phenomena may properly be considered as conformable to the law enunciated by Malthus, which may be stated thus:—If, as time goes on, food-production increases in an arithmetical, while population increases in a geometrical, ratio, the latter must inevitably overtake and surpass the former.

Let us suppose that from any given moment (*i.e.*, for $t = 0$) the (possible) increase of food-production is continually in the ratio $1 + qt$, and that the population increase is continually e^{rt}. Both expressions are unity for $t = 0$. Let $q = Mr$. The factor M, we may suppose to be considerably greater than unity. Thus for small values of t, the value of $1 + qt$ is greater than e^{rt}. To determine the value for t, when the two expressions become equal, we put—

$$(370)\ldots\ldots 1 + qt = 1 + Mrt = e^{rt}.$$

By expanding, subtracting unity from each side of the equation, transposing and dividing by $\frac{1}{2}r$, we get for Y, the number of years when the population will overtake the food supply.

$$(371)\ldots\ Y = \frac{2\,(M-1)}{r} = t\,(1 + \frac{1}{3}rt + \frac{1}{3.4}r^2t^2 + \frac{1}{3.4.5}r^3t^3 + \text{etc.}$$

The ratio r being very small, this equation may be solved for t by successive approximations, for which purpose the equation is preferably written in the form—

$$(371a)\ldots\ldots Y = \frac{2(M-1)}{r} = t\,\{1 + \frac{1}{3}rt\,[1 + \frac{1}{4}\,t(1 + \frac{1}{5}rt + \ldots)\,]\}$$ [1]

The values of r range up to about 0.03. The solution of this leads to a remarkable result, viz., that if the food-supply can be increased in the ratios 1. $1 + Mr$, $1 + 2\,Mr$ + .. $1 + tMr$, as the years pass, then after a relatively small number of years there will be a shortage, though in the interim there will be an excess. The interval of time necessary, with population increasing continually at any given rate, and the food-production increasing by uniform amounts per annum, may be called the "Malthusian equivalent interval."

12. **Malthusian equivalent interval.**—These intervals have been computed, here, for the cases where the annual increase of food supply is either 2, 4, 8, or 16 times that at which the population is continually increasing.

TABLE XL.—Malthusian Equivalent Intervals corresponding to various Rates of Increase.

Number of Times Food Supply Exceeds Needs of Population.*	Number of Years (t) Before Population Overtakes Food-Supply, the former increasing as e^{rt}, the latter as $(1 + Mrt)$.				
M.	rt.	$r = 0.01$	$r = 0.015$.	$r = 0.02$	$r = 0.03$.
2	1.2564	125.6	83.8	62.8	41.9
4	2.3370	233.7	155.8	116.8	77.9
8	3.3150	331.5	221.0	165.7	110.5
16	4.2290	422.9	281.9	211.4	141.0

* Initially.

The above table shews that even if the possibility of increasing the food supply was initially larger, the interval of time elapsing before the increase of population would overtake that of food supply is small compared with historical or geological periods. Thus it will be seen that the conclusions of II., § 34, pp. 30-32 hereinbefore, are supported from a somewhat different point of view; in other words, the general truth of Malthus' proposition is certain. In short, the ordinary rates of population increase, small as they may appear to some investigators, are sufficient

[1]If a table of values of e^x be available, we may rewrite (371) in the form $Mx = e^x - 1$, consequently $M = (e^x - 1)/x$ and this can be solved by trial. The following values will serve for most cases required.

x =	.1	.2	.3	.4	.5	.6	.7	.8	.9	1.0	2.	3.	4.	5.
M =	1.052	1.107	1.166	1.255	1.297	1.370	1.448	1.532	1.622	1.718	3.195	6.362	13.400	29.483

to bring about, in a relatively short time, trouble from over population, at least in countries where the population density is already appreciable. In this connection it may be of interest to mention that the rates for Australia deduced from the populations determined as at the censuses of 1901, 1911, and from the deaths for that intercensal period, shew that for the number of males and females at birth to be each constantly 100,000 per annum the constant populations would have to be as follows—

		For Males.	For Females.
Population	(T_0) ..	5,520,030 ;	5,883,742 ;
Number born	(l_0) ..	100,000 ;	100,000 ;
Expectation of life	(e°_0)	55,200 ;	58,837 ;
Death-rate	$\dfrac{l_0}{T_0} = \dfrac{1}{e^\circ_0}$ =	0.01812 ;	0.01700

Thus if the death-rates and birth-rates were equal, a constant population of 5,701,886, with a crude masculinity of 0.938184, would give 100,000 births, in which the numbers of the sexes would be initially equal. The masculinity of the actual population based on the aggregates for the years 1901 to 1910 inclusive was actually 1.115600. The ascertained increases of population due to excess of births over deaths and of immigrants over emigrants were, however, as follows :—

Population at 31st December, 1900—
Males 1,976,992 ; Females 1,788,347 ; Total 3,765,339

Population at 31st December, 1910—
Males 2,296,308 ; Females 2,128,775 ; Total 4,425,083

Decennial Increase of Population—
Males, 319,316 ; Females, 340,428 ; Total, 659,744.

Rate of continuous increase—
$r_m = 0.0149726$; $r_f = 0.017426$; $r_p = 0.0161435$.

As already shewn this rate of increase could not, of course, possibly be maintained over many centuries.

13. **The Malthusian coefficient and Malthusian gradient.**—Let us suppose that in any community unrestricted fecundity would give a birth-rate B, the actual birth-rate, however, being $\beta = \mathrm{m}B$. The value of m is necessarily less than unity, and is a coefficient of reduction which diminishes in the ratio that the birth-rate diminishes. This coefficient may be called the *Malthusian Coefficient*, since it measures the degree of the restriction of fecundity which characterises the community in question.[1] Thus—

$$(372)\ldots\ldots\ldots \mathrm{m} = \beta/B.$$

[1] See the Essay on the Principle of Population ; or a view of its past and present effects on human happiness ; etc., by T. R. Malthus, A.M. ; Professor of History and Political Economy, East India College. In particular Chap. IX. pp. 506-536, 4° Edit., 1807.

BIRTH-RATES OF VARIOUS COUNTRIES, 1860 to 1913.

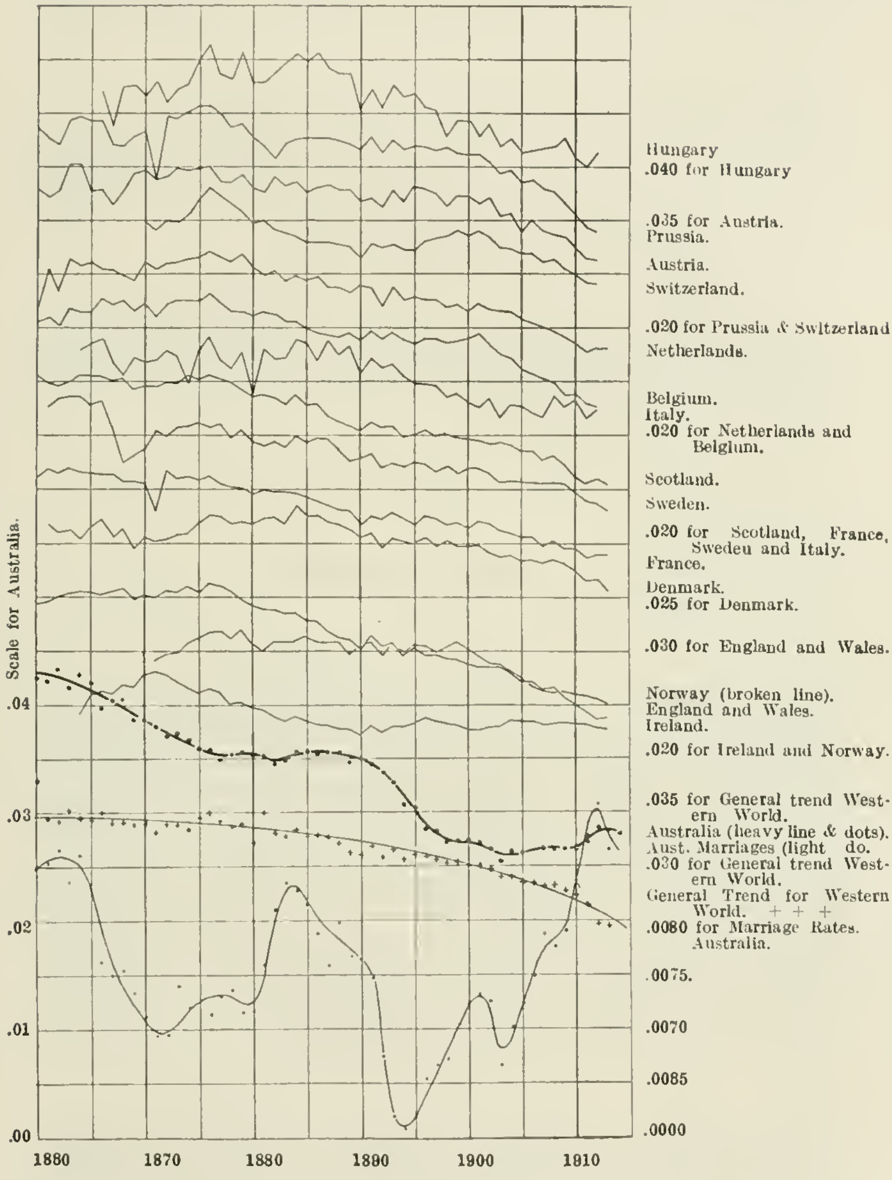


Fig. 53.

The heavy curve with heavy dots is the general trend of the Australian birth-rates, the dots denoting the individual annual rates. The light curve with light dots is the general trend of the Australian marriage-rates in Australia, the dots denoting the individual annual rates. The light curve with vertical crosses + + + is the unweighted average of the various birth-rate curves, the crosses denoting the mean of the annual values.

If $\beta = f(t)$, we shall have also $\mathrm{B}\mathfrak{m} = f(t)$, that is the graphs of the two are of the same form and differ only in scale. The question of the determination of this coefficient will be later considered.

The general change in the values either of the birth-rate or of the Malthusian coefficient, if regarded as characteristic of any country, is best found by taking the means *irrespective of the magnitude of the populations.* A mean so found does not, of course, apply to the aggregate of the populations; a weighted mean would do so however.

The *Malthusian gradient* ($\mathfrak{g}$) may be defined either as the rate of fall of the curve representing the birth-rate, or as the rate of fall of the curve representing the Malthusian coefficient. Until B is determined, see above, the former definition is the more practical. Thus—

$$(373)\ldots\ldots \mathfrak{g} = \frac{d\beta}{dt} = \frac{df(t)}{dt}\ ; \quad \text{or} = \frac{d\beta}{\mathrm{B}dt} = \frac{df(t)}{\mathrm{B}dt}$$

of which the former is, at present, to be preferred.

14. **Reaction of the marriage-rate upon the birth-rate.**—If the marriage-rates were constant in all communities, or were constant at different periods in the one community, the birth-rates would be properly comparable as measures of fecundity. The effect of the marriage-rates depends, however, upon the degree of fecundity characteristic of different ages of life; hence exact comparisons of the fecundity are possible only after a correction is applied depending upon its variation with age and the age distribution of the marriages. This question will be considered hereinafter. It will suffice to observe that there is some slight indication of the correlation of the facts exhibited by the curves representing the birth and marriage rates, Fig. 53, though the effect is easily masked by the economic factors which influence human affairs: see p. 165.

15. **Annual periodic fluctuation of births.**—In order to see whether there was any distinct evidence of a seasonal fluctuation among births, the numbers of births have been compiled, according to the actual date of birth, for the three years 1911-1913, in equalised half-months, and, assuming the rate of increase of population to be uniform, these numbers were corrected for its general increase during the period. The results are as follow:—

TABLE XLI.—Seasonal Fluctuations in the Corrected Frequency of Births. Australia, 1911-1913.

Equalised Months.	January.		February.		March.		April.		May.		June.	
Nuptial ..	14,444	14,350	15,045	14,893	15,766	15,058	15,098	15,218	15,132	15,202	15,347	15,445
Ex-nuptial..	893	806	831	835	896	870	872	857	912	887	905	1,003
Totals ..	15,337	15,156	15,876	15,728	16,662	15,928	15,970	16,075	16,044	16,089	16,252	16,448
Ratio of total to Aggregate for the Year	.9604	.9491	.9942	.9849	1.0434	.9974	1.0001	1.0066	1.0047	1.0075	1.0177	1.0300

Equalised Months.	July.		August.		September.		October.		November.		December.	
Nuptial ..	15,542	15,513	15,367	15,216	15,514	15,715	15,183	14,323	14,787	14,398	14,352	14,773
Ex-nuptial	975	949	986	961	946	903	955	902	872	849	846	867
Totals ..	16,517	16,462	16,353	16,177	16,460	16,618	16,138	15,225	15,659	15,247	15,198	15,640
Ratio of total to aggregate for the year	1.0343	1.0309	1.0241	1.0130	1.0308	1.0406	1.0106	.9534	.9806	9548	.9517	.9794

The semi-monthly means for nuptial, ex-nuptial and total births were 15,070,899, and 15,969 respectively. An examination of the results shews that *a fluctuation certainly exists*, and although the number of ex-nuptial births is both relatively and absolutely very small, the agreement of the fluctuation of nuptial with that of ex-nuptial births is fairly definite. The results for the totals are shewn by curve A, Fig. 54, the curved line denoting the general trend of the fluctuation.

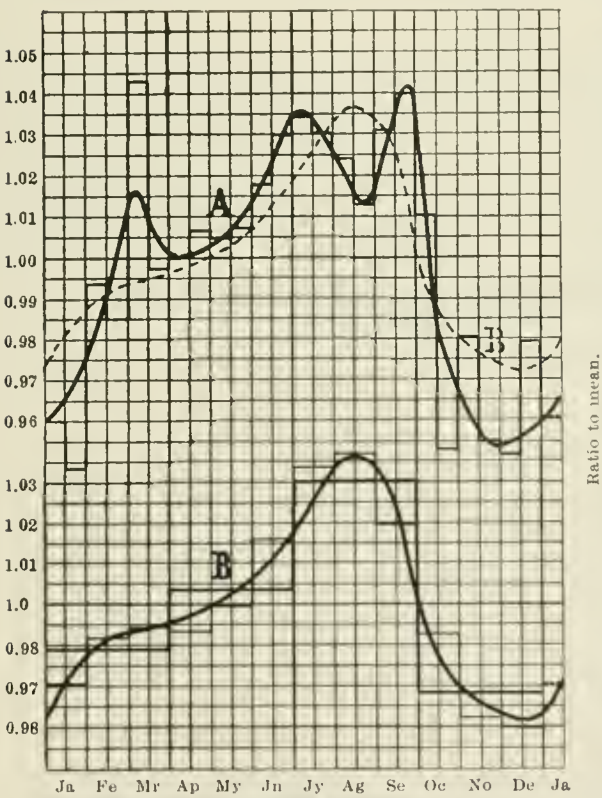


Fig. 54.

In order to further examine the question, the quarterly results of the births, *as registered*, during the eight years 1907-1914 were compiled, and this is done also for population. The interval between birth and registration has, however, shortened since the introduction of a maternity bonus: see pp. 151 and 152.

The following procedure was adopted. The births registered were taken out in the several quarters; these quarters were then equalised, the numbers being corrected to shew what would have been given by a constant population, since it was found that the increase of this last was sensibly at the rate $1 + 0.0247265\ t$. In this way the values shewn in Table XLII. hereinafter were obtained. These quarterly results may be subdivided into monthly values, as explained on the next section, so as to give the monthly values. These results are shewn by the curve B in Fig. 54.

TABLE XLII.—Births Registered. Australia, 1907-1914.

Births as Registered.				Births as Corrected for Equal Quarters and a Constant Population.			
236,462	243,191	254,141	242,860	241,457	244,914	251,457	238,830
—	—	—	—	.98891	1.00307	1.02987	.97815

The values for the individual months may be deduced as explained in the next section, and are as follows:—

1	2	3	4	5	6	7	8	9	10	11	12
.9807	.9916	.9944	.9936	.9996	1.0160	1.0333	1.0366	1.0197	.9924	.9922	.9699

and these monthly results are shewn by the small rectangles in curve B, Fig. 54.

For the greater part of the year, at least, the results are substantially identical for the two sexes, as a compilation made for the four years, 1907-1910, shews. The results were as follows:—

TABLE XLIII.—Seasonal Fluctuations* of Births, according to Sex. Australia, 1907-1910.

Males, Females or Persons.	Jan.	Feb.	Mar.	April	May.	June.
M	.9874	.9169	.9949	1.0152	1.0064	.9978
F	.9903	.9229	.9950	1.0079	1.0069	.9859
P	.9889	.9198	.9949	1.0116	1.0067	.9920
	July.	**Aug.**	**Sept.**	**Oct.**	**Nov.**	**Dec.**
M	1.0321	1.0410	1.0299	1.0378	.9924	.9482
F	1.0170	1.0583	1.0437	1.0465	.9760	.9479
P	1.0249	1.0504	1.0367	1.0420	.9844	.9480

* The registration was on the average 38.2 days after birth for the years 1907-1910.

Reverting to curves A and B, Fig. 54, the curve drawn by lines may be taken as a probable representation of the fluctuation; since there is no reason to suppose that the large oscillations are other than accidental.

As the theory of determining the Fourier curves to fit the group results presents certain special features, it is given hereunder.

16. **The subdivision of results for equalised quarters into values corresponding to equalised months.**—When quarterly results are available, they may (after equalising and also being freed from the annual progression so as to give, as residuals, only the fluctuation elements) be readily resolved into monthly values, which have a high degree of probability. The most convenient form in which to give such results is the *height* of the monthly group. Let the mean of the *heights* of four quarterly groups be denoted by R, with suffixes corresponding to the quarter (viz., 1 to 4), and that of the monthly group by r, with corresponding suffixes (viz., 1 to 12). Then the solution can proceed on one of two possible assumptions, viz. (a) that the amplitudes of the component fluctuations are identical, and the epochs are different, or (b) that the epochs are identical and the amplitudes are different.[1]

That is, we may assume either (a) that—

$$(375)\ldots\ldots y = a + b\,sin.\,(x + \beta) + b\,sin.\,2\,(x + \gamma)\,;$$

or (b), that—

$$(376)\ldots\ldots y = a + b\,sin.\,(x + \beta) + c\,sin.\,2\,(x + \beta).$$

The data are, of course, inadequate in themselves to determine which assumption should be adopted, and the results are to that extent, uncertain. But this uncertainty, in general, is of small moment.

In case (a) we have—

$$(377)\ldots l = -\frac{6}{\pi}\,b\,cos.\,\beta = \frac{3}{2}\,(R_3 + R_4)\,;\ m = \frac{6}{\pi}\,b\,sin.\,\beta = \frac{3}{2}\,(R_1 + R_4),$$

$$(378)\ldots p = -\frac{3}{\pi}\,b\cos 2\gamma = \frac{3}{4}(R_2 + R_4)\,;\ q = \frac{3}{\pi}\,b\sin 2\gamma = \frac{3}{4}\sqrt{-2(R_1R_3 + R_2R_4)}$$

It will be seen that q is not independent of l, m and p, since we must have—

$$(379)\ldots\ldots q^2 = \frac{1}{4}\,(l^2 + m^2 - 4\,p^2)$$

From this last, the value $\frac{3}{4}\sqrt{-2\,(R_2R_4 + R_1R_3)}$ is deduced. Observing that $\frac{1}{2}\sqrt{3} - 1 = -0.1339746$; $\frac{1}{2}\,(1 - \sqrt{3}) = -0.3660254$; $\frac{1}{2}\sqrt{3} = 0.8660254$; we may put the values of r_1 to r_{12} in the following very convenient forms, viz.:—

$$(380)\ldots\ldots r_1 = -0.1340\,l + \frac{1}{2}\,m - \frac{1}{2}\,P + 0.8660\,q.$$

$$(381)\ldots\ldots r_2 = -0.3660\,l + 0.3660\,m - P$$

[1] See Studies in Statistical Representation (Statistical Applications of the Fourier Series), by G. H. Knibbs, Journ. Roy. Soc. New South Wales, Vol. xlv., pp. 76-110, 1911. In particular see pp. 88-89.

$$(382)\ldots\ldots r_3 = -\frac{1}{2}\,l + 0.1340\,m - \frac{1}{2}\,P - 0.8660\,q.$$

$$(383)\ldots\ldots r_4 = -\frac{1}{2}\,l - 0.1340\,m + \frac{1}{2}\,P - 0.8660\,q.$$

$$(384)\ldots\ldots r_5 = -0.3660\,l - 0.3660\,m + P$$

$$(385)\ldots\ldots r_6 = 1.8660\,l - \frac{1}{2}\,m - \frac{1}{2}\,P + 0.8660\,q.$$

$$(386)\ldots\ldots r_7 = +0.1340\,l - \frac{1}{2}\,m - \frac{1}{2}\,P + 0.8660\,q.$$

$$(387)\ldots\ldots r_8 = +0.3660\,l - 0.3660\,m - P$$

$$(388)\ldots\ldots r_9 = +\frac{1}{2}\,l - 0.1340\,m - \frac{1}{2}\,P - 0.8660\,q.$$

$$(389)\ldots\ldots r_{10} = +\frac{1}{2}\,l + 0.1340\,m + \frac{1}{2}\,P - 0.8660\,q.$$

$$(390)\ldots\ldots r_{11} = +0.3660\,l + 0.3660\,m + P$$

$$(391)\ldots\ldots r_{12} = -1.8660\,l + \frac{1}{2}\,m - \frac{1}{2}\,P + 0.8660\,q.$$

In case (*b*) we have—

$$(392)\ldots.l = -\frac{6}{\pi}\,8\,b\cos = \frac{3}{2}\,(R_3 + R_4)\,;\ m = \frac{6}{\pi}\,b\sin\beta = \frac{3}{2}\,(R_1 + R_4).$$

$$(393)\ldots.\ P = -\frac{3}{\pi}\,c\cos 2\beta = \frac{3}{4}\,(R_2 + R_4)\,;\ q = \frac{3}{4}\,\sqrt{\left\{\frac{c^2}{b^2}\left[(R_3 + R_4)^2 + (R_1 + R_4)^2\right] - (R_1 + R_3)^2\right\}}$$

Again, q is not independent of l, m, and p, since we have—

$$(394)\ldots\ldots \frac{4b^2}{c^2} = \frac{l^2 + m^2}{p^2 + q^2}\,;\ \text{or}\ q^2 = \frac{c^2}{4b^2}\,(l^2 + m^2) - P^2.$$

which leads to the value of q above written.

If $c = b$, the last expression for q in (394), reduces to that first given, viz., in (379). It is obvious from this last value for q, that the *ratio* c/b is at our disposal, and provided it be so chosen that the whole expression within the braces is *not* negative, there will be a real value for q. A unique solution will be that which makes the q term zero in the above series of equations for monthly values. This is given by making the expression within the braces in (393) zero. Hence for this we have

$$(395)\ldots\ldots \frac{c}{b} = (R_1 + R_3)\,/\,\sqrt{\left\{(R_3 + R_4)^2 + (R_1 + R_4)^2\right\}}$$

If, therefore, the relation between β and γ, and between b and c are *both* unknown, we may, with advantage, write $q = 0$ in the series of equations 380) to (391). In short, if we assume that $c = b$ then γ is determinate.

If this relation be not assumed, but that $\gamma = \beta$ is assumed, we may, *within certain limits*, still make the ratio of c to b whatsoever we choose, and, if we have no ground for believing that a particular ratio is to be preferred, the simplest solution of the whole problem is, making the epochal angles β and γ identical, to so take the ratio of c to b that the q term will be eliminated from the series of equations for monthly values, viz., formulæ (380) to (391), etc.; that is, we may determine this ratio by (395)[1]. It may be reiterated that the subdivision of the quarterly into monthly values by the preceding formulæ assumes that the fluctuation involves only terms sin. x and sin. $2x$.

17. **Equalisation of periods of irregular length.**—In order to apply the formulæ of the preceding section, it has already been indicated that the crude data must be freed from any annual progression depending on a progression in population numbers and among the births themselves. It is preferable to operate, therefore, on rates, *i.e.*, to divide the number of births (or marriages or deaths, etc.) each month, quarter, or year, as the case may be, by the mean population of the month, quarter, or year itself. Even then a correction is necessary, since for precise results it is still necessary to equalise the period, in fact, if the seasonal fluctuation (or annual period of oscillation) to be determined be small in amplitude, the equalisation is an essential. Both months and quarters differ appreciably in length.[2]

For population-numbers and for birth-numbers, the equalising corrections will necessarily be made in a somewhat different manner. A table of corrections for the ends of the months or quarters is first formed. Numbers such as population-numbers and rate-numbers may be called *continuant*, and those such as numbers of birth, marriages and deaths, etc., *accretional*. For the purpose of *corrections* it may also be assumed that the daily values at the terminals of the unequal periods is the mean of the values for the adjoining periods.[3]

Then, except for the first and final period, there are two corrections. For a single leap-year there is no correction at the end of August, and none at the end of October. The equalised February is always in January, and excepting as above mentioned the terminal of the equalised month is always in the month following.[4]

[1] Such a solution has the further advantage of making the deviations from the averages for the respective quarters a minimum.

[2] The shortest month is no less than 8 per cent. short of the average, and shortest quarter 1.37 per cent.

[3] It is more rigorous, of course, to determine the function, the integral of which gives the result dealt with, but this process is tedious and ordinarily quite unnecessary.

[4] There would have been some advantage if January had had 30 days, instead of 31, and February 30 days in ordinary and 31 in leap years, instead of 28 and 29 days.

Let δT and $\delta' T$ be the small periods to be *added* respectively to the beginning and the end of an unequal period to make it coincide with an equalised period, the length of this last being T_0. Let also the periods preceding and following that to be corrected be denoted by T and T'; and let the period to be corrected be denoted by T_m. Then, the correcting periods δT, etc., being small, we have very approximately, for *continuant numbers*, P, P_m and P', etc., denoting that corresponding to T_0,

$$(397)\ldots . P_0 = P_m + \frac{1}{2T_0}\{(P - P_m)\,\delta T + (P' - P_m)\,\delta' T\}$$

and for *accretional numbers*, N, N_m, N', etc., N_0 denoting that corresponding to the period T_0,

$$(398)\ldots . N_0 = N_m - \frac{1}{2}\left\{\left(\frac{N}{T} + \frac{N_m}{T_m}\right)\delta T + \frac{1}{2}\left(\frac{N_m}{T_m} + \frac{N'}{T'}\right)\delta' T\right\}$$

$$= Nm + \frac{1}{2T_0}\{-(N + N_m)\,\delta T + (N_m + N')\,\delta' T\}$$

The approximate identity of these expressions can readily be established.[1] In regard to the sign of the corrections it may be observed that for continuant numbers the value is to be *increased* when the shift of *either* terminal of the unequalised period towards the terminal of the equalised period is in the direction of higher values. For accretional numbers, the number is increased for an additive shift, diminished for a negative shift.

18. **Determination of a purely physiological annual fluctuation of birth-rate.**—The annual birth-rate fluctuation, as obtained in section 15, by means of the formulae of sections 16 and 17, cannot be regarded as furnishing the variations of the reproductive activity solely due to physiological causes, which variations may be presumed to repeat themselves every year. The distribution of the frequency of marriage, and therefore of birth, throughout the year is affected by the fêtes observed, and particularly by the "movable feasts" (Easter, etc.). The number of years to be included to secure a true mean-determination must embrace the whole cycle of movement. The extent of this cycle has been referred to in a paper on the Statistical Application of the Fourier Series, by the writer.[2] But even when this mean result is obtained, what may be called the physiological fluctuation is not to hand, since the effect of the "movable feast" is *distributed*, not *eliminated*. By a systematic analysis,

[1] The question of corrections of this kind has been dealt with at length by me in a paper read 5th July, 1911, at the Roy. Soc., N.S.W., see its Journ. xlv., pp. 79-85, which treats of the correction of an increasing population, and that for unequal months, quarters, half and whole years.

[2] Vide Journal Royal Soc. N.S.W., Vol. xlv., pp. 76-110.

however, of the results for different years in which the place of the movable feast is as different as possible, the effect of this distribution can be ascertained and corrections applied to eliminate the effect. The difficulty of a perfectly satisfactory solution will be apparent from Fig. 55 hereunder.

19. **Periodicities due to Easter.**—As ecclesiastically defined, Easter Day is the first Sunday after the 14th day of the paschal "Calendar Moon," a fictitious ecclesiastical moon, which is from one to three days later than the real moon. The average position of Easter for the century 1800 to 1899 is April 8.55 days, and for the century 1900 to 1999 is April 8.89 days, or say for the whole period of 200 years April 8.72 days. In Fig. 55 the Easters in each decade are shown on a single line for the years 1800 to 1999 inclusive. An inspection of the figure shews that the points lie approximately on a series of 10 slanting lines, four days apart, these lines progressing at the rate of one half day per decade, and further that they are inversely symmetrical. For lines *a*, *b*, *c*, and *e* and *a′*, *b′*, *c′*, and *e′* the symmetry is perfect; for lines *d* and *d′* however the symmetry is not absolutely perfect. It is evident that no means derived from two decades nor from periods of 19 years, nor from centuries are exactly comparable.

POSITION OF EASTER FOR 200 YEARS.

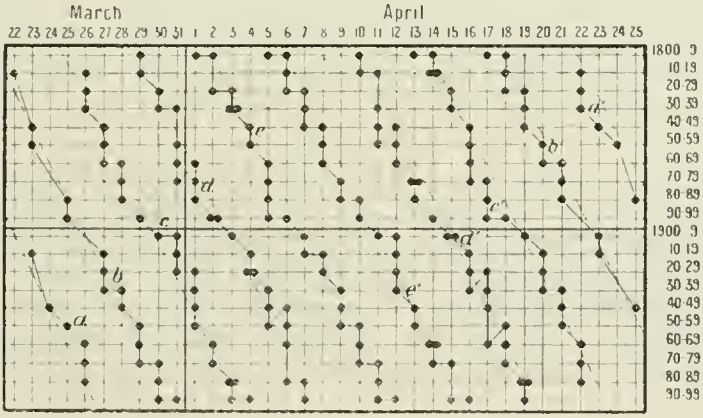


Fig. 55.

Since the tropical year = 365.2422 days and the synodic lunar month = 29.530588 days, the Metonic cycle, *19 tropical years is 6939.6018 days, and 235 complete lunations equal 6939.6882 days, differing only .0864 day from the nineteen years.

The following table exhibits the peculiarities for successive decades.

TABLE XLIV.—MEAN POSITION OF EASTER FOR 200 YEARS.*

1800.					1900.				
Decade.	Mean.	Easters in March.	Mean of March Easters.	Mean of April Easters.	Decade	Mean.	Easters in March.	Mean of March Easters.	Mean of April Easters.
	April.					April.			
0–9	9.46	1	29	9.56	0–9	10.56	2	30.5	13.12
10–19	8.16	3	25.67	13.57	10–19	8.36	3	27	12.86
20–29	8.86	2	28	11.50	20–29	8.06	2	29	10.25
30–39	7.86	3	29	11.43	30–39	9.66	2	27.5	12.62
40–49	9.06	2	25	12.50	40–49	8.96	2	26	12 12
50–59	8.56	3	27	13.57	50–59	7.56	2	27	10.38
60–69	7.96	3	28.67	11.86	60–69	9.66	2	27.5	12.88
70–79	9.36	2	29.5	11.88	70–79	8.66	3	28.33	13.14
80–89	9.46	2	25.5	12.62	80–89	8.66	2	28	11.25
90–99	6.76	2	27	9.25	90–99	8.76	2	30.5	10.88
Means	8.55	2.3	27.48	11.70	Means	8.89	2.2	28.09	11.94

* The complete Easter Cycle, restoring both the day of the week and of the month, is known as the "Dionysian" or "Great Paschal" period. Its length is 4.7.19 = 532 years.

To obtain a normal periodic fluctuation it would be preferable, were it practicable, to combine the results, each for a series of years such as would give Easter an identical distribution. In the period such a series is, however, impracticably long. Hence in the case of marriage, birth-rate, migration, etc., it is necessary to consider the actual effect on the periodic fluctuation studied. In respect of marriages the effect of Easter is to reduce the number of marriages in the Lent period (6 weeks) preceding, and to augment them in the preceding and following periods.

It may be noted that for the fluctuations of annual period in the marriage frequency, the great length of the Lent period, viz., 6 weeks, has the effect of throwing the increase of frequency as far back as February. The migration frequency is often thrown back into March. Thus, as is evident from the preceding table and the diagram, decennial means will clearly be nearly but not *exactly* comparable. The data for a thorough study of periodic fluctuation would in these cases have to be weekly groups.

XII.—NUPTIALITY.

1. **General.**—The phenomena of reproduction have a double aspect, viz., one a sociological and the other a physiological. Thus, from the standpoint of a theory of population, both are important. The women of reproductive age in any community furnish the potential element of reproduction; but the resolution into fact depends also upon social facts as well as upon physiological; for example, the relative proportion of married and single, *i.e.*, the nuptial-ratio, even more profoundly affect the result than physiological variations of fecundity. In Chapter XVIII. of the Census Report (Conjugal Condition), the numbers of married and unmarried females have been given as at 3rd April, 1911, in Australia. These will be considered mainly in regard to the child-bearing age, in dealing later with fecundity.

2. **The Nuptial-Ratio.**—The "nuptial-ratio," j, may be defined as the ratio of the married, J, to the unmarried, U, which latter may be taken generally as including the never married, the widowed, and the divorced. This ratio, J/U may apply to either sex and to any age, or age-group, or to the total for all ages, etc. The nuptial-ratio in any community may be regarded as a measure of the social instinct, and also a measure of the reproductive instinct, modified by social traditions as well as facilitated or hindered by economic conditions. This ratio, for the case of females, is, of course, specially important in relation to fecundity.

The significance of marriage in respect of reproductive activity depends upon the relative frequency of nuptial and ex-nuptial births, as well as upon the relative proportions of the married and unmarried, that is, it depends not merely upon the nuptial-ratio, but also upon nuptial and ex-nuptial fecundity, particularly during the reproductive period of life. The values of

$$(399)\ldots\ldots j = J/U$$

for various countries are given in the following table for women during the reproductive period, and for women of all ages, viz., from age 0 to the end of life.

TABLE XLV.—Ratios of Married Women in various Age-groups to Unmarried Women in the same Groups. Reproductive Ages. Female Nuptial Ratios.

Ages of Women.	Aust. Census, 1911† Metro pol'n.	Aust. Census, 1911† Other.	Aust. Census, 1911† Total.	C'wlth Aust. 1908.	England and Wales. 1901.	England and Wales. 1911.	Scotland. 1901.	Scotland. 1911.	Ireland. 1901.	Ireland. 1911.	Belgium. 1910.	Germany. 1900.	Germany. 1910.
10 to 14 incl.	.0000	.0000	.0000	.0001	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
15 „ 10 „	.0337	.0435	.0395	.0382	.0157	.0121	.0767	.0145	.0075	.0063	.0271	.0161	.0139
20 „ 24 „	.03510	.4892	.4242	.4214	.3731	.3184	0.3040	.2758	.1538	.1538	.04482	.3077	.3959
25 „ 29 „	1.0945	1.6325	1.3613	1.2997	1.8011	1,2645	1.3759	.0615	.8307	.8137	1.6385	1.8172	1.0359
30 „ 34 „	1.8201	2.8810	2.3318	2.4698		2.4565		1.8345			2.8324	3.6381	3.8471
35 „ 39 „	2.2491	3.5996	2.8938	2.9805	3.0124	3.0299	2.2851	2.3418	1.6777	1.7040	3.4697	4.2515	4.4905
40 „ 44 „	2.5045	3.9037	3.1586	3.1159		3.0747		2.4112			3.3632	3.8012	4.0035
45 „ 49 „	2.4617	3.6935	3.0324	3.1068	2.3015	2.6855	1.8199	2.0750	1.4343	1.5443	2.8921	3.0086	3.2188
50 „ 54 „	2.0628	3.1420	2.6634	2.6025		2.1756		1.6795			2.2601	2.1635	2.3415
55 „ 59 „	1.5747	2.3651	1.9470	1.8482	1.3217	1.6585	1.0007	1.3061	.8686	1.0190	1.5909	1.4864	1.5995
60 „ 64 „	1.0622	1.5751	1.3070	1.5815*		1.1481		0.9089			1.0929	.09590	1.0353
0 „ 105 „	.5231	.5198	.5213	..	.5159	.5528	.4293	.4516	.3543	.3765	.5781	.5200	.5465

* Ages 60 to 61 only. † 3rd April, 1911.

The results in the table shew that there are considerable divergences between populations as regards their nuptial constitution, consequently even if the individual fecundity were constant, the birth-rates would differ. The results of the Australian Census of 1911 shew also that there are striking differences between metropolitan and extra-metropolitan communities, the marriage-rate being very much higher for the latter; and they shew also that the nuptiality is very different as regards the sexes. See Vol. I., Chap., XVIII., Conjugal Condition. § 6, of the Census Report.

3. **The Crude Marriage-Rate.**—The lack of homogeneity in populations, illustrated in the last section, renders the crude marriage-rate, viz., the ratio of the *marriages*, J, to the *population*, P, of uncertain significance. The heterogeneity arises largely from divergences of social life and tradition, in respect of the relative frequency of marriage, and the frequency according to age. Inasmuch, however, as ordinarily the constitution of any population does not materially change, the marriage-rates for any particular country and for limited periods are comparable among one another, and their variations may generally be attributed to variations in the *economic* conditions of the population in question. Wars have, of course, a marked effect, see the points marked with asterisks, on Table XLVI., and also Fig. 56, giving the curve of the mean of the marriage-rates of a number of important countries. We shall denote the crude marriage-rate by n; thus—

$$(400) \ldots\ldots n = J/P.$$

In some countries the marriage-rate is the ratio, not of the "marriages," but of the "persons married," to the population. In such cases the rates will be double those shewn in Table XLVI. hereunder, the which gives the marriage-rates for the countries for which in Table XXXIX. the crude birth-rates were given. This also gave the values of the marriage-rate. In Table XLVI., the mean in the final column is merely the unweighted mean, and is therefore *not* the rate for the aggregate of the populations. The trend, thus determined, treats each population as equally important in regard to the revelation of the secular tendency, if any, of the marriage-frequency. For the constitution of a norm a weighted-mean would of course be needed.

Fig. 56 illustrates the movement in the marriage-rate, and shews that movement in its relation to that of the western world generally (excluding America). Although the general trends shewn by broken lines of curves A and B, are by no means similar, there are often very similar fluctuations about this general trend, which appear readily enough if the general trend be regarded as a basic line about which the minor fluctuations may be regarded as moving.

Birth, Marriage and Divorce Rates.

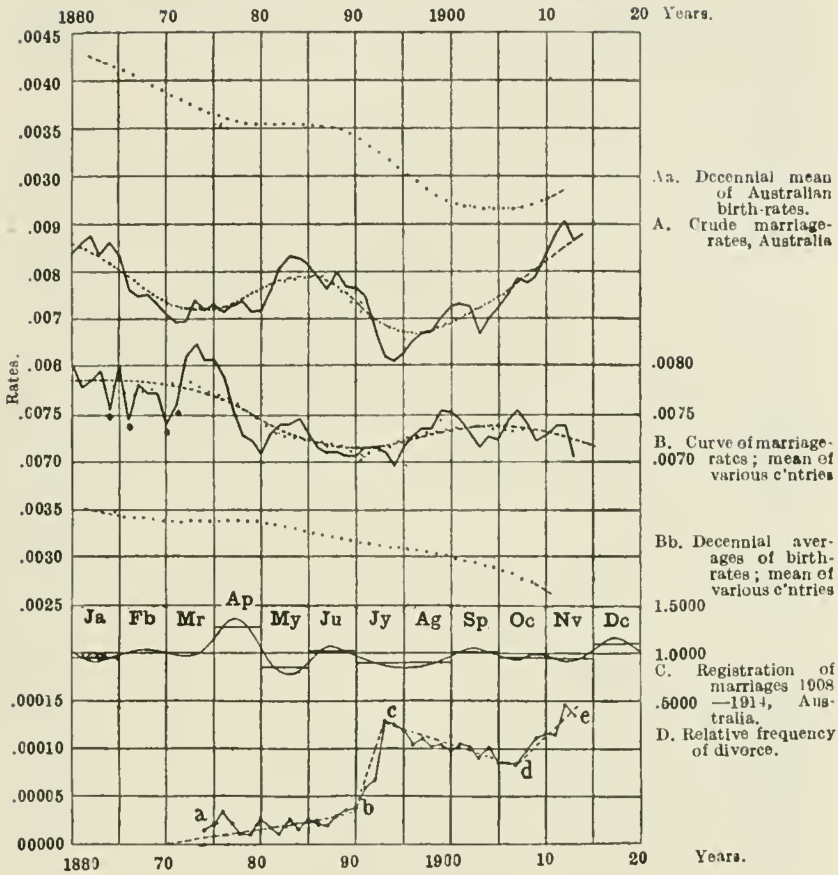


Fig. 56.

Curve Aa shews the successive decennial means of the birth-rates of Australia, the central year being changed one year at a time.

Curve A shews the marriage-rates of Australia by the zig-zag line; the fine dots shew the successive decennial means; the broken line, closely following the decennial means, indicates the general trend.

Curve B shews the mean of the marriage-rates of a series of countries; the fine dots shew the successive decennial means of these; the broken line indicates the general trend of the marriage-rates.

Curve Bb shews the successive decennial averages of the means of the crude birth-rates of a number of countries.

Curve C shews the mean annual fluctuation of the registration of marriages in Australia for the period 1908-1914.

Curve D shews the relative frequency of divorce per unit of population for Australia, the portion a b being prior to acts facilitating divorce; b c being the condition immediately following upon the passing of the facilitating Acts; c d, and d e being the subsequent trends of the relative divorce-frequency.

As regards birth-rates and marriage-rates, it will be observed that here there is some indication of a correlation between the phenomena. This correlation will not, of course, be well-marked, since the aggregate

of "first births" is not large compared with "all births," But the trend of the Australian birth-rate shewn by Curve Aa is strikingly similar to Curve A shewing the marriage-rate, and Curve Bb gives some indication of its connection with Curve B.

TABLE XLVI.

Marriage-rates for Various Countries—1860-1913—per 10,000 of the Population.

Year	Australia.	England and Wales.	Scotland.	Ireland.	France.	Prussia.	Italy.	Switzerland.	Norway.	Sweden.	Denmark.	Netherlands.	Belgium.	Austria.	Hungary.	Means per 100,000
1860	84	86	70	..	79	84	..	..	..	78	..	82	73	85	..	801
1861	86	82	68	..	82	80	..	..	..	73	75	81	71	80	..	778
1862	88	81	67	..	81	85	..	..	..	71	74	79	71	88	..	785
1863	84	84	72	..	80	87	..	..	..	73	75	83	73	85	..	796
1864	86	86	72	48	79	87	80	..	..	70	57*	84	75	83	..	*756
1865	83	88	74	55	79	91	91	..	..	71	89	85	76	78	..	800
1866	76	88	74	54	80	78	57*	..	..	67	84	84	79	65*	82*	*745
1867	75	83	70	54	79	93	68	..	..	61	77	84	78	97	104	781
1868	76	81	67	50	79	89	72	67	..	55	73	77	73	92	137	772
1869	73	80	67	50	83	90	80	72	..	57	74	77	74	104	110	772
1870	71	81	72	53	61*	74*	74	70	..	60	74	80	70	98	98	*740
1871	69	84	72	54	73*	80*	75	73	67	65	73	80	74	95	104	*759
1872	70	87	76	50	98	103	75	79	70	70	75	83	78	93	108	810
1873	74	88	78	48	89	102	79	77	73	73	81	86	78	94	113	822
1874	72	85	76	46	83	97	76	83	77	73	82	84	76	91	107	805
1875	73	84	74	46	82	91	84	90	79	71	85	84	73	86	109	807
1876	71	83	75	50	79	86	82	82	77	71	86	83	72	83	102	788
1877	73	79	72	47	75	80	78	79	76	69	81	81	69	76	94	753
1878	74	76	67	48	75	78	72	74	73	65	74	78	67	76	95	728
1879	72	72	64	44	76	77	76	70	68	63	74	77	68	78	104	722
1880	72	75	66	39	75	77	70	69	67	63	76	75	71	76	92	709
1881	76	76	70	43	75	77	81	69	64	62	78	73	71	80	100	730
1882	81	78	71	43	75	79	78	69	67	64	77	72	70	83	103	740
1883	84	78	71	43	75	80	81	69	66	65	77	71	68	79	105	741
1884	83	76	68	46	76	81	83	70	69	66	78	72	68	80	103	746
1885	82	73	66	43	75	82	80	70	67	67	76	70	68	77	101	731
1886	79	71	63	42	74	82	79	70	65	64	71	70	67	79	97	715
1887	76	72	64	43	73	80	80	71	63	63	70	70	71	79	90	710
1888	80	72	64	42	72	80	79	71	61	59	71	69	71	80	94	710
1889	77	75	67	45	71	82	77	71	63	60	71	70	73	76	82	707
1890	76	78	69	45	70	82	73	70	65	60	69	71	73	76	82	706
1891	75	78	70	46	75	82	75	71	66	59	68	71	74	78	86	716
1892	67	77	71	47	76	81	75	72	64	57	68	72	77	78	92	716
1893	62	74	66	47	75	81	74	72	65	57	70	73	76	80	94	711
1894	61	75	67	47	75	80	75	72	64	58	70	72	75	80	93	696
1895	62	75	68	51	74	80	73	73	65	59	71	74	78	81	85	713
1896	66	79	71	51	76	83	71	76	67	60	73	75	81	80	81	727
1897	67	80	72	51	76	84	72	79	67	61	75	74	83	81	82	736
1898	67	81	74	50	74	85	69	78	70	62	76	73	83	79	84	737
1899	70	83	75	50	77	85	74	78	71	63	75	74	83	83	91	755
1900	72	80	73	48	78	86	72	78	69	62	76	76	86	83	89	752
1901	73	80	70	51	78	83	73	76	66	61	72	77	87	82	88	745
1902	73	80	71	52	76	80	73	74	64	60	71	76	81	78	87	731
1903	67	79	72	52	76	80	72	74	60	58	71	75	79	78	82	716
1904	70	77	71	52	76	81	75	74	60	59	72	74	80	78	92	727
1905	73	77	68	53	77	81	77	75	58	59	72	73	79	78	86	724
1906	75	79	72	52	78	83	79	77	59	62	75	75	81	79	88	743
1907	79	80	72	52	80	82	78	77	60	62	77	75	80	76	100	754
1908	78	76	68	52	80	80	84	76	61	61	75	72	78	77	92	740
1909	79	74	64	52	78	78	78	75	60	60	74	71	77	76	87	722
1910	84	75	65	51	78	78	79	73	62	61	73	72	79	76	87	729
1911	88	76	67	54	78	80	75	74	63	59	72	72	80	76	93	738
1912	91	78	69	53	79	80	76	73	62	59	73	75	80	74	86	739
1913	87	78	71	51	75	..	..	..	63	59	72	78	..	..	..	704
1914	88	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..
M'ns	758	791	698	487	771	831	761	740	661	634	744	761	753	815	949	..

4. **Secular Fluctuation of Marriage-rates.**—Fig. 56, embodying the results on Table XLVI., reveals the fact that the relative frequency of marriage has been increasing in Australia since 1897, although it has tended to diminish recently in the old world. It is apparent from a comparison of the two curves, A and B, that there is no very marked correlation between the two progressions. The factors influencing the relative frequency of marriage probably have a very unequal incidence in different countries. In order to obtain an accurate measure of reaction of the larger economic influences on the rates, statistics covering long periods of time will be required. The characteristics of the longer or secular fluctuations will fully appear only when much more statistical material is available than exists at present.

The period of the larger oscillations in the data shewn amounts to about 22 or 23 years in Australia, and about 30 or 31 years for the aggregate of the populations of the western world. The period of the minor fluctuations is very variable, and is somewhat ill-defined. In Table XLVII. are shewn the values of successive decennial means for the marriage-rates, and also for the birth-rates. These are shewn by dots on Fig. 56.

TABLE XLVII.--Decennial Unweighted Means of Marriage and Birth-rates, 1860 to 1909.

Decade Year.	Marriages per 100,000 of the Population.									
	1860.		1870.		1880.		1890		1900.	
Year.*	A	W	A	W	A	W	A	W	A	W
0 ..	..	..	740	781	758	746	734	712	687	734
1 ..	..	..	729	781	766	739	715	710	697	732
2 ..	..	..	724	786	774	732	701	711	707	737
3 ..	..	..	722	783	777	727	692	714	719	739
4 ..	..	..	721	778	783	725	680	716	729	739
5 ..	812	779	719	773	788	724	673	721	740	735
6 ..	799	773	719	770	793	723	669	726	749	733
7 ..	782	771	726	767	792	722	668	729	764	732
8 ..	764	773	737	760	778	720	673	730	782	733
9 ..	754	776	747	752	756	717	678	731	802	732
Decade Year.	Births per 100,000 of the Population.									
0 ..	..	..	3,894	3,396	3,534	3,382	3,435	3,179	2,743	3,009
1 ..	..	..	3,832	3,397	3,532	3,365	3,382	3,161	2,702	2,984
2 ..	..	..	3,793	3,397	3,526	3,343	3,313	3,142	2,683	2,960
3 ..	..	..	3,739	3,400	3,532	3,326	3,239	3,122	2,669	2,935
4 ..	..	..	3,688	3,396	3,533	3,311	3,155	3,102	2,663	2,915
5 ..	4,141	3,478	3,659	3,397	3,522	3,284	3,082	3,092	2,657	2,890
6 ..	4,102	3,436	3,625	3,381	3,519	3,272	3,005	3,082	2,651	2,862
7 ..	4,059	3,423	3,598	3,401	3,512	3,240	2,932	3,063	2,652	2,827
8 ..	3,997	3,420	3,572	3,393	3,503	3,218	2,863	3,052	2,671	2,776
9 ..	3,955	3,407	3,546	3,382	3,483	3,206	2,788	3,026	2,700	2,732

A denotes the values for the Commonwealth of Australia.

W denotes the values derived from the unweighted means for the series of countries shewn on Tables XXXIX and XLVI.

* The moment of time to which the values apply is the *beginning* of the years 0, 1, 2, etc.........9.

5. **Fluctuation of annual period in the frequency of marriage.**—Social custom in regard to marriage expresses itself in a fluctuation of annual period, but the changes in the date of Easter make the results for any one year not comparable in general to those of any other. The movement of Easter has been already considered, see Part XI., Natality. The following results are for the period 1908-1914, and are corrected for inequality in the length of the month, and for an increasing population. The table gives the crude and the adjusted data.

TABLE XLVIII.

Number of Marriages Registered in the Different Months. Australia, 1908-14.

Period.	Jan.	Feb.	Mar.	April.	May.	June.	July.	Aug.	Sept.	Oct.	Nov.	Dec.
1908–14 ..	21,462	21,106	22,732	28,358	19,714	22,959	20,752	20,733	22,824	22,138	21,140	25,534
Equalised ..	21,060	22,691	22,420	28,653	19,205	23,232	20,357	20,369	23,154	21,750	21,455	25,106
Constant Population	21,325	22,924	22,599	28,817	19,271	23,258	20,434	20,299	23,022	21,579	21,343	24,790
Ratio to Average..	.9490	1.0201	1.0057	1.2824	.8576	1.0350	.9093	.9033	1.0245	.9603	.9498	1.1032

These results are shewn, the reetangles and the probable fluctuation, by curve C, on Fig. 56, and represent the fluctuation of the *registration* of marriage. It is not certain that the returns made to the Registrars of Marriages by those who celebrate them have not also seasonal peculiarities, and consequently the fluctuation shewn is compounded of the two, and in reference to the time scale is in advance of the true position. The components of the curve can be found by applying formulae (90) to (101) of § 5, part III., Determination of Constants, etc.[1]

6. **General.—Conjugal Constitution of the Population.**— The "general conjugal constitution" of a population is defined by the number of persons therein who have never been married ; who are living in the state of marriage ; or of widowhood, etc. ; or who are living in the state of "divorced" persons. The actual unadjusted numbers of males and females in age-groups on the 3rd April, 1911, as indicated by the Census are shewn on the table of § 4, Chapter XVIII., Vol. I., of the Census Report. These are represented on Fig. 57, which shews both the group-values and the curves, which give sensibly the same totals. The results as furnished by the Census are somewhat vitiated by misstatements as to age ; on the whole, however, they give a fair representation of the change in the

[1] See also formulae (375) to (395), § 16, Part XI., Natality.

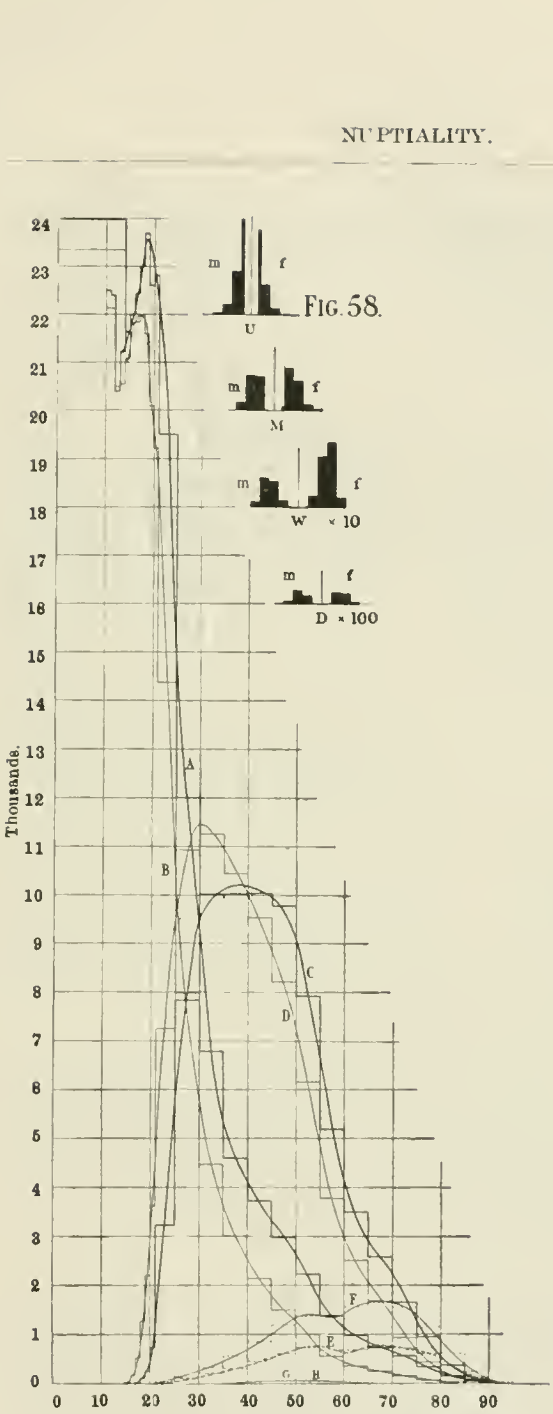


FIG. 58.

Fig. 57—The rectangles shew the total numbers as at the Australian Census of 1911 in 5-year groups, and the curves give approximately the equivalent areas, the heavy curves denoting the results for males, and the light those for females. Curves A and B shew the numbers of the "Never married"; C and D the numbers of the "married" E and F the numbers of the "widowed"; the former being for males; and G and H (which cannot be distinguished) shew the numbers of the divorced."

Fig. 58—The figures, which illustrate Table XLIX., shew the asymmetry of the distribution for the "never married, U; the "married," M; the "widowed," W; and the "divorced." The scale of W is ten times that of U and M, and that of D is 100 times that of U and M.

Fig. 57.

conjugal constitution with age. The general significance can be better grasped from the results shewn in the following table :—

TABLE XLIX.—Proportional Conjugal Constitution of the Australian Population, 3rd April, 1911, per 10,000,000 Total Population (Adjusted Numbers.)

Age-groups.	Proportion per 10,000,000 of Total Population.					Proportion per 1,000,000 of same Sex and Age-groups.			
	Never Married.	Married.	Widowed.	Divorced	Total.	Never Married.	Married.	Widowed.	Divorced
Under 14 M	1,506,806	0	0	0	1,506,806	1,000,000	0	0	0
F	1,467,395	2	0	0	1,467,397	999,998	2	0	0
14 to 20 M	710,197	5,304	34	4	715,539	992,533	7,413	48	6
F	662,798	35,358	184	18	698,358	949,080	50,630	264	26
21 to 39 M	875,496	699,580	14,646	1,731	1,591,453	550,123	439,586	9,203	1,088
F	602,222	862,948	24,658	2,265	1,492,093	403,609	578,347	16,526	1,518
40 to 59 M	231,079	746,217	55,057	2,941	1,035,294	223,201	720,778	53,180	2,841
F	116,157	621,059	107,535	2,229	846,980	137,142	733,263	126,963	2,632
60 to 79 M	58,438	194,935	61.309	595	315,277	185,354	618,297	194,461	1,888
F	18,608	124,159	134,718	285	277 770	66,991	446,985	484,998	1,026
80 & above M	4,507	10,770	12,301	45	27,623	163,161	389,893	445,317	1,629
F	1,129	3,850	20,424	7	25,410	44.431	151,515	803,778	276
All Ages M	3,386,523	1,656,806	143,347	5,315	5,191,992	652,259	319,108	27,609	1,024
F	2,868,309	1,647,374	287,519	4,804	4,808,008	596,569	342,632	59,800	999

The table is based upon 4,455,005 persons, of which 2,313,035 were males, and 2,141,970 were females ; it shews the distribution of 10,000,000 persons on that basis. The ratios in the second part of the table shew the proportional distribution in each age group. This distribution is illustrated in the small diagrams of Fig. 58, in which U denotes the males and females belonging to the class " never married " ; M denotes the " married" males and females ; W denotes the " widowed," of each sex ; and D the divorced of each sex. These small diagrams represent by the rectangular areas on the left of the median line the males, and on the right thereof, the females. The scale of U and M is identical ; that of W is 10 times, and that of D, 100 times as great.

The age at which the married are equal numerically to the unmarried is about 29.49 for males when the proportion of the total at that age is 0.49557, and 25.27 years for females when the proportion at that age is 0.49699. The difference is 4.22 years, and the mean proportion 0.49629 is close to either. This is due to the fact that the number of widowed and divorced is very small at the ages in question.

7. **Relative conjugal numbers at each age.**—The progress of the conjugal constitution with age is completely defined by giving for each sex, the proportion living at each age, and the proportional division of each such number according to conjugal condition. In the following table, which represents the smoothed results for the population of Australia at the Census of 3rd April, 1911, the relative distribution of males and females is shewn in columns II. and III. These numbers multiplied by 0.2313035 in the case of males, and 0.2141970 in the case of females (see the preceding section) give the absolute numbers, smoothed. The distribution of 100,000 of these at each age is given for each conjugal condition, viz., in IV. and V., the unmarried ; in VI. and VII., the widowed ; and so on. Thus at each age a complete comparison is

possible of the conjugal state. Assuming the constancy of the conjugal constitution of the population the results given in columns IV. to XII. are the probability of the number of males or females which will be found characterised as never married, married, widowed or divorced, in a total of 100,000 males or females of each year of age throughout life. Columns II. and III. shew, for the population of 10,000,000, a probable number of males or females living at each year of age throughout the life-period on the assumption of an unchanging constitution according to sex and age. As a matter of fact the Australian populatiou, however, has not reached a "steady" state as regards the constitution of its population.

TABLE L.—Relative Conjugal Numbers at each Age. Australia, 3rd April, 1911.

Age Last Birthday.	Proportion per 10,000,000 of same Sex.		Proportion per 100,000 of any Age in each Conjugal Condition.							
			Never Married.		Married.		Widowed.		Divorced.	
	Males.	Females.	Males.	Females.	Males.	Females.	Males.	Females.	Males.	Females.
I.	II.	III.	IV.	V.	VI.	VII.	VIII.	IX.	X.	XII.
0	253,554	263,314	100,000	100,000	..	..	..	..	..	..
1	236,741	247,352	100,000	100,000	..	..	..	..	..	..
2	227,662	238,776	100,000	100,000	..	..	..	..	..	..
3	221,173	232,426	100,000	100,000	..	..	..	..	..	..
4	216,158	226,689	100,000	100,000	..	..	..	..	..	..
5	211,030	221,422	100,000	100,000	..	..	..	..	..	..
6	205,544	216,147	100,000	100,000	..	..	..	..	..	..
7	199,236	210,605	100,000	100,000	..	..	..	..	..	..
8	193,611	205,675	100,000	100,000	..	..	..	..	..	..
9	189,232	201,852	100,000	100,000	..	..	..	..	..	..
10	186,115	199,135	100,000	100,000	..	..	..	..	..	..
11	184,835	197,118	100,000	100,000	..	..	..	..	..	..
12	184,813	196,086	100,000	100,000	..	..	..	..	..	..
13	185,860	196,417	100,000	99,998	..	2	..	..	..	..
14	188,588	198,425	99,993	99,958	7	42	..	..	..	..
15	192,846	202,463	99,982	99,783	18	215	..	2	..	..
16	196,742	206,660	99,945	99,207	55	789	..	4	..	..
17	200,105	209,910	99,842	97,445	156	2,547	2	8	..	..
18	202,552	212,020	99,507	94,363	491	5,621	2	15	..	1
19	203,839	212,575	98,803	90,089	1,191	9,878	4	27	2	6
20	202,932	211,646	96,862	84,638	3,111	15,290	23	59	4	13
21	201,908	209,144	93,784	77,311	6,156	22,547	54	121	6	21
22	200,256	205,554	88,724	70,131	11,172	29,634	93	204	11	31
23	197,226	200,885	82,292	61,261	17,537	38,393	153	302	18	44
24	192,582	195,288	76,334	54,327	23,403	45,192	236	418	27	63
25	186,746	189,284	70,235	48,343	29,402	51,018	326	555	37	84
26	180,702	183,033	64,175	44,043	35,349	55,149	426	701	50	107
27	174,619	177,047	58,423	40,529	40,975	58,487	535	857	67	127
28	168,700	171,165	53,325	37,220	45,937	61,599	654	1,036	84	145
29	163,041	165,339	49,526	34,316	49,586	64,280	782	1,242	106	162
30	157,732	159,615	45,773	31,703	53,169	66,641	918	1,477	140	179
31	152,938	154,153	42,050	29,253	56,732	68,827	1,060	1,726	158	194
32	148,316	149,297	38,623	27,299	59,998	70,490	1,210	2,001	169	210
33	144,192	144,913	35,755	25,742	62,703	71,732	1,365	2,303	177	223
34	140,534	141,029	33,532	24,593	64,757	72,523	1,523	2,648	188	236
35	137,417	137,532	32,018	23,352	66,100	73,362	1,687	3,038	195	248
36	134,594	134,166	30,608	22,188	67,326	74,092	1,860	3,462	206	258
37	132,387	131,164	29,495	21,236	68,252	74,546	2,041	3,951	212	267
38	130,491	128,344	28,536	20,209	69,012	75,059	2,233	4,456	219	276
39	128,870	125,725	27,727	19,392	69,612	75,368	2,436	4,957	225	283
40	127,499	123,036	27,035	18,423	70,089	75,785	2,645	5,504	231	288
41	126,085	120,006	26,296	17,585	70,601	76,023	2,867	6,100	236	292
42	124,753	116,766	25,596	16,697	71,060	76,272	3,098	6,737	246	294
43	123,297	113,820	24,815	16,111	71,584	76,108	3,345	7,486	256	295
44	121,810	111,075	24,004	15,481	72,121	75,913	3,606	8,311	269	295

Relative Conjugal Numbers at each Age. Australia, 3rd April, 1911.—*Continued.*

Age Last Birthday.	Proportion per 10,000,000 of same Sex.		Proportion per 100,000 of any Age in each Conjugal Condition.							
			Never Married.		Married.		Widowed.		Divorced.	
	Males.	Females.	Males.	Females.	Males.	Females.	Males.	Females.	Males.	Females.
I.	II.	III.	IV.	V.	VI.	VII.	VIII.	IX.	X.	XII.
45	120,227	108,316	23,205	14,831	72,620	75,875	3,887	9,000	288	294
46	118,632	105,580	22,540	14,203	72,959	75,593	4,195	9,911	306	293
47	116,738	102,742	22,062	13,632	73,124	75,231	4,503	10,846	311	291
48	114,201	99,455	21,696	13,064	73,159	74,888	4,831	11,760	314	288
49	110,863	95,281	21,573	12,592	72,924	74,455	5,191	12,668	312	285
50	106,112	90,165	21,317	12,216	72,747	73,959	5,627	13,545	309	280
51	99,890	84,049	20,762	11,745	72,755	73,183	6,172	14,798	311	274
52	93,841	78,320	20,065	11,362	72,858	72,254	6,767	16,118	310	266
53	86,985	73,129	19,593	11,034	72,763	71,200	7,336	17,511	308	265
54	80,868	68,376	19,471	10,676	72,387	70,127	7,837	18,961	305	236
55	74,798	63,726	19,317	10,315	72,094	69,009	8,300	20,484	289	192
56	68,840	59,053	19,105	9,966	71,915	67,531	8,704	22,342	276	161
57	62,865	54,585	18,994	9,653	71,433	65,772	9,291	24,427	282	148
58	57,474	50,631	18,896	9,316	70,806	63,932	10,012	26,611	286	141
59	53,237	47,302	18,694	9,031	70,294	62,064	10,728	28,770	284	135
60	49,602	44,622	18,757	8,754	69,406	60,052	11,558	31,063	279	131
61	46,433	42,424	18,920	8,434	68,371	58,108	12,439	33,332	270	126
62	43,873	40,678	18,940	8,156	67,491	55,959	13,313	35,763	256	122
63	44,724	39,258	18,879	7,741	66,677	53,939	14,206	38,202	238	118
64	39,870	37,904	18,781	7,556	65,853	51,569	15,138	40,761	228	114
65	38,149	36,588	18,382	7,256	65,254	49,263	16,160	43,371	204	110
66	36,376	35,276	17,709	6,879	64,547	47,147	17,542	45,868	202	106
67	34,574	33,782	17,457	6,690	63,524	44,855	18,832	48,353	187	102
68	32,771	32,078	17,520	6,409	62,309	42,859	20,000	50,634	171	98
69	30,912	30,164	17,790	6,068	61,063	40,966	20,979	52,872	168	94
70	29,096	28,194	18,276	5,861	59,406	38,856	22,140	66,194	178	89
71	27,341	26,359	18,817	5,668	57,938	36,880	23,087	57,367	158	86
72	25,460	24,608	19,358	5,422	56,563	34,905	23,943	59,592	186	81
73	23,562	22,890	19,817	5,185	55,468	33,008	24,587	61,730	128	77
74	21,669	21,121	20,152	5,131	53,990	31,012	25,738	63,784	120	73
75	19,861	19,281	20,026	4,858	52,024	29,141	27,841	65,932	109	69
76	18,123	17,400	19,323	4,759	50,334	27,327	30,248	67,849	95	65
77	16,459	15,453	18,125	4,649	48,595	25,442	33,202	69,848	78	61
78	14,639	13,441	16,539	4,558	47,549	23,500	35,824	71,885	88	57
79	12,568	11,545	15,480	4,466	46,784	21,890	37,668	73,591	68	53
80	10,817	9,762	14,788	4,371	43,965	20,214	41,167	75,366	80	49
81	9,023	8,189	14,950	4,291	42,166	18,556	42,789	77,109	95	44
82	7,263	6,830	15,833	4,236	41,072	16,882	42,976	78,842	119	40
83	5,824	5,640	16,704	4,150	39,347	15,183	43,875	80,667	74	..
84	4,630	4,650	17,647	4,080	37,348	13,495	44,912	82,425	93	..
85	3,662	3,842	18,654	4,012	35,419	12,063	45,809	..	118	..
86	2,832	3,142	18,473	3,960	33,588	10,288	47,786	..	163	..
87	2,166	2,516	17,964	3,891	31,936	8,809	49,900	..	200	..
88	1,634	1,975	17,196	3,830	31,746	7,754	50,794	..	264	..
89	1,258	1,536	15,808	3,770	31,615	7,264	52,234	..	343	..
90	968	1,186	14,732	3,720	30,357	7,204	54,464	..	447	..
91	722	878	15,569	3,681	28,743	7,521	55,688	..	..	..
92	532	616	16,260	3,640	26,016	8,581	57,724	..	..	..
93	363	416	17,857	3,601	23,810	9,882	58,333	..	..	..
94	233	280	20,370	3,563	22,222	11,537	57,408	..	..	..
95	169	210	20,513	3,534	20,513	13,044	58,974	..	..	..
96	125	159	20,690	3,501	24,138	14,746	55,172	..	..	..
97	99	117	21,739	3,470	26,087	16,630	52,174	..	..	..
98	78	84	22,222	3,446	27,778	18,276	50,000	..	..	..
99	61	61	21,429	3,427	28,571	20,050	50,000	..	..	..
100	43	42	20,000	3,411	30,000	21,811	50,000	..	..	..
Total	10,000,000	10,000,000	..	..	..	..	..	..	..	..

8. **The curves of the conjugal ratios.**—The smoothed results for each sex, representing the ratios which the "never married," the "married," the "widowed" and the "divorced" bear to each other (given in Table L) are graphed in Fig. 59, and are represented respectively by the curves U_m and U_f, M_m and M_f, W_m and W_f, and D_m and D_f.

Conjugal Ratios, Australia, 1911.

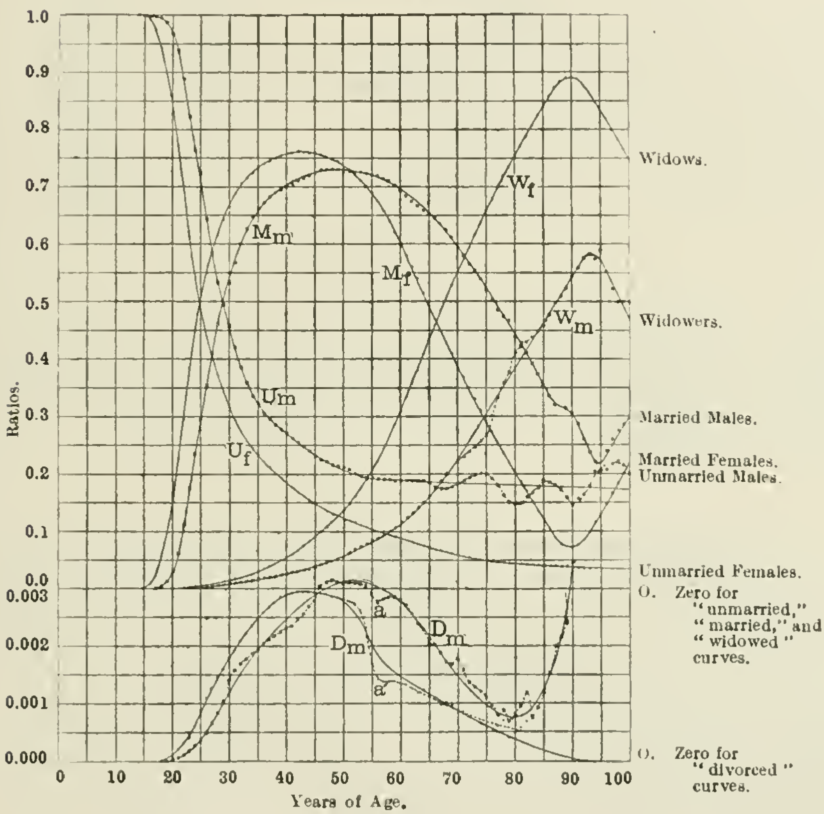


Fig. 59.

These curves shew merely the proportion of the unmarried, married, widowed, and divorced at each age, the number at *each* age being unity for males, and also unity for females. They thus shew the distribution for each age according to age, but not between one age and another.

The results for males are shewn by small crosses in the figures; those for females by small dots. The curves for the "never married" are somewhat of the type e^{nx^p}, where p is large. The critical features of these curves can be best shewn in a tabular form, and are as in the following table :—

TABLE LI.—Critical features in the frequencies of conjugal conditions.

Australian Census, 3rd April, 1911.

Character of Critical Feature.	Exact Age.	Proportion of Total Age-group.
Maximum proportion married, males	49.5	0.73100
,, ,, ,, females	43.0	0.76160
Minimum proportion married, males	95.0	0.00217
,, ,, ,, females	90.0	0.00063
Equal frequency married and unmarried males	29.49	0.49556
,, ,, ,, ,, females ..	25.27	0.49699
Maximum proportion widowed, males	90.5	0.89100
,, ,, ,, females	93.7	0.58400
Equal frequency unmarried and widowed, males ..	67.5	0.18600
,, ,, ,, ,, females ..	49.5	0.12600
Maximum proportion of divorced, males	52.0	0.003115
,, ,, ,, females	44.0	0.00295

In general these results are for the smoothed curves represented in Fig. 59, as may be seen by a reference thereto. The ratios among one another of the various ratios given in Table L follow no simple law, and an examination of them was found to lead to no important results.

9. **The norms of the conjugal ratios.**—It is eminently desirable that a series of curves based upon the *aggregate* of all populations to be compared, should be tabulated and constructed on some such model as that indicated here for the population of Australia. Such a norm, representing the relative frequency of the never-married, the married, the widowed and divorced for the entire aggregate would constitute the best possible bases for comparisons of the position of individual nations and peoples. The international issue of graph paper on which such curves were already drawn, preferably in faint colour, would enable the statistician to see instantly the position of his own country in regard to the larger average in respect of the particular character compared.

10. **Divorce and its secular increase.**—The frequency of divorce is of sociological interest. The effect of the Divorce Act (55 Vict., No. 37) of New South Wales, and of Victoria (53 Vict., No. 1056), which came into force on 6th August, 1892, and 13th May, 1890, respectively, have had a conspicuous influence in increasing its frequency. In the former State the frequency was more than quadrupled for about three years; in the latter it was tripled, as the result of the operation of these Acts. Table

LII. shews the frequency of divorce per 10,000,000, for the several States of the Australian Commonwealth for which they were available up to 1886, and for the whole Commonwealth from 1887 onward.

The populations up to 1886, used to compute the divorce-rate, correspond to the number of States for which the divorce results were available, and the number of divorces include the judicial separations. The results for the successive years are as follows :—

TABLE LII.—Relative Frequencies, per 10,000,000 population, of Divorces and Judicial Separations. Australia, 1874 to 1913.

Year of Decade.	Rates* in Decades.					Proportion† of Judicial Separations.	
	1870.	1880.	1890.	1900.	1910.	Period.	Proportion.
0	..	237	377	981	1,066	1874–1879	.020
1	..	179	594	1,052	1,154	1880–1884	.052
2	..	113	684	1,024	1,464	1885–1889	.062
3	..	274	1,293	909	1,347	1890–1894	.043
4	140	176	1,261	1,014	..	1895–1899	.038
5	220	269	1,194	862	..	1900–1904	.042
6	350	229	1,039	860	..	1905–1909	.043
7	210	205	1,113	854	..	1910–1913	.023
8	140	297	1,024	997	..	1874–1913	.0381
9	120	361	1,043	1,163	..	..	..

* Number per 10,000,000 of population. † Ratio of judicial separations to judicial separations and divorces together.

The total number of divorces and judicial separations were 10,194 and 404 respectively, the total thus being 10,598. The relative frequencies, tabulated above, are shown by the bottom curve in Fig. 56, viz., curve D. The proportions which judicial separations bear to the totals appear also in the table. Apparently divorce was increasing at first approximately at the rate 0.00000165 per unit of the population per annum, so that the number of divorces (V.) from 1781 to 1890 would be represented roughly by

$$(401)\ldots\ldots V = 0.00000165\, P\, (t - 1870),$$

in which formula t denotes the year for which the number is required, and P the population at the middle of the year.

The values according to this formula are denoted by the dotted line a b on Fig. 56. The relative frequency then rises in 3 years from, say 0.0000330 to the value 0.0001293 ; that is at the rate 0.0000321 per

person per annum — the line b c on the figure. The relief afforded through the change in the divorce acts, having apparently been secured in the short time mentioned, the relative frequency of divorce fell fairly regularly until about 1907, viz., at the rate of 0.00000333 per person per annum. Hence for this period the relative frequency is about

$$(402)\ldots\ldots V = - 0.00000333\ P\ (t - 1893).$$

This is the line c d on the graph. The relative frequency of divorce then rapidly increases to about 0.0000100 per person per annum. This is denoted by line d e on the graph.

11. **The abnormality of the divorce curve.**—Owing to the change in the divorce law being, as shewn, instantly followed by a large increase in the number of cases, the curve of frequency cannot be regarded as normal for the larger ages. For the purpose of estimating the rate of increase, previous to the legal change, the results for a few years before the change can be used. Similarly the results after the change can be carried backward to some common year in the changing period. This gives the following results :—

TABLE LIII.—Shewing Influence of Divorce Acts on Number of Divorces. Australia.

State.	Average Increase per Annum (Number).		Number as at Change per Year. (1892).		Factor of Increase
	Before Legal Change	After Legal Change	Before Change.	After Change.	
N.S. Wales ..	(1884-1891) 5.6	(1893-1895) 0.0	69.7*	306.7*	4.4
Victoria ..	(1881-1889) 1.9	(1891-1893) 0.0	32.5	91.7	2.8
Commonwealth	(1881-1888) 7.4	(1893-1907) 5.6	116.6	436.6	3.7

* Divorces and judicial separations together.

In view of the fact that, as shewn, the change consequent upon the operation of the Divorce Acts is very marked in the frequency of divorce between 1890 and 1893, say 21 to 18 years before the Census of 1911, and that there is a remarkable decrease in the proportion of " divorced" for

ages about 55, see the points marked a and a′ in Fig. 59 (which would correspond to ages of about 35 in the year 1891), it seems more than probable that the left-hand branch of the divorce curves belongs to the later, and the right-hand branch belongs to the earlier divorce regime. To obtain the true tendency to divorce according to age of the parties, these irregular frequencies would, of course, have to be eliminated. Hence it is desirable to include in the statistics of divorce the age of petitioners and respondents. See later.

12. **Desirable form of divorce statistics.**—From what has preceded, it is evident that for divorce statistics to be of high value from the standpoint of sociology, they should fulfil the following requirements, viz., they should include the numbers both of petitions for judicial separation and for divorce, and should shew for each :—(1) The date and the ground of the petition ; (2) The action resulting therefrom (granting, refusal, or other action), together with the date of such action ; (3) The date of birth both of petitioner and respondent ; (4) and the date of their marriage. Statistics so kept would furnish results showing frequency-according-to-age and age-differences and according to duration-of-marriage. The sociological value of such statistics is self-evident, for it would throw light upon the influence of age *per se*, of difference of age, and of duration of marriage, and thus would *expose the conditions which are of danger from the standpoint of social stability.*

13. **Frequency of marriages according to pairs of ages.**—The frequency of marriage according to pairs of ages can be well determined only for a considerable number of instances. For example, if assigned to groups, according to age last birthday, there are, between the ages 12 and 95 for brides, and 15 and 99 for bridegrooms, no less than 7140 groups. As for the last eight years the average number of marriages per annum was only 37,740, this gives a little over 5 per group on the average, a number insufficient to indicate the characteristics of the frequency. For this reason eight years marriages were taken, viz., 301,918, or the marriages of 603,836 persons, who were married during the years 1907 to 1914 inclusive. Of these marriages the ages of 57 brides were not stated, though the ages of the bridegrooms were given ; the ages of 19 bridegrooms were not furnished, though those of the brides were given ; and in 54 cases neither the age of bride or bridegroom was given. That is, there were 130 cases (or about 1 in 2322, or the 0.00043058th part) defective. These are disregarded.

For single year groups the numbers of marriages are shown in Table LIV.

TABLE LIV.—NUMBER OF MARRIAGES* ARRANGED ACCORDING

* The figures denote the num

AGES OF BRIDES.

Ages of Bride-grooms	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40
15	..	..	..	..	1	..	..	..	1	2	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..
16	..	..	2	2	8	13	8	6	3	4	3	..	..	1	..	1	..	..	..	..	..	..	..	..	..	..	..	..	..
17	..	..	..	12	44	59	51	30	16	10	8	5	..	1	3	..	..	..	..	..	..	..	..	..	..	..	..	..	..
18	..	1	3	28	113	233	309	195	112	78	58	30	12	11	6	5	2	6	..	1	..	..	1	1	..	..	..	..	..
19	..	1	2	38	208	426	778	740	395	327	194	90	47	42	24	7	11	4	5	2	4	3	1	..	1	..	..	1	
20	..	..	8	37	214	602	1033	1261	1075	891	489	320	160	124	176	42	28	29	17	3	5	7	5	3	3	2	..	2	..
21	..	..	10	65	362	1158	2076	2527	2301	3764	1845	1185	738	465	289	190	118	73	58	34	24	15	19	19	8	4	5	5	
22	..	..	16	68	266	870	1703	2342	2384	3960	3008	1869	1269	744	526	307	226	134	78	51	59	17	20	16	10	4	6	4	
23	..	2	6	40	233	742	1494	2152	2550	4114	3468	3128	2032	1247	835	513	309	231	179	82	84	63	30	30	20	13	22	11	
24	..	..	2	32	195	603	1170	1885	2267	3672	3302	3249	2989	1776	1311	769	545	395	228	132	94	61	72	41	34	27	21	15	
25	..	..	7	38	157	460	971	1458	1824	3170	2981	3006	2821	2664	1616	1089	832	450	360	206	162	78	70	75	35	26	25	15	1
26	..	..	1	25	158	405	875	1197	1627	2630	2555	2683	2540	2382	2200	1351	948	584	424	239	190	124	100	64	48	30	26	19	1
27	..	..	2	16	97	311	607	985	1238	2171	2050	2255	2141	2073	1844	1723	1098	730	490	298	253	154	123	86	55	76	35	24	2
28	..	..	6	19	80	247	567	775	1036	1858	1783	1873	1957	1736	1668	1532	1320	879	590	360	289	203	166	114	82	66	53	31	3
29	..	..	2	10	52	182	339	606	766	1364	1353	1431	1484	1457	1317	1242	1121	1037	709	409	314	260	168	132	88	66	60	38	2
30	..	..	1	7	46	133	287	439	609	1022	986	1088	1145	1172	1183	982	944	830	742	423	365	229	204	153	100	76	61	33	4
31	..	..	..	6	27	94	186	311	402	727	737	769	786	820	841	752	754	703	599	555	339	230	182	157	100	76	46	48	2
32	..	..	..	9	29	78	154	259	344	604	710	668	727	767	790	621	673	636	555	472	497	280	225	173	126	94	67	42	4
33	..	..	..	2	22	54	114	193	261	462	513	553	512	549	563	536	541	517	461	355	342	317	193	180	130	90	64	50	3
34	..	..	..	5	12	50	121	157	215	334	366	422	551	455	432	449	449	407	384	264	341	275	298	197	157	95	95	66	3
35	..	..	..	7	18	26	81	126	174	304	323	313	364	477	375	398	387	354	377	270	288	246	235	233	164	103	94	87	6
36	..	..	1	6	13	37	76	93	129	241	272	273	300	323	368	337	324	333	362	239	258	228	223	190	188	126	115	75	6
37	..	..	1	3	7	27	40	90	101	188	171	259	236	242	263	327	254	250	255	167	227	194	178	153	153	165	111	103	5
38	..	..	1	1	8	22	47	74	75	181	178	205	259	250	216	215	354	240	243	193	216	216	178	187	172	160	189	120	8
39	..	..	..	4	8	18	28	57	58	125	113	128	164	177	204	190	197	243	208	160	168	169	174	154	147	142	139	151	10
40	..	..	..	5	9	13	18	39	71	90	110	112	138	157	139	161	172	180	208	121	152	132	128	173	118	124	126	124	11
41	..	..	..	1	2	6	14	21	26	70	53	87	87	85	122	109	116	108	119	142	117	115	93	106	101	73	76	86	7
42	..	..	1	1	3	1	22	26	34	52	70	64	88	84	84	112	114	106	131	107	135	110	98	132	124	86	111	97	7
43	..	..	..	..	2	3	6	18	21	39	42	72	59	64	68	93	91	81	110	84	105	101	97	96	86	88	85	81	7
44	..	..	..	..	3	3	10	11	17	41	36	49	48	57	62	64	65	70	74	66	96	80	90	72	76	68	80	84	6
45	..	..	..	1	4	5	14	12	12	35	23	37	45	72	45	51	72	79	92	75	61	72	86	93	95	77	73	78	6
46	..	..	..	..	3	4	5	13	15	38	29	32	37	43	52	44	50	56	66	57	63	62	57	88	87	48	66	54	6
47	..	..	..	1	..	6	6	8	14	30	21	21	24	26	29	48	52	36	49	38	46	49	60	65	64	74	69	47	4
48	1	..	..	..	1	3	10	7	14	16	15	37	32	30	27	30	49	44	41	42	47	49	48	44	55	66	81	60	6
49	..	..	1	..	1	2	4	5	4	11	20	15	21	27	26	32	28	29	23	31	39	19	41	48	58	50	58	55	4
50	..	..	..	..	1	4	6	8	6	6	9	15	23	28	26	24	22	31	37	32	34	38	27	52	40	41	54	57	6
51	..	..	..	..	..	..	2	2	6	7	8	5	8	8	16	10	9	20	20	24	13	26	23	28	38	20	36	34	2
52	..	..	..	..	1	4	3	..	5	8	11	7	11	10	9	21	12	12	16	25	25	20	18	32	24	22	40	34	3
53	..	..	..	..	2	2	1	3	3	2	6	9	4	14	11	11	12	19	13	19	17	14	14	22	21	23	26	27	2
54	..	..	..	..	..	..	2	..	1	3	7	5	8	7	18	9	12	13	12	9	19	19	24	19	24	14	20	20	2
Ages.	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40
55	..	..	..	..	..	..	4	2	1	2	5	4	3	3	9	4	6	7	13	14	15	14	15	17	8	23	17	21	2
56	..	..	..	..	..	..	1	1	4	4	3	5	5	9	4	3	6	8	17	6	5	9	9	14	18	18	17	15	2
57	..	..	..	..	1	..	..	1	..	2	3	3	1	2	7	3	5	12	3	5	5	3	13	14	10	9	14	11	
58	..	..	..	..	..	..	..	..	2	3	4	5	2	4	4	7	7	5	6	4	6	8	6	10	13	11	11	13	1
59	..	..	..	..	..	..	..	1	2	..	1	4	5	2	4	2	5	1	5	4	5	5	2	2	6	7	12	3	
60	..	..	..	..	1	..	1	1	..	1	2	2	2	1	3	5	5	11	6	3	3	6	4	9	10	12	14	11	
61	..	..	..	..	..	1	..	..	1	..	1	4	1	2	..	2	6	..	2	1	4	2	3	3	1	7	10	10	
62	..	..	..	..	..	..	..	1	..	..	1	2	3	2	3	..	3	1	3	1	7	1	1	3	6	2	6	7	
63	..	..	..	..	..	..	..	..	1	..	..	1	3	3	2	3	3	2	5	2	4	2	7	6	4	3	4	4	
64	..	..	..	..	1	..	..	..	..	..	1	1	1	3	3	5	2	1	1	..	6	3	2	1	1	3	5	10	
65	..	..	..	..	..	..	..	..	1	1	..	..	1	1	..	..	2	1	3	3	2	..	8	3	2	8	4	4	
66	..	..	..	..	..	..	..	..	2	1	..	1	..	..	2	..	1	2	..	2	..	2	1	1	5	2	2	1	
67	..	..	..	..	..	..	..	1	..	1	..	..	3	..	1	..	1	..	1	3	1	3	2	1	2	3	1	5	
68	..	..	..	..	..	..	..	..	2	..	1	..	..	2	1	1	1	3	..	2	..	1	3	8	..	4	3	2	
69	..	..	..	..	..	..	..	..	..	..	..	1	..	..	1	..	3	1	2	2	..	2	..	2	2	2	3	1	
70	..	..	..	..	..	..	..	..	..	..	..	..	2	..	1	1	2	3	2	1	2	1	2	1	..	5	2	4	
71	..	..	..	..	..	..	..	..	..	..	..	..	..	1	..	..	..	1	..	1	1	..	1	..	1	2	2	1	
72	..	..	..	..	..	..	..	..	..	..	..	..	..	2	..	..	1	..	..	1	..	..	1	..	2	..	2	1	
73	..	..	..	..	..	..	..	..	..	1	..	..	2	..	1	..	..	..	..	..	..	2	..	1	..	..	1	1	
74	..	..	..	..	..	..	..	..	..	..	..	..	1	..	..	2	1	..	..	..	..	..	2	2	..	..	2	..	
75	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	1	..	1	1	..	..	..	..	..	
76	..	..	..	..	..	..	..	..	1	..	..	..	..	..	1	..	..	..	..	2	..	..	1	..	1	..	1	..	
77	..	..	..	..	..	..	..	..	1	..	..	..	..	..	..	..	..	1	..	..	..	..	..	..	..	1	..	..	
78	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	1	..	..	..	1	..	..	1	..
79	..	..	..	..	..	..	1	..	..	..	..	..	..	1	..	..	..	..	..	1	..	..	..	..	..	..	..	1	..
80	..	..	..	..	..	..	..	..	..	2	..	..	..	1	..	..	..	1	..	..	..	..	1	..	..	1	..	..	..
81	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..
82	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..
83	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	1	..	..
84	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	1	..	..	..	..	..	..	..	..	..
85	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..
86	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..
87	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	1	..	..	..	..	..	..	..	..	..	..	..	..
88	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..
89	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..
90	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..
95	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..
99	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..
N.S.	..	..	..	..	..	..	1	1	1	4	2	..	1	1	..	2	..	..	1	1	..	1	..	..	..	..	..	..	..
Total	1	4	73	489	2412	6907	13246	18140	20231	32673	27950	26402	23903	20707	17731	14440	12372	10010	8405	5848	5558	4341	3854	3521	2924	2438	2369	2005	168

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couples: not of persons.

AGES OF BRIDES.

43	44	45	46	47	48	49	50	51	52	53	54	55	56	57	58	59	60	61	62	63	64	65	66	67	68	69	70	71	72	73	74	75	76	77	78	79	80	81	82	83	84	95	Not stated	Total.	Ages of Bride-grooms
..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	4	15
..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	51	16
..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	239	17
..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	1205	18
..	1	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	3353	19
1	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	1	6438	20
3	1	3	..	..	..	..	1	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	1	17374	21
1	2	1	3	..	1	1	..	..	1	..	..	..	..	1	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	2	19977	22
3	1	1	1	1	2	1	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	2	23655	23
1	3	3	2	..	1	1	2	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	2	24918	24
5	2	7	1	2	2	2	1	..	..	..	1	..	..	..	1	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	2	24650	25
7	3	2	3	2	4	2	2	..	1	1	1	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	4	23494	26
9	4	2	8	3	1	2	2	..	..	2	1	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	2	21012	27
8	6	3	4	1	5	3	3	1	..	1	..	..	..	..	..	..	..	..	1	..	..	..	1	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	2	19384	28
7	15	6	4	3	2	4	1	..	1	..	1	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	4	16113	29
6	10	11	3	6	5	2	3	1	2	..	..	..	1	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	2	13392	30
10	6	10	2	3	5	1	1	..	1	1	..	..	..	1	..	..	..	..	1	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	2	10340	31
12	14	11	6	7	3	2	..	1	3	..	1	..	1	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	4	9745	32
19	10	7	9	3	1	..	4	2	1	..	..	..	..	1	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	1	7712	33
27	20	16	12	6	9	6	3	1	1	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	2	6796	34
30	21	23	7	8	6	3	7	5	..	..	1	1	..	..	..	..	1	..	..	..	..	..	..	..	1	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	2	6066	35
24	15	12	6	7	7	7	2	5	..	3	2	..	..	1	2	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	1	5345	36
27	18	20	8	10	12	4	3	2	1	..	..	3	1	1	1	..	..	..	..	1	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	2	4411	37
38	27	20	20	13	24	4	2	2	4	1	3	..	1	1	..	..	..	..	..	..	..	..	..	..	1	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	2	4530	38
37	35	40	21	14	8	9	11	2	4	3	2	2	..	..	1	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	2	3737	39
33	30	33	15	19	13	10	10	3	5	2	3	..	..	2	1	1	1	1	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	1	3252	40
62	28	31	23	18	13	6	6	6	3	3	1	..	..	2	2	..	1	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	2	2336	41
42	42	34	15	22	20	10	6	7	9	1	3	1	1	..	1	..	..	..	1	..	1	1	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	2	2437	42
73	46	29	19	26	14	12	8	4	3	2	..	5	2	2	..	1	1	..	..	..	1	1	..	..	..	1	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	1	2058	43
56	53	56	32	12	13	15	13	2	3	6	3	2	..	2	1	..	..	..	..	1	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	1	1745	44
78	61	58	33	27	13	20	19	4	5	5	7	6	2	2	..	..	..	1	1	1	..	1	2	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	1	1847	45
58	40	55	51	29	31	12	15	16	13	5	5	2	5	3	2	1	2	..	2	1	..	..	1	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	1575	46
50	49	49	42	34	27	10	10	7	12	5	8	4	3	2	3	1	3	..	..	..	2	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	1	1350	47
39	41	42	38	38	43	31	24	15	8	4	5	4	..	7	4	1	3	1	..	1	3	1	..	1	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	1	1363	48
42	36	41	37	40	23	35	30	18	18	7	4	7	7	4	2	2	7	2	4	..	2	..	..	2	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	1	1146	49
30	30	50	35	27	36	40	32	12	15	10	9	5	7	6	4	2	2	1	1	1	1	2	1	1	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	1113	50
27	31	29	18	29	29	32	24	16	16	14	9	2	8	3	6	1	3	3	..	1	..	..	1	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	748	51
23	40	27	34	35	29	22	25	20	24	8	9	3	9	1	3	2	4	1	1	..	..	2	2	..	..	..	1	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	795	52
37	25	18	18	19	23	15	17	13	19	15	12	9	7	6	2	1	1	3	3	2	..	1	2	..	1	1	..	..	..	1	..	..	..	..	..	..	..	..	..	..	..	..	..	647	53
24	30	21	26	21	26	16	19	17	19	16	16	7	10	4	8	1	4	1	2	1	1	4	..	1	..	1	..	..	..	..	..	1	..	..	..	..	..	..	..	..	..	..	..	623	54
43	44	45	46	47	48	49	50	51	52	53	54	55	56	57	58	59	60	61	62	63	64	65	66	67	68	69	70	71	72	73	74	75	76	77	78	79	80	81	82	83	84	95		Total.	Ages.
22	17	20	20	26	21	19	15	16	14	11	10	12	12	3	7	5	1	1	4	4	2	5	1	..	..	..	..	..	..	..	..	..	..	..	..	1	..	..	..	..	..	..	2	545	55
16	22	15	29	19	29	15	19	12	13	8	6	10	12	12	8	1	4	2	1	..	..	2	..	..	..	1	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	489	56
13	16	19	19	22	21	9	12	9	9	8	12	11	8	14	10	6	8	1	2	2	1	2	..	3	..	1	..	..	2	..	..	..	..	..	..	..	..	..	..	..	..	..	..	400	57
15	12	22	17	13	22	17	14	6	20	15	10	6	6	10	11	6	3	3	2	4	2	2	2	2	..	..	1	..	..	..	..	1	..	..	..	..	..	..	..	..	..	..	..	414	58
9	11	6	12	13	9	11	15	7	11	9	8	6	9	9	14	10	5	1	7	3	2	1	1	1	2	..	..	..	..	..	1	..	..	..	..	..	..	..	..	..	..	..	..	286	59
6	11	14	9	11	16	11	17	10	7	11	12	9	14	7	8	10	8	2	2	2	2	3	3	2	3	..	2	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	347	60
4	7	13	9	12	9	8	11	11	9	4	3	3	6	6	3	5	6	4	3	1	..	5	2	3	1	2	..	1	..	..	1	..	..	..	..	..	..	..	1	..	..	..	..	236	61
7	6	4	4	5	10	6	12	7	15	5	9	9	3	8	5	3	4	1	8	6	6	5	..	1	2	2	1	1	1	1	..	..	..	..	..	..	..	..	..	..	..	..	..	227	62
8	11	9	3	10	9	7	10	7	7	5	6	10	6	5	1	1	6	6	7	5	3	4	1	2	2	..	2	2	..	1	..	..	1	..	1	..	..	..	..	..	..	..	..	229	63
4	8	6	5	5	6	6	7	3	5	9	6	2	4	2	8	6	5	5	4	7	4	6	3	3	3	2	2	..	1	..	..	..	..	..	..	..	..	..	..	..	..	..	..	203	64
3	7	12	8	7	6	8	3	7	11	7	7	6	5	4	5	7	6	2	5	4	4	8	3	4	4	2	3	2	..	1	..	..	1	..	..	..	..	..	..	..	..	..	2	219	65
5	3	8	7	6	2	5	6	4	4	5	4	9	4	5	3	6	9	2	6	7	5	7	6	4	2	1	1	1	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	170	66
5	2	5	2	4	6	8	3	9	5	3	4	5	4	11	4	1	5	1	6	5	2	2	3	5	5	3	2	1	..	..	..	1	..	..	..	1	..	..	..	..	..	..	..	163	67
5	4	2	2	6	7	6	5	4	5	5	4	3	7	3	3	2	6	..	1	2	5	3	3	6	4	4	1	1	..	..	1	..	..	2	..	..	..	..	1	..	..	..	..	154	68
..	3	1	5	2	4	4	6	1	4	2	4	2	5	..	7	2	6	2	3	6	6	4	2	4	4	4	1	2	1	..	1	1	..	..	..	1	..	..	..	..	..	..	..	122	69
1	1	2	3	4	6	..	5	3	4	6	4	2	6	..	3	1	5	3	3	4	5	3	4	1	1	2	4	1	..	..	..	2	..	..	..	..	..	..	..	..	..	..	..	134	70
..	..	4	1	..	2	..	2	5	6	1	2	1	2	2	1	2	3	..	1	..	1	2	1	1	2	2	5	1	1	..	..	..	..	1	..	..	..	..	..	..	..	..	..	70	71
1	2	2	..	..	3	4	3	..	2	1	..	2	1	..	1	2	1	1	4	3	..	4	2	..	3	2	3	..	1	2	1	..	..	..	2	..	..	..	..	..	..	..	..	70	72
..	2	2	..	1	1	1	1	5	2	1	6	..	2	2	2	1	3	1	2	1	4	7	3	2	5	3	1	1	1	..	..	..	..	..	..	..	..	..	..	..	..	..	..	77	73
..	1	3	..	2	3	3	1	1	1	2	1	2	..	3	1	2	2	1	..	1	1	3	..	..	2	4	1	1	3	1	..	..	..	..	1	..	..	..	..	..	..	..	..	64	74
..	..	1	..	1	2	..	1	1	..	..	1	2	3	4	2	4	1	..	2	1	1	4	1	2	..	3	..	..	3	2	2	3	1	..	..	..	..	..	..	..	..	..	..	53	75
..	..	1	..	..	1	..	..	1	..	..	..	1	1	1	2	..	..	2	..	1	..	..	3	2	..	..	..	1	3	1	..	1	..	..	1	..	..	..	..	1	..	..	..	33	76
1	1	..	..	4	..	1	2	2	2	3	..	1	1	1	1	..	..	..	1	1	..	1	..	..	1	1	1	..	..	2	..	..	..	..	..	..	..	..	..	..	..	..	..	33	77
..	1	..	1	..	1	..	..	..	..	1	..	..	..	1	1	1	..	..	1	..	..	1	1	..	..	..	2	1	1	2	..	..	..	..	1	..	1	..	1	..	..	..	..	22	78
1	..	..	..	..	..	..	1	..	1	..	1	..	1	..	3	..	1	1	..	..	1	..	..	..	1	..	..	1	2	..	1	..	1	..	..	..	1	..	..	..	..	..	..	23	79
..	4	..	..	3	..	..	1	..	..	..	..	1	1	..	1	..	..	..	1	..	..	..	1	..	1	..	2	2	1	..	1	..	..	1	..	..	..	..	..	..	..	..	..	28	80
..	..	..	..	..	1	1	..	1	..	..	1	..	..	..	..	..	3	..	..	..	..	..	..	..	..	1	..	..	..	..	..	..	..	1	..	..	..	..	..	..	..	..	..	10	81
..	..	1	..	1	..	1	..	..	..	..	1	1	..	..	..	..	1	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	1	..	..	..	..	..	..	..	..	..	9	82
..	..	1	..	..	..	..	..	..	1	..	..	..	..	..	..	..	..	1	..	1	1	..	..	1	..	..	..	..	..	..	..	..	..	..	..	1	..	..	..	..	..	..	..	8	83
..	..	1	..	..	..	..	1	..	1	..	..	..	..	..	..	..	1	..	..	..	..	..	..	..	..	..	..	..	..	..	2	..	..	..	..	..	..	..	..	..	..	..	..	7	84
..	..		..	1	..	..	1	..	..	..	..	..	1	..	..	..	..	..	..	..	..	..	..	..	..	..	..	1	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	6	85
..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	1	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	1	..	..	..	1	..	3	86
..	..	..	..	..	..	1	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	1	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	3	87
..	..	..	..	..	1	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	1	88
..	..	..	..	..	..	1	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	1	89
..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	90
..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	1	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	1	95
..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	1	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	1	99
..	..	1	..	..	..	..	1	..	..	1	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	64	73	N.S.
1065	948	946	712	663	649	493	481	308	351	238	229	179	189	165	159	98	136	58	93	81	68	95	58	54	51	43	36	21	21	14	11	10	4	6	6	4	2	1	3	1	..	1	111	301918	Total

This table exhibits the various irregularities in the data. The numbers are not quite trustworthy about the ages 21, for reasons which will appear later, as it is certain that in some cases misstatements are made by persons marrying under that age. This table is suitable for the analysis of the frequency at the lower groups of ages only. For the analysis of the frequency at the more advanced age groups, a second table of five-year groups has been prepared. (Table hereinafter).

The frequencies exhibited by this large group of marriages can, without sensible error, be referred to the beginning of the year 1911 (*i.e.*, to 1911.0), as the moment which they can be regarded as true, and from which any secular change may be reckoned, or they may be regarded as contemporaneous with the Census of 3rd April, 1911.

14. **Numbers corresponding to given differences of age.**—The mode of tabulation in Table LIV., though satisfactory in respect of shewing the grouping according to age-groups for single years, is by no means perfectly satisfactory for the purpose of very accurately determining the frequency of conjugal-groups according to various *differences of age*. It is obvious that when all bridegrooms, whose age was say x last birthday, and brides whose age was say y last birthday (x and y being integers), are grouped, the group contains brides who are one-half year older than the difference $x—y$, as well as brides one-half year younger than this difference. This can be readily seen from the nature of the table itself. To obtain some rough idea of the defect of such a mode of grouping, we may first divide the numbers (having regard to second differences) into four parts, so as to get the probable numbers attributable to each half of the age-period analysed. These quarter (or half-year) groups, however, will evidently not agree with what would have been given by an original compilation into half-year groups, for the reason indicated above; this will appear more clearly hereinafter.

To properly determine the law of nuptial frequency according to specified differences of age the only perfectly satisfactory compilation would be one in which, for small age-groups of bridegrooms (say) the tabulation was according to a series of increasing age-differences (of the age of the bride), positive and negative, and (for complete analysis) a similar tabulation for small ranges of the age of the brides, with a series of increasing differences, positive and negative, of the age of the bridegroom. These two tabulations *would not give identical results*, but if the age-groups were small, they would be approximately identical. The data of the table are nevertheless of value, and give a result which is of high precision in regard to the characteristic features of the surface representing the relative frequency of marriages for given pairs of ages.

The results given in Table LIV. are for 301,918 marriages occurring in Australia during eight years, and are drawn from populations (mean annual), which aggregated to nearly 36 millions. The marriage rates were thus as shewn in Table LV., p. 193.

TABLE LV.—Marriage Rates, Australia, Total Period, 1907-1914.

Males ..	18,614,557	Females	17,206,457	Persons	35,821,014
Rates, Males	0.0162195	,,	0.0175468	,,	0.0168570

These rates may consequently be regarded as representing the probability of a marriage occurring in a population of males, females, or persons, constituted as the average for the eight years, 1907 to 1914, both inclusive, in Australia. The probability of a marriage occurring among the never-married, the widowed, and the divorced, cannot be so well ascertained.

By excluding the unspecified, the probability of marriage for any pair of ages can be ascertained roughly by dividing the numbers in Table LIV. by 301,864 ; the quotient is the *chance* of the marriage occurring in the group of the pair of ages in question, provided that the proportions to the whole population of the males and females in each group is unchanged. Denoting this probability by p_{xy}, the marriage-rate by r_m, and the population by P, the number of marriages, N_{xy}, to be expected of bridegrooms whose age last birthday was x, with brides whose age last birthday was y, is :—

$$(403) \ldots N_{xy} = P\, r_m\, p_{xy}\ ;\quad N_{xy} = P' r'_m\, p_{xy}\ ;\quad N_{xy} = P'' r''_m\, p_{xy}\ ;$$

P, P' and P'' denoting the population of persons, males, and females, respectively ; and r_m, r'_m and r''_m similarly denoting the marriage rates based upon persons, males, or females, respectively. The numbers of the table would roughly give the chance according to "alleged age," not according to "actual age" unless the alleged is also the actual age. We shall proceed to examine this question.

15. **Errors in the ages at marriage.**—Before analysing the data giving the protogamic surface, it is desirable to determine the error of statement at ages earlier than 21. Here it may be mentioned[1] that the curves of apparent frequency of birth at different ages from say 17 to 22 shew that the numbers are doubtless erroneous. The same fact is suggested by the peculiar irregularities in the numbers graphed in Fig. 60, which shews the numbers of brides and bridegrooms at all ages ; see curve A in the figure shewing the result for brides and curve B shewing that for bridegrooms. The explanation is unquestionably that the group "21 years last birthday" contains a number of persons whose real age was 18, 19 or 20, or possibly even younger than 18. From an investigation of birth-frequency during the seven years, 1908 to 1914, both inclusive, it was found that the numbers given at ages 18 to 21 needed to be multiplied by the factors 1.05701, 1.07918, 1.17022, and 0.82704 respectively. (This applies to females only. There is doubtless also an error for males). Correcting these factors so as to obtain the same totals, the figures in line (4) below are obtained ; these are the probable correcting factors to be applied to the

[1] The matter is dealt with fully hereinafter.

numbers furnished directly. That is let M' be the true number of marriages for brides of any given age, and let M be the alleged number: then m being the factor of correction, we shall have :—

$$(404) \ldots\ldots M' = m\,M,$$

hence, if the error occur solely through misstatements by persons of 18, 19, 20, and 21 years of age we should have, for each age of bridegroom, to form corrections of the type :—

$$(405)\,..\,(M'_{18}+M'_{19}+M'_{20}+M'_{21}) = (m_{18}M_{18}+m_{19}M_{19}+m_{20}M_{20}+m_{21}M_{21})$$

This would be the appropriate scheme of correction[1] if corrections for only one sex were needed. The result would then be as follows in Table LVI. hereunder :—

TABLE LVI.—Correction of Numbers of Brides of Alleged Ages, 18 to 21. Australia, 1908-1914.

(1) Age of Bride ..	18.	19.	20.	21.	18–21.
(2) Number of Brides ..	13,246	18,140	20,231	32,673	Total, 84,290
(3) Ratio to Total for Ages 18-21 ..	0.1572	0.2152	0.2400	0.3876	Total, 1 0000
(4) Factor of Correction	1.0572	1.0794	1.1704	0.8272	Mean, 1.03355
(5) Product of (3) & (4)	0.16619	0.23229	0.28090	0.32062	Total, 1.00000

These figures imply that there are 5.72 per cent. more brides of 18, 7.94 per cent. more brides of 19, 17.04 per cent. brides of 20, and 17.28 per cent. less of brides of 21 than admit that they are the ages in question.

Misstatement of Ages.

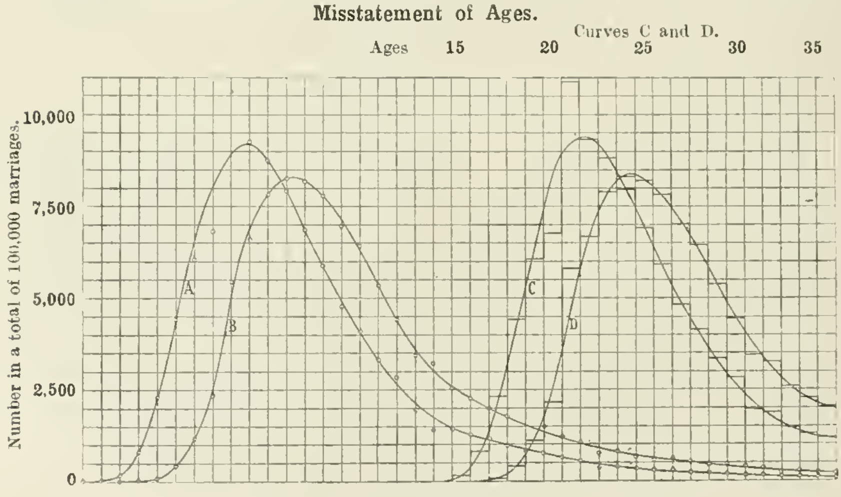

Fig. 60. Fig. 60a.

The curves A and B denote the number out of a total of 100,000 marriages of brides and bridegrooms respectively; married at given ages. The dots and circles represent the original data, the curves themselves being the smoothed result.

Curves C. and D.—The areas of the rectangles shew the numbers of brides and bridegrooms, respectively, married at the given alleged ages. The true numbers are the areas to the curves, which furnish the smoothed results.

[1] If in any example the result needed a small correction to balance, it should be made proportional to these last $m\,M$-quantities.

An attempt has also been made to ascertain, by smoothing, the probable misstatement on the part of bridegrooms as well as on that of brides. For the sake of comparison the factors for converting the crude data into the smoothed results are given for both bridegrooms and brides, and for males and females from the smoothing of the results of the 1911 Census. The actual smoothing and its effect is shewn on Fig. 60a, see curves C and D, the former being the curve for brides, the latter that for bridegrooms. The areas to the curves give the smoothed results, the areas of the rectangles themselves shew the crude data. In this way the results (1) and (2) are obtained.

TABLE LVII.—Correction-Factors for Males and Females of Alleged Ages, 18 to 21. Australia, 1911.

Factor of Correction for—	How Obtained.	18.	19.	20.	21.
(1) Males ..	Smoothing of Curve shewing Number of Bridegrooms	1.211	1.137	1.262	0.831
(2) Females	Smoothing of Curve shewing Number of Brides ..	0.962	1.054	1.228	0.844
(3) Females	Smoothing of Fecundity Curves	1.0572	1.0794	1.1704	0.8272
(4) Females	Mean of (2) and (3) ..	1.010	1.067	1.199	0.836
(5) Males ..	Smoothing of Census of Population, 1911 ..	0.9843	1.0273	0.9955	1.0283
(6) Females	Smoothing of Census of Population	0.9924	1.0217	0.9902	1.0504

The indications from the smoothing of the number of brides, with those from the smoothing of the fecundity curves (see later) are in substantial agreement, so far as the ages of 19, 20, and 21 are concerned; see lines (2) and (3) in the table above. It will be observed, however, that they are not in agreement with the Census deduction. An agreement was not, however, to be expected in the latter case, for the misstatements occur in regard to the age at marriage, an occasion on which there is not infrequently a motive for the misstatement.[1]

16. **Adjustment numbers for ages 18 to 21 inclusive.**—The actual adjustment of a table of numbers according to pairs of ages, however, involves the deduction of a number of brides and bridegrooms, which shall be equal for each group. It is evident that, inasmuch as the factors for the two are disparate, different results are obtained if we first correct by the factors for one sex and then by those of the other, or correct independently and take means, etc. For this reason the following method, though not ideally satisfactory, was adopted.

Denoting the correction-factor for bridegrooms (males) of age x by m_x, and that for brides (females) of age y by f_y, the composite factor (μ)

[1] Chiefly, but not wholly, owing to the attempt, by persons under 21 years of age, to avoid the legal requirements.

for the group of brides and bridegrooms of the respective ages, may be taken as :—

$$(406)\ldots\ldots\mu_{xy} = \sqrt{(m_x f_y)}.$$

that is, it is regarded as the geometric mean of the two. If we decide to make the totals of the groups 18 to 21 unchanged, we shall have to apply a small correction to these factors. Let g_{xy} denote a group of marriages for the ages in question. If the sum of the products μg be equal to the sum of the original groups, no correction will be required. If it be not equal, then the correction can be distributed in the ratio of the groups themselves. That is, ξ denoting the correction, the new values (g') of the groups will become :—

$$(407)\ldots\ldots g' = g + \xi = g\ \{1 + (G - \Sigma\,\mu g)\ /\ G\}$$

G denoting the sum of the groups, that is to say, $G = \Sigma g$. This method of correcting leaves the entire aggregate unaffected, though it adjusts its component groups. The results are shewn in the table hereunder. The ξ correction necessary was very small, amounting to only 18 in 17,862. See Table LVIII.

TABLE LVIII.—Correction of Numbers of Marriages for Ages 18, 19, 20, 21. Australia, 1907 to 1914.

	Crude Results.					Factors of Correction.					Corrected Results.				
	18	19	20	21	Totals.		18	19	20	21	18	19	20	21	Totals
						Males	1.211	1.211	1.211	1.211					
18	309	195	112	78	694	Females	1.010	1.067	1.199	0.836	343	223	136	79	781
						Means	**1.1059**	**1.1367**	**1.2049**	**1.0062**					
						Males	1.137	1.137	1.137	1.137					
19	778	740	395	327	2,240	Females	1.010	1.067	1.199	0.836	837	819	463	320	2,439
						Means	**1.0716**	**1.1015**	**1.1676**	**0.9750**					
						Males	1.262	1.262	1.262	1.262					
20	1,033	1,261	1,075	891	4,260	Females	1.010	1.067	1.199	0.836	1,171	1,469	1,328	919	4,887
						Means	**1 1290**	**1.1604**	**1.2301**	**1.0271**					
						Males	0.831	0.831	0.831	0.831					
21	2,076	2,527	2,301	3,764	10,668	Females	1.010	1.067	1.199	0.836	1,910	2,389	2,306	3,150	9,755
						Means	**0.9161**	**0.9416**	**0.9982**	**0.8335**					
T'tls	4,196	4,723	3,883	5,060	17,862						4,261	4,900	4,233	4,468	17,862

The effect at the dividing ages of this regrouping is to change the groups $\frac{2{,}022}{6{,}897} \Big| \frac{912}{8{,}031}$ into $\frac{2{,}222}{6{,}939} \Big| \frac{998}{7{,}703}$; hence the five-year groups $\frac{3{,}302}{23{,}130} \Big| \frac{1{,}395}{56{,}029}$ become $\frac{3{,}502}{23{,}172} \Big| \frac{1{,}481}{55{,}701}$. The totals for brides require that the original figures in Table LIV. should be corrected by + 65, + 177, + 350, and − 592, and the totals for bridegrooms corrected by + 87, + 199, + 627 and − 913.

TABLE LIX.—Shewing the Number per 100,000 Bridegrooms, and per 100,000 Brides Married at Given Ages. Australia, 1907-1914.†

Age.	Crude Results.		Adjusted Results.	
	Bridegrooms.	Brides.	Bridegrooms.	Brides.
(i.)	(ii.)	(iii.)	(iv.)	(v.)
12 ..	..	1	0.0	0.5
13 ..	..	1	0.1	1.5
14 ..	..	24	0.2	24
12–14 ..	..	26	0.3	26
15 ..	1	162	0.8	162
16 ..	17	799	16.9	799
17 ..	79	2,288	79	2,288
18 ..	428*	4,409*	428	4,409
19 ..	1,176*	6,067*	1,176	6,600
15–19 ..	1,701	13,725	1,700.7	14,253
20 ..	2,340*	6,817*	2,542	8,020
21 ..	5,452*	10,626*	4,997	8,920
22 ..	6,615	9,257	6,868	9,200
23 ..	7,834	8,745	7,834	8,745
24 ..	8,253	7,917	8,253	7,917
20–24 .	30,940	43,362	30,494	42,802
25 ..	8,165	6,858	8,190	6,819
26 ..	7,782	5,873	7,782	5,843
27 ..	6,960	4,783	7,120	4,897
28 ..	6,420	4,098	6,290	4,078
29 ..	5,337	3,315	5,337	3,297
25–29 ..	34,664	24,927	34,719	24,934
30 ..	4,436	2,784	4,383	2,670
31 ..	3,428	1,937	3,603	2,155
32 ..	3,228	1,841	3,003	1,760
33 ..	2,554	1,438	2,603	1,470
34 ..	2,251	1,277	2,278	1,260
30–34 ..	15,897	9,277	15,870	9,315
35 ..	2,009	1,166	1,995	1,143
36 ..	1,770	968	1,748	1,003
37 ..	1,461	808	1,533	873
38 ..	1,501	785	1,346	753
39 ..	1,238	664	1,183	643
35–39 ..	7,979	4,391	7,805	4,415
40 ..	1,077	560	1,040	547
41 ..	774	373	912	465
42 ..	807	426	800	397
43 ..	682	353	713	343
44 ..	580	314	649	303
40–44 ..	3,920	2,226	4,114	2,055
45 ..	612	313	589	271
46 ..	522	236	527	241
47 ..	447	220	468	213
48 ..	452	215	413	187
49 ..	380	163	363	163
45–49	2,413	1,147	2,360	1,075
50 ..	369	159	319	138
51 ..	218	102	282	118
52 ..	263	116	251	102
53 ..	214	79	225	90
54 ..	206	76	203	80
50–54 ..	1,300	532	1,280	528

Age.	Crude Results.		Adjusted Results.	
	Bridegrooms.	Brides.	Bridegrooms.	Brides.
(i.)	(ii.)	(iii.)	(iv.)	(v.)
55 ..	181	59	184	72
56 ..	162	63	167	66
57 ..	133	55	151	60
58 ..	137	53	136	54
59 ..	95	32	122	48
55–59 ..	708	262	760	300
60 ..	115	45	109	42
61 ..	78	19	97	36
62 ..	75	31	86	31
63 ..	76	27	76	27
64 ..	67	23	68	24
60–64	411	178	436	160
65 ..	73	32	60.0	21.8
66 ..	56	19	53.0	19.5
67 ..	54	18	47.0	17.1
68 ..	51	17	42.0	15.0
69 ..	41	14	37.5	12.6
65–69 ..	275	100	239.5	86.0
70 ..	44	19	34.0	10.5
71 ..	23	7	31.0	8.5
72 ..	23	7	28.0	6.7
73 ..	26	5	25.0	5.1
74 ..	21	4	21.5	3.6
70–74 ..	137	35	139.5	34.6
75 ..	17	3	17.2	2.8
76 ..	11	1	13.6	2.1
77 ..	11	2	10.9	1.6
78 ..	7	2	8.7	1.2
79 ..	7	1	7.0	0.9
75–79 ..	53	9	57.4	8.6
80 ..	9	1	5.6	0.7
81 ..	3	0	4.5	0.6
82 ..	3	1	3.6	0.5
83 ..	3	0	2.9	0.4
84 ..	2	0	2.3	0.3
80–84	20	2	18.9	2.5
85 ..	2	0	1.8	0.2
86 ..	1	0	1.4	0.1
87 ..	1	0	1.0	0.0
88 ..	0	0	0.6	0.0
89 ..	0	0	0.3	0.0
85–89	4	0	5.15	0.3
90 ..	0	0	0.25	
91 ..	0	0	0.15	
92 ..	0	0	0.1	
93 ..	0	0	0.05	
94 ..	0	0	0.0	
90–94 ..	0	0	0.55	
Unspecified ..	24	37	Nil	Nil

* These have been partially corrected for misstatement of age. † See Section 17, hereinafter, for description of Table.

17. **Probability of marriage of bride or bridegroom of a given age, to a bridegroom or bride of any (unspecified) age.**—The correction of the data, as indicated in the preceding section, admits of the construction of a table shewing in say 100,000 marriages the number occurring for bridegrooms of any given ages, and for brides of any given ages, the age of the other partner to the union being unspecified. In columns (ii.) and (iii.) of Table LIX., hereinbefore, the data are given the corrections referred to having been applied: columns (iv.) and (v.) are the smoothed results. The original data are shewn by dots on Fig. 60, the smoothed results by the curve, the ordinates to which represent throughout the probability of a marriage occurring within one half-year either side of any given age: that is, they are the values of the integrals:—

$$K\int_{x-\frac{1}{2}}^{x+\frac{1}{2}} x\,d\,x \quad \text{and} \quad K'\int_{y-\frac{1}{2}}^{y+\frac{1}{2}} y\,d\,y\,;$$

see section 19 hereinafter.

18. **Tabulation in 5-year groups.**—So small a number as 300,000 does not give sufficient data for the determination of the averages for single years, at the higher ages. Before 25 is reached over one-fourth of the marriages have been consummated, and before 30, over two-thirds (exactly 0.277921, and 0.691744 respectively). This leaves for groups of over 30 years of age only about 93,069 among 6500 groups or an average of about 14 per group. It is thus necessary to form 5-year groups. These are shewn in Table LX. hereunder. The corrections, referred to in last section, change these numbers as follows:—

Original Data.			Adjusted Data.		
3,302	1,395	4,852	3,502	1,481	5,138
23,130	56,029	92,354	23,172	55,701	92.068
41,193	13,1151	Totals.	41,135	130,909	Totals.

The numbers given in the table itself are the uncorrected data. It will be seen that they are still small for the higher ages. To determine the critical features of the surface representing the frequency of marriage both Tables LIV. and LX. are required. Were these two tables smoothed they would give the probabilities of a marriage occurring within the year groups of specified ages or specified quinquennia. None of the groups is perfectly regular, but the greater regularity of the larger groups exists only for a limited range of years. The matter will be dealt with more fully hereinafter, viz., in § 23.

TABLE LX.—Number of Marriages Arranged According to Age at Marriage in Five Year Groups. Australia, 1907-14.

Bridegrooms' Ages.	Brides' Ages. 10 to 14.	15 to 19.†	20 to 24.†	25 to 29.	30 to 34.	35 to 39.	40 to 44.	45 to 49.	50 to 54.	55 to 59.	60 to 64.	65 to 69.	70 to 74.	75 to 79.	80 to 84.	Total,* 10 to 84.	Ratio Bridegrooms to Total.
15–19† ..	9	!3,302	1,395	124	17	3	2	..	..	..	..	..	..	..	..	4,852	1,608
20–24† ..	**44**	**23,130**	**!56,029**	11,302	1,437	325	60	22	4	1	..	..	..	..	..	92,354	30,602
25–29 ..	18	10,637	!50,597	**34,896**	6,739	1,369	282	78	20	1	1	1	..	..	..	**104,639**	34,673
30–34 ..	1	2,795	15,513	!17,366	**9,130**	2,476	525	146	26	4	1	..	..	..	..	47,983	15,900
35–39 ..	3	917	5,134	!7,298	5,672	**3,621**	1,038	!313	65	15	2	2	..	..	..	24,080	7,979
40–44 ..	1	237	1,576	2,564	!2,811	2,473	**1,502**	510	112	26	8	1	..	..	..	11,821	3,917
45–49 ..	2	115	598	1,077	1,313	!1,653	1,279	**859**	263	74	36	8	..	..	..	7,277	2,411
50–54 ..	..	41	183	384	538	!768	!754	675	**406**	117	37	20	2	1	..	3,926	1,301
55–59 ..	..	11	73	129	197	313	360	!445	289	**218**	65	26	4	2	..	2,132	706
60–64 ..	..	6	28	71	79	152	162	!207	!208	144	**106**	60	10	2	1	1,242	412
65–69 ..	..	1	15	24	43	66	80	!133	122	113	105	**97**	19	7	1	826	274
70–74 ..	..	..	6	16	17	30	50	47	!65	41	50	59	**28**	6	..	415	138
75–79 ..	..	1	2	3	8	6	11	13	17	!31	14	21	25	**8**	**4**	164	54
80–84 ..	..	..	2	2	2	2	8	!10	7	4	!9	4	!8	**4**	..	62	21
85–90 ..	..	..	..	1	..	..	1	!4	1	1	1	1	1	..	1	12	4
Total*	78	41,193	131,151	75,257	28,003	13,257	6,114	3,462	1,605	790	435	300	103	30	7	301,785	100,000
Ratio of Brides to Total	26	13,650	43,459	24,937	9,279	4,393	2,026	1,147	532	262	144	99	34	10	2	100,000	.3313617‡

* Brides over 85 and bridegrooms over 95, and unspecified cases are omitted. The heavy faced type denotes the maximum on the vertical lines; the mark of exclamation (!) denotes the maximum on the horizontal lines.

† The values corrected for misstatement of ages, 18, 19, 20, and 21 give the following results:—For 3,302 and 1,395, 3,502 and 1,481; and for 23,130 and 56,029, 23,172 and 55,701. In the totals 41,193 and 131,151 become 41,435 and 130,909; and 4,852, and 92,354 become 5,138 and 92,068. The ratios 13,650 and 43,459 become 13,730 and 43,378; and 1,608 and 30,602 become 1,703 and 30,508.

‡ Factor of reduction to 100,000.

19. **Frequency of marriage according to age representable by a system of curved lines.**—Frequency according to pairs of ages (bride and bridegroom) can best be represented by a surface, the vertical height of which, above a reference plane, is the frequency for any pair of ages denoted by x, y co-ordinates. The numbers marrying in any given period, whose ages range between $x - \frac{1}{2}k$ and $x + \frac{1}{2}k$ (for bridegrooms), and between $y - \frac{1}{2}k$ and $y + \frac{1}{2}k$ (for brides), as ordinarily furnished by the data, are denoted by Z, the height of the parallelepiped. This frequency may, of course, be expressed as for the *exact age*, or it may be for the *age-groups*. When k is not infinitesimally small, the difference between the two is *sensible* and *important*. We shall assume for the present that the frequency varies only with age (not with time). The exact (instantaneous) age-frequency denotes the frequency which would exist if the persons were all of the exact age (x) in question, instead of being of various ages between $x - \frac{1}{2}k$ and $x + \frac{1}{2}k$. The age-group frequency denotes the frequency with the ages distributed between the limits referred to. For most practical purposes the latter is the more important. Suppose the exact frequency, z, for the population P, to be :—

$$(408)\ldots\ldots \frac{z}{P} = F(x \,.\, y)$$

then we shall have for any group-value :—

$$(409)\ldots\ldots Z = P \int\int F(x.y)\, dx\, dy$$

The group-values usually furnished are for single-year groups, hence the limits of the integral are $x \pm \frac{1}{2}$, $y \pm \frac{1}{2}$. It may sometimes be more convenient to use a series of functions of the form :—

$$(410)\ldots\ldots \frac{Z}{P} = F_Y(x)\,;\ \text{or } F_X(y)$$

in which case the fixed value of Y or of X will be the middle of the range $y \pm \frac{1}{2}$, or $x \pm \frac{1}{2}$. Then we shall have :—

$$(411)\ldots\ldots Z = P \int F_Y(x)\,dx\,;\ \text{or} = P \int F_X(y)\,dy$$

These last expressions, with fixed values either of Y or of X, are thus appropriate for representing the vertical or horizontal columns of figures in Tables LIV. and LX. by means of equations. For the vertical columns the abscissa is x, the age of the bridegrooms ; for the horizontal columns the abscissa is y, the age of the brides ; and the constants of the equations relate only to a particular range of y in the first case, and of x in the second, as many equations being required as there are ranges taken. We consider the matter more fully in a later section. This scheme of representation is practically more convenient than a more generalised system, it shews for each age of bridegroom (or of bride) the frequency of marriage with a bride (or a bridegroom) of a given age. (See part V., § 10, formulæ 211 to 216.)

20. **The error of adopting a middle value of a range.**—In dealing with group-ranges, in the manner referred to in the preceding section, the results are not strictly attributable to the middle age of the range, nor is the error of such an attribution by any means always wholly negligible. The function represents the value of a *range of values* of the argument, *i.e.*, for example, *all* bridegrooms whose age last birthday was x, x being an integer, or the group of bridegrooms whose age last birthday was say, between 20 and 24, etc. Suppose, for example, that the progression of a series of numbers, representing numbers at successive ages is approximately :—

$$(412)\ldots\ldots y = a + mx\,;\ \text{so that } xy = ax + mx^2\,;$$

then the true value of the product of the numbers into the ages is given by the integral :—

$$(413)\ldots\int_x^{x+1} xy\,dx = a\,(x + \tfrac{1}{2}) + m\,(x^2 + x + \tfrac{1}{3})$$

Consequently where we require the weighted mean-age, it is necessary to compare this value with that arising from the supposition that all may be regarded as of age $x + \frac{1}{2}$. If we make this last assumption,then we should have for the product of the numbers into the age supposed common to all,

$$(414)\ldots\ldots a\,(x + \tfrac{1}{2}) + m\,(x^2 + x + \tfrac{1}{4}).$$

The former expression is algebraically greater than this latter one by the difference of $m/3$ and $m/4$, that is $m/12$, which is sensibly equivalent to a

shift (ϵ) of the central position of the amount $m/12y$. Thus, instead of the central value of the range of ages we should take the "weighted mean" xa, which is given by :—

$$(415) \ldots\ldots x_a = x + \tfrac{1}{2} + \epsilon = x + \tfrac{1}{2} + \frac{m}{12y}$$

In applying this we may take m as indicated by the mean of the differences of the groups adjoining on either side. Thus if the groups for the ages 20 (and less than 21), 21, and 22 were respectively 76,132, and 224, then, instead of taking 21.5 as the mean age-value, *i.e.*, the middle age of the range 21 (which include everyone whose age last birthday was 21), we could take m as the mean of 132–76 and 224–132, that is, $m = \frac{1}{2}(56 + 92)$; or, as is obvious, $\frac{1}{2}(224–76)$, *i.e.*, 74. Consequently by the rule above, viz. (415), we have $x_a = 21.5 + 74 / (12 \times 132) = 21.54\dot{6}7\dot{1}$.

A curve which would give the group-results indicated is $60 + 20\xi + 18\xi^2$, the origin of abscissæ being $x = 20$, so that $\xi = 1$ for $x = 21$, and so on. The integral of the curve is $60\,\xi + 10\,\xi^2 + 6\xi^3$. If we put $\xi = x - 20$ we obtain the curve $y = 6860 - 700\,x + 18x^2$ with the origin at $x = 0$, hence the integral between the limits $x = 21$ and $x = 22$ is $3430\,x^2 - 233\frac{1}{3}\,x^3 + 4\frac{1}{2}\,x^4$, which gives the result $2844\frac{1}{6}$ as the sum of the xy products. Dividing this by 132, the number in the group, the average age is found to be $21.54\dot{6}7\dot{1}$ as before. Let three successive groups for equal ranges of the variable be denoted by A, M, and B; and let x_m be the middle point on the range of abscissæ of the middle group, M; then the mean value required (*i.e.*, in the case under review, the average age of the persons in the group) is :—

$$(416) \ldots\ldots x_a = x_m + \frac{1}{24}\,k\,\frac{B - A}{M}$$

in which k is the range of the variable common to the three groups. If the curve of instantaneous values be of the second degree, this last formula is rigorously accurate. By means of it, the average values can, as a rule, be written in by inspection, and it can be ascertained where the correction $\epsilon = \frac{1}{24}\,k\,(B - A)\,/\,M$ is sufficiently large to be taken into account.

21. **General theory of protogamic and gamic surfaces.**—The ages of husbands being adopted as abscissæ, and those of wives as ordinates, the infinitesimal number dM in an infinitesimal group of married couples, consisting of husbands, whose ages lie between x and $x + dx$, and their wives, whose ages lie between y and $y + dy$, will be :—

$$(417) \ldots\ldots dM = Z\,dx\,dy = kF\,(x, y)\,dx\,dy.$$

Thus $Z = kF\,(x, y)$ is representable by a co-ordinate vertical to the xy plane. Since Z denotes an actual number of persons in a double age-group, between say the earliest age of marriage and the end of life, viz., (x_1 to x_2) and (y_1 to y_2), it is necessary, if we desire to institute comparisons between different populations, that Z should be expressed as a *rate*, z say : that is, z = either Z/P; or Z/M; that is to say, the

vertical height will represent the relative frequency of married couples whose ages are, in the order of husband and wife, x and y, in either the whole popuation P, or the married portion of it M. Thus we shall have

$$(418)\ldots\ldots P, \text{ or } M = k \int\int F(x, y)\, dx\, dy.$$

If the value of the double integral be taken for the limits denoting the range of ages of the married, say about 11 to 105, we shall have either M/P, or unity, as the result; according as we denote by the frequency in reference to the total population or to the total married.

Thus the marital or gamic condition of a community is completely specified by the gamic surface $F(x, y, z)$, the unique mode of which is the summit of the conoidal solid represented by (418) above. Its first principal meridian is the line joining the modes of the curves x = a constant, or y = a constant, passing therefore through the unique mode. The curves, z = any constant less than its maximum value, are necessarily closed curves, and may be called *isogamic contours*. The *orthogonal trajectory* passing through the unique mode is the second principal meridian of the surface. The values of x, y, and z for the unique mode of the surface may be called the *gamic mode of the "population,"* or of the "*married population*," according as the constant k, in (418) above, gives M/P, or unity for the value of the double integral between the widest age limits.

The *gamic characteristics* of a population are more briefly, and of course less completely, defined by the two principal meridians which we may call its *gamic meridians*, and the position (and magnitude) of the *gamic mode*. Reducing these to their simplest numerical expression we have, for the briefest possible statement of the gamic characteristics of any community the values of x_m, y_m, and z_m; and of the *skewness of the profiles of the first and second principal meridians*. The sign of the skewness may be determined by always making the right hand branch of the curve that for increasing age for the first principal meridian, and increasing age of the husband for the second principal meridian.

A surface representing the frequency of marriage at particular pairs of ages we shall call a *protogamic surface*, and one representing the number of persons of particular pairs of ages living together in the state of marriage we shall call simply a *gamic surface*.

Curves of equal frequency on these two surfaces, we shall call *isoprotogamic* and *isogamic contours*, respectively, or more briefly, *isoprotogams* and *isogams*, and curves cutting such contours orthogonally will be called *protogamic* and *gamic meridians*.[1]

Let s denote a distance measured along a slope, so that ds is an element thereof. Then when—

$$(419)\ldots\ldots dz/ds = \sin \zeta$$

[1] The word "isogamy" has already been appropriated in a different sense in biology, viz., to denote the union of two equal and similar "gametes" in reproduction. This, however, will obviously lead to no confusion. The isogamy of a people might be regarded as of two kinds, *initial* or *nuptial* isogamy (isoprotogamy), and *characteristic* or *marital* isogamy (or simply isogamy).

= a maximum or a minimum, the element ds is an element of a meridian ; such meridians are the *principal meridians* above referred to ; *i.e.*, the principal meridians are the *lines of greatest and least slope*.

22. **Orthogonal Trajectories.**—The general theory of orthogonal trajectories may be stated as follows :—Let the co-ordinates of a system of curves (isogams or equal marriage frequency in the case considered) be denoted by x and y, and those of the trajectory, cutting the system orthogonally, by ξ and η : then, although for any point of intersection of the two $x = \xi$ and $y = \eta$, dy/dx is not the same as $d\eta/d\xi$. Since the tangents to the two curves are at right angles, we have the geometric relation $dy/dx = - d\xi/d\eta$ or

$$(420)\ldots\ldots 1 + \frac{dy}{dx} \cdot \frac{d\eta}{d\xi} = 0$$

For any system of curves we have then

$$(421)\ldots f\ (x,\ y,\ a)\ = 0\ ;$$

where a is a constant ; then, employing $\delta/\delta x$ and $\delta/\delta y$ to denote partial differentiation with respect to x and y, we have also

$$(422)\ldots\ldots \frac{\delta f}{\delta x} + \frac{\delta f}{\delta y} \cdot \frac{dy}{dx} = 0,$$

an equation by means of which a may be eliminated, so that a relation may be obtained between x, y and dy/dx. Let this relation be denoted by :—

$$(423)\ldots\ldots \psi\ (x,\ y,\ \frac{dy}{dx}\ ) = 0$$

This last expression is the differential equation of the system of curves we require.

For orthogonal trajectories we have $\xi = x$, $\eta = y$ and $dy/dx = - 1/(d\eta/d\xi)$, hence the differential equation of the system of orthogonal trajectories is :—

$$(424)\ldots\ldots \psi\ (\xi,\ \eta,\ - \frac{1}{\frac{d\eta}{d\xi}}\ ) = 0$$

In the system we are considering, the curves (isogams) do not conform to any simple specification, hence the present imperfect data do not indicate any unique system of curves of a simple character. If they did, it would be preferable to deduce the principal meridians of the surface by means of the general equation thereto. An examination of the surface, however, shews that there is no practical advantage in attempting to express it analytically.

23. **Critical characters on the protogamic surface.**—A review of the figures in Tables LIV. and LX. reveals the fact that, in general, if we regard the numbers of marriages corresponding to any given age for brides (the columns), there is a clearly-defined maximum value ; but that if we regard those corresponding to any given ages for bridegrooms (the rows), there are in many cases two or even three maximum values.

In this latter case, too, the maximum is often less clearly defined. The positions of these maximum points and the numbers (frequency) corresponding thereto, are important, as they disclose the characteristics of the surface. There are two ways of estimating the position and frequency at the maximum (or any other point). One is to ascertain the position and frequency for the maximum of the frequency integral taken over the range $x - \frac{1}{2}$ to $x + \frac{1}{2}$, or over the range $y - \frac{1}{2}$ to $y + \frac{1}{2}$; the other is to determine those elements for the maximum instantaneous frequency ; that is to ascertain the point when the frequency for an indefinitely small range is a maximum (expressed, however, per unit of age-difference, say one year). The latter only will be ascertained.

By applying formulæ (292) to (294), see Part VII., § 11, p. 92, the position and value of these maximum points (viz., those on the surface for ages of brides constant that of bridegrooms being variable, or for ages of husbands constant and that of brides variable), may be obtained. In this way the results given in the two following tables are deduced, viz., Tables LXI., and LXII., and in connection therewith it is to be remarked (a) that for results of high precision, the quinquennial grouping can be used only for the small groups at higher ages ; and (b) that the grouping in fives, not only tends to obliterate characteristics readily discernible in year-groupings, but gives a frequency of the order of about 25 times the magnitude of those groupings. Thus for very young ages and for the older age-pairs, the large grouping gives the best indication.[1]

[1] The values are obtained in the following way :—The position of the maximum of one group (say of bridegrooms) corresponding to the range of another group (say of brides) is found from the succession of the group-totals of the first, for any one range of the second, and is attributed to a mean age of the second, computed from the progression of numbers in the series of group totals of the second. By way of illustration consider the group of 59, for the age-group 65-69 of brides, and 70 to 74 of bridegrooms ; viz., the following figures :—

Instances in Group.	Adjoining Group Totals.
	60
3 . 4 . 1 . 1 . 2	105.97.19
2 . 1 . 1 . 2 . 2	
4 . 2 . 0 . 3 . 2	41.50.**59**.28.6
7 . 3 . 2 . 5 . 3	
3 . 0 . 0 . 2 . 4	14.21.25
	4

The surrounding group-totals are as shewn. If the ages of brides be taken as at the middle of the years, *i.e.*, as 65½, 66½, etc., and of the bridegrooms as 70½, 71½, etc., the actual weighted-mean ages (deduced from the individual numbers) are as shewn hereunder. Slightly different results are obtained if the ages are deduced from the vertical and horizontal columns, viz., 97, 59, 21 ; 50, 59, 28 ; and from the diagonal totals, viz., 105, 59, 25 ; and 19, 59, 14. These different results are for bride and bridegroom respectively :—

	Middle Values of Groups.		Actual Weighted Group-means.		Computed from Vertical Groups, etc.		Computed from Diagonal Groups.	
Years	67.5	72.5	67.35	72.64	67.48	72.45	67.46	72.40

This series of results shows that the error of assuming that the entire group is representable by the middle ages is not ordinarily considerable.

TABLE LXI. Critical Positions on the Protogamic Surface for Year-groups. Marriages in Australia, 1907-1914. (Greatest frequency for various combinations of Age at Marriage).

Mean Age of Brides in Maximum Group.	Age of Bridegroom for Maximum Frequency.	Difference of Age.	Maximum Frequency.	Proportion of All Brides of same Age-Group.
13.5	21.2	7.7	1	0.250
14.7	22.4	7.7	17	0.233
15.7	21.6	5.9	69	0.141
16.6	21.6	5.0	372	.1504
17.6	21.7	4.1	1203	.1742
18.5	21.7	3.2	2164	.1621
			1986‡	.1492‡
19.5	21.9	2.4	2600	.1434
	21.8	2.5	2500‡	.1364
20.5	23.4	2.9	2573	.1272
				.1256
21.5	23.3	1.8	4156	.1266
				.1295
22.5	23.7	1.2	3511	.1256
23.5	24.3	1.2	3269	.1239
24.5	24.6	0.1	3040	.1272
25.5	25.7	0.2	2744	.1325
26.5	26.6	0.1	2247	.1276
27.5	27.7	0.2	1753	.1214
28.5	28.5	0.0	1328	.1073
29.5	29.5	0.0	1045	.1046
30.5	30.7	0.2	768	.0913
31.5	31.6	0.1	565	.0966
32.5	32.5	0.0	510	.0916
33.5	33.5	0.0	320	.0737
34.5	34.6	0.1	305	.0791
35.5	35.5	0.0	236	.0670
36.5	36.5	0.0	190	.0650
37.5	37.9	0.4	167	.0685
38.5	38.6	0.1	194	.0801
39.5	39.5	0.0	153	.0765
40.5	40.3	−0.2	121	.0717
41.5	41.2	0.3	74	.0657
42.5	43.1	+0.6	94	.0730
43.5	45.2	+1.7	80	.07512
44.5	45.3	0.8	63	.0664

In determining any critical point, however, the ages deduced as shewn above are not what is required. What is definitely sought is the position and value of the maximum frequency, referred to a mean-age of bridegrooms (x), (or of brides (y)); that is the value of y (or of x, respectively) at which the maximum value occurs. The data from which these are deduced are the series of parallelepipeds the heights of which may be taken as the group-totals. Thus, the horizontal series of group-numbers 50, 59 and 28, treated as ordinate-values bounded by a curve, gives 66.13 years as the *age* of brides, corresponding to a maximum frequency of 62.18. If the 41 group be included, the maximum will be changed to age 67.50 years, and the frequency to 60.29. The mean age of the bridegrooms should be ascertained on the vertical line 67.50 for brides, but without incurring sensible error it may be taken as 72.50 − 5 (97 − 21) ÷ (24 [illegible] 59) [illegible] 72.23, see this part, section 20, formulæ (412) to (415); the factor 5, however, appearing because the unit is 5-years. Respecting the computation of the critical positions, it may be mentioned that the curve is found to be y [illegible] $44\frac{1}{4} - 19\frac{5}{12}X$ [illegible] $23\frac{1}{4}X^2$ [illegible] $5\frac{1}{6}X^3$, the origin of X being at age 55.00, and the unit of X being 5 years. This gives $x^2 - 3x$ [illegible] say 1. Hence (with sufficient approximation) X_m = $\frac{1}{2}$ and $\frac{5}{2}$ or in years five times these amounts, or $2\frac{1}{2}$ and $12\frac{1}{2}$. This gives [illegible] and [illegible] as the frequencies at the maximum and minimum.

Mean Age of Bridegrooms in Maximum Group.	Age of Bride for Maximum Frequency.	Difference of Age.	Maximum Frequency.	Proportion of all Bridegrooms of same Age-Group.
15.5	16.5	1.0	1	0.250
16.5	17.5	1.0	14	.274
17.5	17.7	0.2	60	.250
18.5	18.4	—0.1	318	.264
	18.5‡	0.0	352‡	.272‡
19.5	18.9	—0.6	820	.2416
			897‡	.2554‡
20.5	19.5	—1.0	1279	.1986
			1496‡	.2117‡
21.5	19.7	—1.8	2558	.1472
	19.7‡	—1.8‡	2410‡	.1465‡
21.5	21.4‡	—0.1‡	3250‡	.1968‡
22.5	21.6	—0.9	4110	.2057
	21.8‡	—0.7	3424	.1714
23.5	21.7	—1.8	4250	.1839
	22.1‡	—1.4‡	3508‡	
24.5	21.8	—2.7	3766	.1511
	22.8‡	—1.7‡	3333‡	
25.5	21.5§	—4.0§	3276§	.1329§
	23.3‡	—2.2‡	3026‡	.1225‡
	21.9	—3.6	3342	
26.5	21.6§	—4.9§	2710§	.1158
	23.4‡	—3.0‡	2694‡	.1147‡
	21.9	—4.6	2774	
27.5	22.6	—4.9	2271	.1080
	20.6	—6.9	2230	.1061
	21.8	—5.7	2293	
28.5	24.3	—4.2	1977	.1199
	21.9	—6.6	1973	
29.5	24.7	—4.8	1492	.0932
	22.0	—7.5	1458	
30.5	26.0	—4.5	1195	.0892
	21.9	—8.6	1080	
31.5	26.2	—5.3	849	.0820
32.5	26.1	—6.4	809	.0830
	22.7	—9.8	719	
33.5	26.3	—7.2	565	.0733-
	23.4	—10.1	557	.0722
34.5	24.5	—10.0	560	.0823
	34.2?	— 0.3?	309?	.0455?
35.5	25.5	—10.0	486	.0800
36.5	26.5	—10.0	371	.0694
37.5	27.4	—10.1	332	.0753
	37.2	— 0.3	171	.0388
38.5	28.5	—10.0	364	.0804
	38.3	— 0.2	195	.0430
39.5	29.5	—10.0	246	.0658
	39.2	— 0.3	157	.0420
40.5	30.3	—10.2	217	.0667
41.5	31.5	—10.0	144	.0617
	41.2	— 0.3	76	.0325
42.5	32.5	—10.0	137	.0562
	42.3	— 0.2	90	.0361
43.5	32.9	—10.6	108	.0525
	42.9	— 0.6	94	.0457
44.5	32.7	—11.8	99	.0567
	43.5	— 1.0	57	.0326

‡ The results include corrections for misstatements of age. § These maxima disappear altogether when corrections are applied for misstatements of age.

TABLE LXII.—Critical Positions on the Protogamic Surface, for 5-Year Groups. Marriages in Australia, 1907-1914.

Maximum age-group of brides	10–14	15–19	20–24	25–29	30–34	35–39	40–44
Mean age of brides in maximum group ..	*14.3 ?	*18.3 ?	*21.6 22.5	*26.6 27.3	*32.2 32.1	? 37.2	? 42.2
Age of bridegroom for maximum frequency	22.9	23.1	†23.8	27.5	32.1	37.5	43.4
Difference of age ..	8.6	4.8	2.2 1.3	0.9 0.3	-0.1 −0.0	0.3	1.2
Maximum frequency ..	46.7	24685 ‡24727	†72500 ‡72170	36722	9397.6	3716.5	1541.1
Proportion of all brides of same age-group	0.600	0.599 ‡0.601	0.553 ‡0.551	0.188	0.336	0.280	0.251
Maximum age-group of bridegrooms ..	15–19	20–24	25–29	30–34	35–39	40–44	45–49
Mean age bridegrooms in maximum group	*18.4	22.3	27.3	32.2	37.1	42.2	47.3
Age of bride for maximum frequency ..	17.8	22.1	23.6	25.9	27.9	32.1	37.4
Difference of age ..	0.6	0.2	3.7	6.3	9.2	10.1	9.9
Maximum frequency ..	3800 ‡4000	59496 ‡59166	51865	18290	7465.5	2837.0	1683.8
Proportion of all bridegrooms of same age-group	0.783 0.779	0.644 0.643	0.496	0.381	0.310	0.240	0.231

Maximum age-group of brides	45–49	50–54	55–59	60–64	65–69	70–74	75–79	80–84
Mean age of brides in maximum group ..	? 47.3	? 52.2	? 57.3	? 62.3	*67.3 67.3	*71.5 72.1	*76.50 76.8	*81.8 82.2
Age of bridegroom for maximum frequency	48.3	52.7	57.9	64.9	67.5	73.7	78.0	77.5
Difference of age ..	1.05	0.5	0.6	2.6	0.2 0.2	2.2 1.6	1.5 1.2	4.3 −4.7
Maximum frequency ..	887.5	417.1	225.8	111.0	100.1	28.5	8.3	4.3
Proportion of all brides of some age-group	0.254	0.260	0.286	0.255	0.334	0.277	0.280	0.610
Maximum age-group of bridegrooms ..	50–54	55–59	60–64	65–59	70–74	75–79	80–81	85–89
Mean age bridegrooms in maximum group	52.1	57.3	62.3	67.3	72.2	77.3	82.4	87.4
Age of bride for maximum frequency ..	39.6	46.8	48.8	45.2	62.8 66.1	57.5 72.5	47.4 62.5 72.5	47.5
Difference of age ..	12.5	10.5	13.5	22.1	9.4 6.1	?	?	?
Maximum frequency ..	785.5	457.9	213.9	139.0	66.6 62.2	32.3 25.9	10.2 9.6 8.7	4.3
Proportion of all bridegrooms of same age-group	0.200	0.215	0.172	0.168	0.160 0.150	0.197 0.158	0.165 0.155 0.140	0.360

* Calculated from yearly group results. † It is impossible from the data to determine these values with precision. ‡ With partial corrections for misstatements of age.

Fig. 61 shews the graphs of the maximum values. It is evident from these graphs that the greatest frequency of marriage is not well-defined according to alleged ages. The surface shews ridges on the lines Aa, Ab, Acde, Afg and Ah. The highest point is for the group bridegrooms about 23.4, and brides 21.6 years of age, the frequency attaining to about 4,200, or about one seventy-second part (0.013911) of all the marriages.

The maximum group is 4114, or 0.13626 of the marriages. These figures are, however, somewhat uncertain, for reasons which will be pointed out in the next section.

24. **Apparent peculiarities of the protogamic frequency.**—Fig. 61 shews, by dots, the positions of maxima on the (vertical) columns, that is according to the ages of brides; and, by dots with circles, the positions of the maxima on the (horizontal) rows, that is according to the ages of bridegrooms. If the ages have been correctly given there is no unique mode on the horizontal lines; and this is a matter which demands special consideration. In Part X., § 6, Fig. 42, p. 115, it is shewn that the number of under-statements by women amounting to 10 years, is quite abnormal; it does not follow the progressive diminution which characterises understatements amounting from 1 to 11 years. In the figure the line bAde would be the characteristic summit if the greatest frequency of marriage was in the case of parties of the same age. The line f g would be the characteristic if a large number of men married wives 10 years younger than themselves: while for the line Af to hold good, very large numbers of men of ages 22 to 31 must marry women of 21 years of age, irrespective of the disparity of age. To give the line of maxima Ah, a considerable number of men must marry women whose difference of age is one-half their age above 22. Such characters in a protogamic surface, are, *a priori*, extremely improbable. They would also characterise the apparent protogamic surface, if a considerable number of women, really of ages 22 to 32, all gave their ages as 22, when marrying men of from 22 to 32 years of age, and if a considerable number of women of 32 and upwards understated their ages by 10 years. This explanation probably does not differ very materially from the fact. Hence Tables LIV and LX must be regarded as of inferior value. It is, of course, much to be regretted that social organisation does not admit of the social-psychological fact of conjugal frequency at equal and disparate ages being accurately ascertained.

25. **The contours of the protogamic surface.**—The tedium of a rigorous analysis of a surface, when the measure of uncertainty is so large as is the case with the protogamic surface for Australia, is not warranted. A rough smoothing of the 5-year groups was, therefore, effected, and attributing the smoothed values to the centre points of the groups, and a series of contours for the proportions of 5, 10, 20, 40, etc., in a million of total marriages of all ages, were inserted by graphic methods. These gave fairly smooth contours. Regular curves being drawn, so as to ignore the minute undulations of the contours the results shewn on Fig. 61 are obtained. These represent with considerable precision the actual data from which they were derived, and will enable such data to be reproduced. They disclose the frequency distribution, for all combinations of ages.

Curves of Equal Marriage Frequency.—The Protogamic Surface.

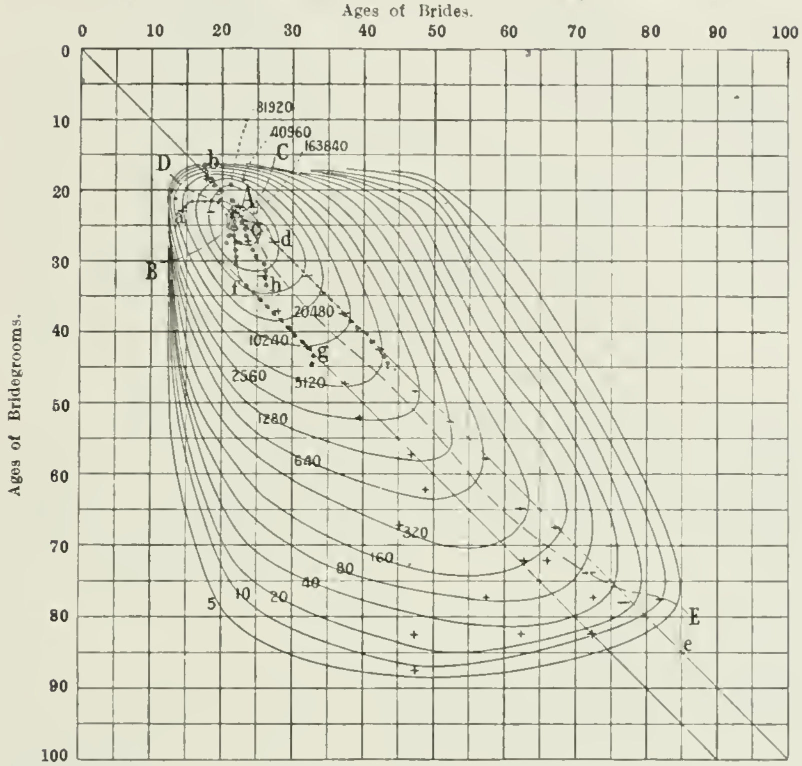

Fig. 61.

NOTE.—The pairs of ages which give equal frequency of marriage are found by following the course of any isoprotogam. The frequency indicated is per million marriages of all ages. The co-ordinates of any two points, whatsoever, on any isoprotogam are equivalent age-pairs, that is pairs of ages which are characterised by the same frequency of marriage.

The protogamic surface, indicated by the family of curves or isogamic contours, is not the surface of frequency for indefinitely small ranges of age, but the surface for 5-year ranges of age; see hereunder. These contours or "isogams" are numbered 5, 10, 20, etc., denoting the doubling of the frequency. The point denoted by an asterisk near A, is the summit of this surface, *i.e.*, its ordinates are the centre of the 5-year ranges of age for which the frequency of marriage is greatest. From the summit it falls most rapidly in the directions A, B and A, C, and least rapidly in the directions A, B and A E, the directions being shewn by broken lines.

The values on the protogamic surface can be thus interpreted:—Assuming that the frequency of marriage for given pairs of ages, is as in Australia during the eight years, 1907-1914, in every 1,000,000 marriages of brides and bridegrooms of all ages, the number to be expected in any 5-year group over the range of $2\frac{1}{2}$ years earlier to $2\frac{1}{2}$ years later than the ordinates of the point taken, in the case of both bride and bridegroom, will be that shewn by the corresponding isogam, along which there will be equal frequency of marriage. Thus, for example, following the variation with age contour corresponding to 10,240 marriages out of a total of 1,000,000, the frequency indicated will be very approximately that for the 5-year ranges, the middle values of which are brides 20 with bridegrooms 37; brides $24\frac{1}{2}$ with bridegrooms 40; brides 30 with bridegrooms 42; brides 35 with bridegrooms either $41\frac{1}{2}$ or 29; brides 37 with bridegrooms either 40 or 33; and so on. The contours thus shew the centre values of a 5-year range of age, at which there is equal frequency of marriage within the range. That is, if the co-ordinates of any point on a contour be x and y, the frequency of marriage is for the ages bridegrooms $x - 2\frac{1}{2}$ to $x + 2\frac{1}{2}$, with brides $y - 2\frac{1}{2}$ to $y + 2\frac{1}{2}$. Hence if M be the total number of marriages, the actual number will be the number on the contour divided by 1,000,000 and multiplied by M.

Characteristics of the Protogamic Surface.

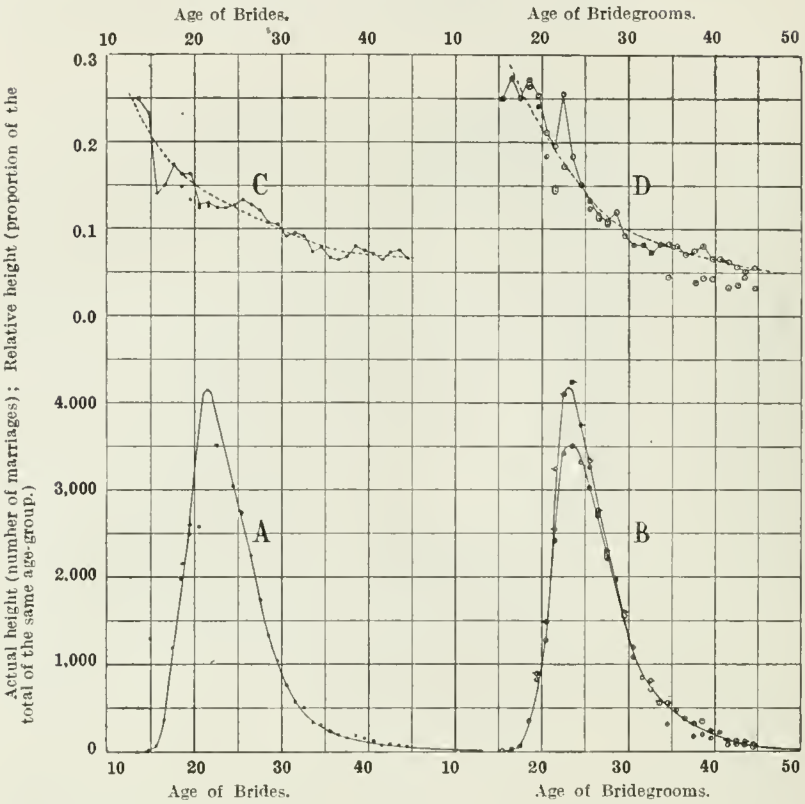

Fig. 62.

Curves A B C and D shew the various vertical features of the protogamic surface. Of these :—

Curve A shews the projection of the profile on the y or age-of-brides axis, the dots indicating the values according to the data, and the continuous line shewing the probable true position of the surface profile.

The outer Curve B shews the projection of the profile on the x or age-of-husbands axis, the dots and circles indicating the positions according to the data. The inner curve indicates the position of a series of second and fairly well-defined maxima. All the points shewn are maxima of some kind.

Curve C shews by dots, and a zig-zag line joining them, the proportion which the frequency at the various maxima bears to the totals for the same age-groups of brides. The general trend of this frequency as a function of age is shewn by a broken line.

Curve D shews by dots with circles and by a zig-zag line, the proportion which the frequencies at the various maxima bear to the total for the same age-groups of bridegrooms. The broken line shews their general trend.

Each contour is twice the height of the contour immediately outside it; thus the surface rises with great rapidity, and is very steep on the top, and also the left hand side in the figure. The proportion per million marriages for a 5-year group, ranging between $x \pm 2\frac{1}{2}$ and $y \pm 2\frac{1}{2}$ is defined by the numbers written along the contours. The *projection* on the $y-axis$ of the ridge running from the top left-hand corner to the

bottom right-hand corner is shewn by curve A, Fig. 62; and its projection on the $x-axis$ is shewn by Fig. B. The proportion which the frequency at the maximum bears to the total for the same age-group of brides is shewn by curve C, and for the same age-group of bridegrooms by curve D. In these two last curves the zig-zag lines shew the successive principal maxima, and the dotted lines the general trend. It is probable that in a large population, when the ages at marriage are correctly given, the results would yield regular curves of the types drawn. The contours do not indicate curves of great regularity, but that is doubtless due (at least in part) to the inexact statement of age and the paucity of the numbers for higher ages.

27. **Relative marriage frequency in various age-groups.**—For sociologic purposes, a table shewing the relative marriage frequency in various age-groups is of obvious importance. Given an Australian population, constituted as to numbers of married and unmarried in age-groups as was its population during 1907 to 1914, 1,000,000 marriages are found to be distributed as follows :—

TABLE LXIII.—Relative Frequency of Marriage in Various Age-Groups. Australia, 1907-1914.

Age-group of Bride-grooms	Age-group of Brides.																All Ages.*
	10–14	15–19	20–24	25–29	30–34	35–39	40–44	45–49	50–54	55–59	60–64	65–69	70–74	75–79	80–84	85–89	
15–19	30	11,605	4,920	411	56	12	7	3	2	1	..	..	..	..	..	..	17,048
20–24	146	76,788	184,576	37,452	4,762	1,077	199	73	13	3	1	..	..	..	..	..	305,080
25–29	60	35,249	167,668	115,639	22,331	4,537	935	259	66	6	3	2	..	..	..	..	346,765
30–34	10	9,262	51,407	57,547	30,255	8,205	1,740	484	86	13	7	3	..	..	..	..	159,019
35–39	7	3,039	17,013	24,184	18,795	11,999	3,440	1,037	215	50	13	5	..	..	..	..	79,797
40–45	5	785	5,222	8,496	9,315	8,195	4,978	1,690	371	86	30	10	1	..	..	..	39,183
45–49	4	381	1,982	3,569	4,351	5,477	4,239	2,827	872	245	80	27	3	..	..	..	24,058
50–54	3	136	607	1,273	1,783	2,499	2,545	2,237	1,346	388	166	53	7	3	..	..	13,046
55–59	2	43	182	414	686	978	1,293	1,425	1,027	697	215	99	17	6	2	..	7,086
60–64	1	20	93	209	331	457	547	686	689	524	351	199	50	9	3	..	4,169
65–69	1	7	43	88	143	219	265	365	431	431	315	182	63	13	5	..	2,571
70–74	1	5	23	40	66	99	146	186	215	215	166	113	73	21	7	1	1,377
75–79	1	3	7	13	20	28	38	48	64	85	92	73	47	27	11	1	558
80–84	..	1	5	9	10	14	22	28	33	29	23	13	8	1	2	1	202
85–89	..	..	1	1	2	3	5	10	8	5	3	2	1	1	..	..	42
All Ages*	271	137,324	433,750	249,345	92,906	43,799	20,398	11,358	5,438	2,778	1,465	781	270	84	30	3	1,000,000

* These totals are about ten times those in the final columns of Table LX., p. 199. Though in substantial agreement they are not absolutely identical because these results have been slightly smoothed.

The above table is founded upon the results given by a slight smoothing of the actual numbers, and gives the roughly adjusted relative-frequency of marriage according to age-groups, based upon the marriages of the 8-year period, 1907 to 1914 inclusive, the 1911 Census being regarded as giving a sufficient indication of the relative numbers of married and unmarried for the computation of any derivative relations. The middle point of time would be Jan. 0, 1911, while the Census is April 3rd, 1911. The total marriages were 301,922, or about 37,740 annually; half of them had occurred by about April 28, 1911, that is 25 days after the Census, hence a correction is not required.

28. **The numbers of the unmarried and their masculinity.**—The smoothed results of the Census give the following numbers of unmarried at each age, viz., those shewn in Table LXIV. From these the ratios of the males to the females (M/F) have been computed ; they are shewn opposite the letters " Mas." in the Table. From the numbers given the masculinities can be computed of the various age-groups, which are required hereinafter for the computation of the probability of marriage according to pairs of ages.

TABLE LXIV.—Number of Unmarried Males and Females and the Masculinity (M/F) at each Year of Age. Australia, 3rd April, 1911.

Year of Decennium in Age.	Ages.									
	0	10	20	30	40	50	60	70	80	90
0 M	58,648	43,049	45,466	16,700	7,973	5,232	2,152	1,230	370	33
F	56,401	42,654	38,370	10,839	4,987	2,340	830	360	92	8
Mas.	1.03984	1.00926	1.18493	1.54073	1.5987	2.2359	2.593	3.417	4.0	4.1
1 M	54,759	42,753	43,799	14,875	7,669	4,797	2,032	1,190	312	26
F	52,982	42,222	34,634	9,659	4,623	2,127	760	320	80	6
Mas.	1.03354	1.01258	1.26462	1.54001	1.6588	2.2553	2.674	3.719	3.9	4.3
2 M	52,659	42,748	41,097	13,250	7,386	4,332	1,922	1,140	266	20
F	51,145	42,001	30,878	8,730	4,226	1,938	715	280	69	4
Mas.	1 02960	1.01779	1.33094	1.51775	1,7477	2,2353	2,688	4,071	3.85	5.0
3 M	51,158	42,990	37,541	11,925	7,077	3,942	1,822	1,080	225	15
F	49,785	42,072	26,360	7,835	3,940	1,780	690	240	58	3
Mas.	1.02758	1.02182	1.42418	1.52201	1.7962	2.2146	2.641	4.50	4.05	5.0
4 M	49.998	43,618	34,003	10,900	6,763	3,642	1,732	1.010	189	11
F	48,556	42,484	22,725	7,278	3,707	1,549	650	205	48	2
Mas.	1.02970	1.02669	1.49628	1.49766	1.8244	2.3512	2.665	4.927	3.94	5.5
5 M	48,812	44,598	30,338	10,177	6,453	3,342	1,622	920	158	8
F	47,428	42,273	19,600	6,791	3,441	1,363	600	180	38	2
Mas.	1.02918	1.03062	1.54785	1.49860	1.8753	2.4519	2.703	5.11	4.16	4.0
6 M	47,543	45,482	26,823	9,529	6.185	3,042	1,490	810	121	6
F	46,298	43,915	17,267	6,319	3,212	1,248	550	155	28	1
Mas.	1.02689	1.03568	1.55342	1.50799	1.9256	2.4375	2.709	5.23	4.32	6.0
7 M	46,084	46,212	23,597	9,032	5,957	2,762	1,396	690	90	5
F	45,111	43,813	15,370	5,910	3,000	1,145	500	135	20	1
Mas.	1.02157	1.05475	1.53526	1.52826	1.9856	2.4122	2.792	5.11	4.5	5.0
8 M	44,783	46,620	20,808	8,613	5,731	2,512	1,328	560	65	4
F	44,055	42,854	13,646	5,630	2.783	1,032	450	120	14	1
Mas.	1.01652	1.08788	1.52484	1.52984	2.0593	2.4341	2.951	4.67	4.6	4.0
9 M	43,770	46,470	18.677	8,265	5,522	2,302	1,272	450	46	3
F	43,236	41,020	12,153	5,303	2,570	910	405	105	10	1
Mas.	1.01235	1.13286	1.53682	1.55855	2.1486	2.5297	3.141	4.28	4.6	3.0

100 and over—Males, 2 ; Females. 1. Totals under 13, 662,764, 611,873 = 1.08317.

NOTE.—The masculinity is for the year-groups, and may be assumed to be the masculinity at age $x + \frac{1}{2}$, where x is the tabular age, viz., the " age last birthday."

The change of masculinity with age follows no simple law, as will be seen from curve A on Fig. 63. The irregularities after 80 are due to the relatively small numbers on which the curve is based, and must be regarded as accidental. The masculinity diminishes in the earlier years, because of the greater mortality among males. Its constancy between the ages 25 and 37 is remarkable, as also is the sudden increase commencing at 66 years of age, and continuing to 76.

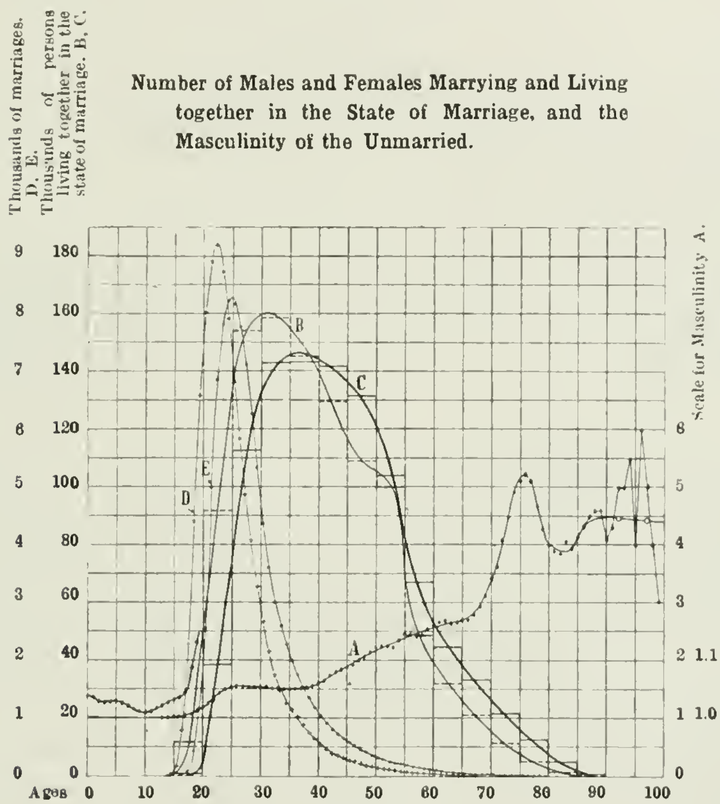

Fig. 63.

Curve A denotes the variation with age of masculinity (M/F) of the unmarried. The small lozenge-shaped dots are the values according to the data; the continuous line shews the general trend. The scale for the masculinity up to nearly 20 years has also been plotted on ten times the scale. See Table LXIV., p. 212.

Curve B denotes the number of married females of marriages living with their husbands in a total of 1,000,000 couples. See Table LXVIII., p. 224.

Curve C denotes the number of married males of various ages living with their wives, in a total of 1,000,000 couples. See Table LXVIII., p. 224.

Curve D shews the adjusted number of females of various ages, per 100,000 marriages, occupying between 1907 and 1914. See Table LIX., p. 197.

Curve E shews the adjusted number of males of various ages, per 100,000 marriages, occurring between 1907 and 1914. See Table LIX., p. 197.

29. **The theory of the probability of marriages in age-groups.**—The data do not exist for a definite and rigorous determination of the probability of marriage in age-groups ; nevertheless a fairly accurate estimate is possible by means of a somewhat empirical theory, which will now be indicated. The deduced results are shewn in Tables LXVI. and LXVII., see pp. 219 to 222.[1]

For convenience the adjusted numbers from the Census are given in Tables LXIV. and LXV. hereinafter ; the corresponding numbers of marriages occurring in each age-group are also given. The values of q given in the tables enable the number of marriages likely to occur in each age-group to be computed when the numbers of unmarried males and females in the group are known. Thus, q being the tabular number, the number of marriages, N, may be computed by means of formula (431) or formula (434) hereinafter. (See next section.)

Suppose that in any age-group there are M unmarried males and F unmarried females ; and that in a unit of time N pairs of these marry. The probability *with F females in the group*, of a particular marriage occurring among the M males is obviously N/M ; and *with M males in the group*, the probability of a particular marriage occurring among the F females is similarly N/F. Such a statement of probability, however, lacks generality. To obtain a more general one, an expression is needed which, given a definitive tendency towards the conjugal state in males and in females, though not necessarily of the same strength (or potential) in each sex, and not necessarily independent of the relative numbers of the sexes, nor even independent of the lapse of time, will give the number of marriages occurring in a group, constituted in any manner whatever in regard to the numbers of either sex. We shall call the tendency to marry the *conjugal potential* under a given condition. In the case of males let the conjugal potential be denoted by γ, and in the case of females by γ'; γ and γ' vary with age, doubtless also with time, and (we may assume) with the relative frequency of M and F. Without asserting it to be *exactly* the law of variation, we may suppose that the conjugal potential varies somewhat as some constant, multiplied into some power of the ratio of the numbers of the unmarried of each sex. Put p for the constant in the case of males, p' for the constant in the case of females, then the conjugal potentials are of the type $p. f\left(\frac{M}{F}\right)$, which function can, for all practical purposes, probably take the form

$$(425)\ldots\ldots \gamma = p\left(\frac{M}{F}\right)^r; \text{ and } \gamma' = p'\left(\frac{F}{M}\right)^s$$

formulae in which r and s are indices to be ascertained by experiment.

[1] These results are on the basis of 10 million males, and the same number of females. Hence if they are multiplied by one ten-millionth of 1,508,623, and 1,277,259 respectively, they will give the absolute numbers, since these were the number of unmarried males and of unmarried females respectively, on 3rd April, 1911.

Thus $\gamma = p$ and $\gamma' = p'$ when the numbers of unmarried of either sex are equal; ordinarily they do not differ sensibly therefrom. Again, if the number of females be large, the γ potential is doubtless smaller; and if the number of males be large the γ' potential is smaller. This appears to be confirmed by experience. The expressions (425) can be made to fit the facts by appropriately determining r and s.

From (425) we have at once for the ratio of the conjugal potentials

$$(426)\ldots\ldots \frac{\gamma}{\gamma'} = \frac{p}{p'}\left(\frac{M}{F}\right)^{w},$$

where $w = r + s$, from which it is evident that it is not necessary to ascertain r and s individually, but only their sum, w. And if the conjugal potential vary with age, it could be ascertained only by comparing a series of results for the one age-group when the numbers of males and females were very divergent; all other circumstances promoting marriage remaining constant. For this reason, with the limitations of existing data, we must assume (which doubtless, as already indicated, is not *exactly* true), that, when the numbers of the unmarried of each sex are equal, the conjugal potential and probability of marriage vary in the same way. That is

$$(427)\ldots\ldots \gamma / \gamma' \propto p/p';$$

or the probability of marriage is the effective measure of the conjugal potential; or in other words (subject to what has been said above) we may suppose that, with equal numbers of unmarried males and females, the frequency of marriage is a normal measure of the conjugal potential.

If we make still another assumption, viz., that indicated hereunder (in the passages in italics), a crude type of solution becomes possible, and the problem may then be envisaged as follows:—

If there be M males in any age-group and F females in any other age-group, it is obvious that there can be MF marriages of particular pairs among these groups: and if a group out of these of N males and N females be taken, it is similarly self-evident that they can form NN marriages of particular pairs. *Consequently assuming that the marriage of particular pairs is equally probable, and that the relative magnitude of M and F does not influence the probability, p, then the chance* of N marriages occurring is

$$(428)\ldots\ldots p_{xy} = N_x N_y / (M_x F_y)$$

x and y denoting the age-groups referred to. The value of p cannot possibly become unity unless $M = F = N$. This probability does not, however, enable us to compute the likelihood of N marriages occurring with particular values for M and F, since obviously N is not $\sqrt{p}.\sqrt{(MF)}$, although that is a solution of equation (428).[1] Subject to the assumptions

[1] For example, given M constant, N would depend upon $\sqrt{F}$, which is certainly not correct if M be large and F small. In this case N would evidently vary as F, not as $\sqrt{F}$.

made, the function representing the chance of N marriages occurring must clearly vary approximately as $\sqrt{(MF)}$, when they are sensibly equal, and must vary sensibly as F (or M) when M (or F) is relatively very large.

In order to obtain an expression that will readily fulfil the necessary conditions, we may observe that if we put

$$(429)\ldots\ldots N_{xy} = q_{xy} \,.\, \phi\,(M_x) \,.\, \psi\,(F_y)$$

and for ready computation assume that the functions ϕ and ψ may, with sufficient precision, take the form M^{ξ} and F^{η} ; then ξ and η must fulfil the following conditions, viz. :—

(a) In order to give kN, when the numbers of males and females are kM and kF (at the same time), we must have $\xi + \eta = 1$, so that $k^{\xi} \,.\, k^{\eta} = k$.

(b) As a consequence of this condition, viz., (a), it follows that when $M = F$, $\xi = \eta$, and each must be $\frac{1}{2}$.

(c) In order that, when M is relatively very great (or small) as compared with F, N shall vary as F (or M), ξ (or η) must become in such a case sensibly zero, and η (or ξ) must become sensibly unity.[1]

(d) The fundamental assumptions require also that the expressions shall be symmetrical in regard to M and F, *i.e.*, one can be had from the other by mere interchange.

In practical examples we may have the ratio of M to F varying from about 0.1 to about 10.0 through a wide range of important ages, so that a formula, to be of the widest application, should at least embrace this range. Up to 40 years of age M ranges from about F to $8F$. A function that will fulfil the required condition as above indicated is

$$(430)\ldots\ldots N \propto M^{\frac{F}{M+F}} \,.\, F^{\frac{M}{M+F}}$$

Consequently we may write instead of (429) :—

$$(431)\ldots N_{xy} = q_{xy} \,.\, M^{\frac{F}{M+F}} \,.\, F^{\frac{M}{M+F}} = q_{xy} \,.\, M\phi^{\frac{1}{1+\phi}} = q_{xy} \,.\, F\mu^{\frac{1}{1+\mu}}$$

and to find q from the results furnished in Tables LXIV. and LXV. we have,

$$(432)\ldots\ldots q_{xy} = \frac{N_{xy}}{M\phi^{\frac{1}{1+\phi}}} = \frac{N_{xy}}{F\mu^{\frac{1}{1+\mu}}} \text{ ; or}$$

$$(432a)\ldots\ldots \log q_{xy} = \log N_{xy} - \frac{1}{1+\mu}\log M - \frac{1}{1+\phi}\log \mathrm{F},$$

x and y denoting the central values of the age-groups, *i.e.*, $x \pm \frac{1}{2}k$, $y \pm \frac{1}{2}k$ where k is the range of the group. The application of this formula can be greatly facilitated in the following way :—Let $S_{xy} = M_x + F_y$, that is, let

[1] ξ is the quantity denoted by ϕ_2, and η that denoted μ_2 on page 132 hereinbefore.

S_{xy} denote the total number of single persons in the groups of males of age x and females of age y, and let the masculinity (or the femininity) of S be denoted by M/F (or F/M); then assuming that the probability is identical for A males and B females, with that for B males and A females (which, however, though by no means certain, is not determinable from existing data), we may compute the value of the ratio

$$(433)..\ R_\mu = R_\phi = (M^{\frac{F}{M+F}} . F^{\frac{M}{M+F}}) \Big/ \tfrac{1}{2}(M+F) = F\mu^{\frac{1}{1+\mu}} \Big/ \tfrac{1}{2}S = M\phi^{\frac{1}{1+\phi}} \Big/ \tfrac{1}{2}S$$

which depends merely upon the masculinity, μ (or the femininity ϕ), and is independent of the absolute value of S, or of M and F. Consequently with a table of values of R arranged according to the argument μ (or ϕ), we have, by simply dividing M by F, (or F by M) and entering the table,

$$(434)......\ N_{xy} = \tfrac{1}{2}\, S_{xy}\ .\ R_\mu\ .\ q_{xy} = \text{say}\ \tfrac{1}{2}\, S_{xy}\ .\ Q_{xy}$$

Q itself could be tabulated but for the fact that the masculinity in age-groups may differ appreciably with the lapse of time. We require, therefore, two tables, viz., one for R depending upon the masculinity (or femininity), and one for q depending on the frequency of marriage for the age-groups in question. After preparing a table of the values of R, Table LXV., those of q can readily be calculated. In using the following table of the values of R, it is, of course, a matter of indifference whether it be entered with the argument "masculinity" or "femininity."

TABLE LXV.

Values of $R = (M^{\frac{F}{M+F}} . F^{\frac{M}{M+F}}) / \frac{1}{2}S$, for computing the effect of unequal numbers of unmarried males and females on the frequency of marriage.

M/F and F/M are interchangeable.

VALUES OF—

M/F	F/M	10000 R	M/F	F/M	10000 R	M/F	F/M	10000 R	M/F	F/M	10000 R	M/F	F/M	10000 R	M/F	F/M	10000 R	M/F	F/M	10000 R
1.0	1.0	10000 34	2.0	.5000	8,399 203	3.0	.3333	6,580 152	4.0	.2500	5,278 107	5	.2000	4,359 668	15	.0667	1,481 96	60	.0167	351
1.1	.9091	9,966 90	2.1	.4762	8,196 200	3.1	.3226	6,428 147	4.1	.2439	5,171 102	6	.1667	3,691 503	16	.0625	1,385 84	70	.0143	299
1.2	.8333	9,876 130	2.2	.4545	7,996 195	3.2	.3125	6,281 142	4.2	.2381	5,069 100	7	.1429	3,188 388	17	.0588	1,301 76	80	.0125	261
1.3	.7692	9,746 159	2.3	.4348	7,801 191	3.3	.3030	6,139 136	4.3	.2326	4,969 96	8	.1250	2,800 309	18	.0556	1,225 66	90	.0111	231
1.4	.7143	9,587 178	2.4	.4167	7.610 186	3.4	.2941	6,003 132	4.4	.2273	4,873 93	9	.1111	2,491 249	19	.0526	1,159 61	100	.0100	207
1.5	.6667	9,409 193	2.5	.4000	7,424 180	3.5	.2857	5,871 127	4.5	.2222	4.780 90	10	.1000	2,242 207	20	.0500	1,098 227	200	.0050	102
1.6	.6250	9,216 200	2.6	.3846	7,244 174	3.6	.2778	5,744 123	4.6	.2174	4,690 87	11	.0909	2,035 173	25	.0400	871 151	400	.0025	51
1.7	.5882	9,016 204	2.7	.3704	7,070 169	3.7	.2703	5,621 118	4.7	.2128	4,603 84	12	.0833	1,862 146	30	.0333	720 186	700	.0014	29
1.8	.5556	8,812 207	2.8	.3571	6,901 163	3.8	.2632	5,503 115	4.8	.2083	4,519 81	13	.0769	1,716 126	40	.0250	534 111	1000	.0010	20
1.9	.5263	8,605 206	2.9	.3448	6,738 158	3.9	.2564	5,388 110	4.9	.2041	4,438 79	14	.0714	1,590 109	50	.0200	423	2000	.0005	10

In the columns "10,000 R," the "differences" are also shewn.

From the values in the above table, a working table may readily be constructed so as to avoid tedious calculations of the function R.

When, however, the value of M is large, and that of F is small (or *vice versa*), the value of N depends mainly on F (or on M). In this case it is preferable to use a table of the values of $\mu^{\frac{1}{1+\mu}}$ (or of $\phi^{\frac{1}{1+\phi}}$) with the argument μ (or ϕ); see formulæ (431) or (432) just given. A table such as LXVa. will then be required.

The formula to be used will be

$$(435)\ldots\ldots N_{xy} = F \, . \, R'_{\mu} \, . \, q_{xy} = M \, . \, R'_{\phi} \, . \, q_{xy} \, ;$$

in which R'_{μ} is the tabular value $\mu^{\frac{1}{1+\mu}}$ and R'_{ϕ} is the tabular value $\phi^{\frac{1}{1+\phi}}$, the q quantities being as before.

TABLE LXVa.

Values of $R' = 2\mu^{\frac{1}{1+\mu}}$, for computing the effect of unequal numbers of unmarried males and females on the frequency of marriage.

M/F or F/M	R′	M/F or F/M	R′	M/F or F/M	R′	M/F or F/M	R′
10	1.2328	60	1.0694	200	1.0267	700	1.0094
20	1.1533	70	1.0616	300	1.0192	800	1.0084
30	1.1159	80	1.0556	400	1.0150	900	1.0076
40	1.0958	90	1.0507	500	1.0124	1,000	1.0069
50	1.0797	100	1.0467	600	1.0107	2,000	1.0038

The table shews very clearly that as the unmarried females (or males) become relatively fewer the number of marriages varies more nearly in the proportion of the number of females (or males).

30. **Masculinity of the unmarried in various age-groups.**—The results embodied in Table LXIV., make it possible to compute the masculinity of the unmarried for any combined age-groups, since this affects the number that may be expected to marry. The masculinities are shewn in two tables, viz., Table LXVI. and Table LXVII., the former giving the results for 2-year age-groups for ages 15 to 44 for bridegrooms, and ages 13 to 44 for brides; and the latter the results for 5-year age-groups for ages 15 to the end of life for bridegrooms, and 10 to the end of life for brides. From the values of M/F, $= \mu$, (or F/M, $= \phi$,) the values of $F/(M+F)$ and of $M/(M+F)$ may be readily computed if required. Thus[1]

$$(436)\ldots \frac{F}{M+F} = \frac{1}{1+\mu} = \frac{\phi}{1+\phi} = \phi_2 \, ; \quad \frac{M}{M+F} = \frac{1}{1+\phi} = \frac{\mu}{1+\mu} = \mu_2$$

[1] For other definitions of masculinity and femininity see Part X., § 3, (333) to (335), and Table XXI., pp. 132, 133 hereinbefore.

TABLE LXVI.—Shewing the Masculinity of the Unmarried in 2-year Age-groups (M/F), and the Probability-function 1,000,000 $_2q_{xy}$ for calculating the Number of Marriages in the Two-year Age-groups indicated.

AGE OF BRIDES AND NUMBER OF UNMARRIED FEMALES.

AGE OF BRIDEGROOMS AND NUMBER OF UNMARRIED MALES

Age Group.	13–14 84,574	15–16 87,188	17–18 86,667	19–20 79,390	21–22 65,512	23–24 49,085	25–26 36,867	27–28 29,016
15–16 90,080	1.065 3	1.033 15	1.039 30	1.135 15	1.375 13	1.835 4	2.443 3	3.104 2
17–18 92,832	1.098 6	1.064 273	1.071 905	1.170 515	1.417 253	1.891 96	2.518 54	3.199 23
19–20 91,936	1.087 16	1.054 691	1.061 3,959	1.158 5,081	1.403 3,134	1.873 1,257	2.494 691	3.168 286
21–22 84,896	1.004 38	0.974 1,100	0.979 8,420	1.069 14,490	1.296 21,328	1.729 10,494	2.303 5,304	2.926 2,742
23–24 71,544	0.846 16	0.821 795	0.826 6,391	0.901 14,652	1.092 26,492	1.458 24,774	1.941 13,917	2.466 7,057
25–26 57,161	0.676 15	0.656 696	0.660 4,999	0.720 11,579	0.873 23,148	1.164 26,102	1.551 25,173	1.970 14,407
27–28 44,405	0.525 13	0.509 473	0.512 3,867	0.559 9,172	0.678 18,819	0.905 2,1970	1.205 22,699	1.530 20,553
29–30 35,377	0.418 8	0.406 312	0.408 2,552	0.446 6,632	0.540 13,383	0.721 15,776	0.960 17,670	1.219 16,710
31–32 28,125	0.333 8	0.323 238	0.325 1,719	0.354 4,436	0.429 9,529	0.573 10,650	0.763 12,658	0.969 12,193
33–34 22,825	0.270 8	0.262 169	0.263 1,398	0.287 3,408	0.348 6,950	0.465 8,707	0.619 9,068	0.787 9,683
35–36 19,706	0.233 7	0.226 211	0.227 1,055	0.248 2,497	0.301 5,450	0.402 6,074	0.534 7,830	0.680 7,804
37–38 17,645	0.209 6	0.202 102	0.204 733	0.222 1,823	0.269 3,831	0.359 5,156	0.479 5,392	0.608 6,775
39–40 16,238	0.192 5	0.186 153	0.187 453	0.205 1,317	0.248 2,542	0.331 3,153	0.440 4,035	0.560 4,477
41–42 15,055	0.178 3	0.173 45	0.174 275	0.190 678	0.230 1,538	0.307 2,041	0.408 2,390	0.519 2,977
43–44 13,840	0.164 1	0.159 35	0.160 154	0.174 465	0.211 1,082	0.282 1,551	0.375 1,726	0.477 2,427

See Note to the continuation of the Table on the next page.

TABLE LXVI.—Shewing the Masculinity of the Unmarried in 2-year Age-groups (M/F), and the Probability-function 1,000,000 ${}_2q_{xy}$ for calculating the Number of Marriages in the Two-year Age-groups indicated. (*Continued*).

Age of Brides and Number of Unmarried Females.

Age of Bridegrooms and Number of Unmarried Males.

Age Group.	29–30 22,992	31–32 18,389	33–34 15,113	35–36 13,110	37–38 11,540	39–40 10,390	41–42 8,849	43–44 7,647
15–16 90,080	3.918 ..	4.899 ..	5.960 ..	6.871 ..	7.806 ..	8.670 ..	10.18 ..	11.78 ..
17–18 92,832	4.038 20	5.048 13	6.142 8	7.081 4	8.044 ..	8.935 ..	10.49 ..	12.140 ..
19–20 91,936	3.999 225	4.999 88	6.083 83	7.013 52	7.967 34	8.849 19	10.39 11	12.022 7
21–22 84,896	3.692 1,405	4.617 865	5.618 450	6.476 392	7.357 161	8.170 143	9.594 102	11.102 93
23–24 71,544	3.112 4.241	3.891 2.010	4.734 1,422	5.457 905	6.199 694	6.886 375	8.085 167	9.356 105
25–26 57,161	2.486 7.574	3.108 4,090	3.782 2,318	4.360 1,600	4.953 881	5.502 571	6.460 372	7.475 218
27–28 44,405	1.931 11,620	2.415 6,269	2.938 4,043	3.387 2,421	3.848 1,877	4.274 1,000	5.018 441	5.807 339
29–30 35,377	1.539 15,147	1.924 8,170	2.341 5,493	2.699 3,430	3.066 2,152	3.405 1,287	3.998 714	4.626 471
31–32 28,125	1.223 12,319	1.529 10,652	1.861 6,073	2.145 4,133	2.437 2,354	2.707 1,409	3.178 789	3.678 517
33–34 22,825	0.993 9.604	1.241 8,016	1.510 7,563	1.741 5,146	1.978 2,948	2.197 1,750	2.579 1,100	2.985 939
35–36 19,706	0.857 8,384	1.072 6,901	1.304 6,836	1.503 6,247	1.708 3,874	1.897 2,745	2.227 1,349	2.577 1,124
37–38 17,645	0.767 6,209	0.960 5,546	1.168 5,870	1.346 5,558	1.529 5,695	1.698 3,571	1.994 1,775	2.307 1,354
39–40 16,238	0.706 5,565	0.883 4,343	1.074 4,793	1.239 5,104	1.407 4,966	1.563 4,968	1.835 2,837	2.123 1,725
41–42 15,055	0.655 3,242	0.819 3,783	0.996 3,430	1.148 4,119	1.305 3,323	1.450 3,354	1.701 3,314	1.969 2,123
43–44 13,840	0.602 1,819	0.753 2,792	0.916 3,170	1.056 3,049	1.199 3,185	1.332 3,194	1.564 2,833	1.810 2,871

Note.—The upper figures denote the masculinity of the group, that is, the ratio of the number of all the unmarried males of the 2-year groups of ages (15 and 16 to 43 and 44), to the number of all the unmarried females of the 2-year groups (13 and 14 to 43 and 44). The lower figures are the values of the probability-function, ${}_2q_{xy}$, for the double 2-year groups, by means of which the number of marriages occurring annually in the indicated age-groups may be calculated by formulæ (431) and (432), (434), and (435).

TABLE LXVII.—Shewing the Masculinity (M/F) of the Unmarried in 5-year Age-groups (M/F), and the Probability-function, 1,000,000 ${}_5q_{xy}$, for calculating the number of Marriages in the 5-year Age-groups indicated.

AGE OF BRIDES AND NUMBER OF UNMARRIED FEMALES.

AGE OF BRIDEGROOMS AND NUMBER OF UNMARRIED MALES.

Age Groups.	10-14 211,433	15–19 214.875	20–24 152,967	25–29 78,036	30–34 44.341	35–39 29,953	40–44 21.483	45–49 15,006	50-54 9,734
15–19 229.382	1.085 5	1.067 1,964	1.500 1,027	2.939 150	5.173 37	7.658 11	10.677 9	15.29 7	23.57 7
20–24 201.906	0.955 26	0.940 13,850	1.320 40,184	2.587 13,821	4.553 3,068	6.741 1,054	9.398 280	13.45 152	20.74 44
25–29 120,243	0.569 15	0.560 8,932	0.786 47,073	1.541 46,906	2.712 14,445	4.014 4,309	5.597 1,239	8.013 512	12.35 212
30–34 67,650	0.320 4	0.315 3,897	0.442 22,202	0.867 29,880	1.526 21,665	2.259 8,006	3.149 2,305	4.508 921	6.950 260
35–39 45,616	0.216 5	0.212 1,906	0.298 10,601	0.585 16,323	1.029 15,687	1.523 12,726	2.123 4,724	3.040 1.970	4.686 633
40–44 36,868	0.174 4	0.176 618	0.241 4.033	0.472 6,800	0.832 8,721	1.231 9,356	1.716 7,127	2.457 3.259	3.787 1.084
45–49 29,858	0,141 4	0.139 376	0.195 1,907	0.383 3,439	0.673 4,664	0.997 6,875	1.390 6,452	1.990 5,656	3.067 2,551
50-54 21,945	0.104 4	0.102 188	0.144 812	0.281 1,647	0.495 2.415	0.733 3,746	1.022 4,399	1.462 4,794	2.255 4,041
55–59 13,960	0.066 5	0.065 98	0.091 401	0.179 858	0.315 1,399	0.466 2,062	0.650 2,932	0.930 3.700	1.434 3,415
60-64 9,660	0.046 6	0.045 68	0.063 306	0.123 642	0.218 980	0.322 1,715	0.450 1.658	0.644 2,243	0.992 2,668
65-69 7,108	0.034 6	0.033 31	0.046 198	0.091 379	0.160 584	0.237 876	0.331 1,141	0.474 1.514	0.730 1,992
70–74 5.650	0.027 6	0.026 28	0.037 137	0.072 221	0.127 349	0.189 507	0.263 734	0.376 944	0.580 1,172
75–79 3,430	0.016 7	0.016 27	0.022 67	0.044 127	0.077 181	0.115 247	0.160 321	0.229 399	0.352 536
80–84 1.362	0.006 9	0.006 45	0.009 123	0.017 238	0.031 247	0.046 335	0.063 519	0.091 635	0.140 703
85–89 480	0.002 ..	0.002 26	0.003 51	0.006 75	0.011 124	0.016 195	0.022 382	0.032 697	0.049 565
90–94 105	.0005 ..	.0005 ..	.0007 ..	.0013 ..	.0024 ..	.0035 115	.0049 173	.0070 229	.0108 339

See Note to the continuation of the Table on the next page.

TABLE LXVII.—Shewing the Masculinity (M/F) of the Unmarried in 5-year Age-groups, and the Probability-function, 1,000,000 $_5q_{xy}$, for calculating the number of Marriages in the 5-year Age-groups indicated. (*Continued*)

Age of Brides and Number of Unmarried Females.

Age of Bridegrooms and Number of Unmarried Males.

Age Group.	55–59 5,698	60–64 3,645	65–69 2,505	70–74 1,405	75–79 695	80–84 347	85–89 110	90–94 23
15–19 229382	40.26 6	62.93 ..	91.57 ..	163.3 ..	330.1 ..	661.0 ..	2085 ..	9973 ..
20–24 201906	35.43 18	55.39 10	80.60 5	143.7 ..	290.5 ..	581.9 ..	1835 ..	8778 ..
25–29 120243	21.10 32	32.99 31	48.00 23	85.58 ..	173.0 ..	346.5 ..	.1093 ..	5228 ..
30–34 67,650	11.87 72	18.56 59	27.01 43	48.15 ..	97.34 ..	194.9 ..	615.0 ..	2941 ..
35–39 45,616	8.006 260	12.51 113	18.21 64	32.47 8	65.63 ..	131.5 ..	414.7 ..	1983 ..
40–44 36,868	6.470 442	10.11 249	14.72 126	26.24 16	53.05 ..	106.2 ..	335.2 ..	1603 ..
45–49 29,858	5.240 1,236	8.192 651	11.92 328	21.25 77	42.96 16	86.05 ..	271.4 ..	1298 ..
50–54 21,945	3.851 1,934	6.021 1,321	8.760 636	15.62 150	31.58 145	63,24 34	199.5 ..	954.1 ..
55–59 13,960	2.450 3,030	3.830 1,680	5.573 1,147	9.936 338	20.09 290	40.23 164	126.9 ..	607.0 ..
60–64 9,660	1.695 2,835	2.650 2,769	3.856 2,257	6.875 1,039	13.90 405	27.84 319	87.82 107	420.0 ..
65–69 7,108	1.248 2,572	1.950 2,585	2.838 2,081	5.059 1,287	10.23 582	20.48 467	64.62 212	309.0 ..
70–74 5,650	0.992 1.425	1.550 1.437	2.256 1.315	4.021 1.476	8.129 896	16.28 610	51.36 314	245.6 ..
75–79 3,430½	0.602 764	0.941 979	1.369 957	2.441 970	4.935 1,094	9.885 958	31.18 406	149.1 465
80–84 1,362½	0.239 610	0.374 475	0.544 295	0.969 207	1.960 171	3.925 163	12.38 281	59.22 ..
85–89 480	0.084 341	0.132 205	0.192 99	0.342 59	0.691 45	1.383 31	4.364 ..	20.87 ..
90–94 105	0.184 221	0.288 161	.0419 104	.0747 ..	.1511 ..	.3026 ..	.9545 ..	4.565 ..

Note.—The upper figures denote the masculinity of the group, that is, the ratio of the number of all the unmarried males of the 5-year groups of ages (15 to 19) to (90 to 94), to the number of all the unmarried females of the 5-year groups (13 to 14) to (90 to 94). The lower figures are the values of the probability-function, 1,000,000 $_5q_{xy}$, for the double 5-year groups, by means of which the number of marriages occurring annually in the indicated age-groups may be calculated by formulae (431) and (432), (434), and (435).

31. **The probability of marriage according to pairs of ages.**—Assuming that the "conjugal potential" does not change in any community, the number of marriages likely to occur among groups of the unmarried of given ages can be computed by means of formula (434), at least if the masculinity is at all similar to that shewn in Tables LXVI. and LXVII. These tables give also the values of q_{xy} as well as the masculinity.

If the conjugal potentials are the same for A males and B females as for B males and A females, and the law of variation is, as by hypothesis,

$$(437)\ldots\ldots(\gamma + \gamma') \propto M^{\phi_2} \, . \, F^{\mu_2} = M\phi_1^{\mu_2} = F\mu_1^{\phi_2}$$

then the qualification as to the masculinity being approximately identical disappears.[1] It is not unimportant, however, to remember that the fundamental assumption would have to be very erroneous (and that would seem to be impossible) in order to seriously prejudice the precision of the result obtained by the application of the formula (434). The error in any real application of the formula can be a differential one only, and if the constitution as regards numbers of the population be approximately therefore that from which it was derived, any defect in the theory of variation with relative numbers of the sexes, formula (430), has no sensible effect.

32. **The relative numbers of married persons in age-groups.**—The Census of 1911 disclosed the fact that the number of married persons living together on the night of the 3rd April, 1911, was 623,720. The number of wives absent from their husbands was 112,129, and husbands absent from their wives 110,053. There were 616,738[2] (out of a total of about 734,000 married couples) whose ages were fully specified, and who were living together.

This may not be a perfect sample of the entire population, for although the date of the Census, viz., 3rd April, is well chosen, the number of spouses of each age apart at a given moment is probably not sensibly proportional to the total number. As the totals, however, are only about one-fifth greater than the number for which the information is complete, the 616,738 may be taken as fairly representing the population. The results are shewn upon Table LXVIII.

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[1] μ_1 and ϕ_1 are the same as μ and ϕ above ; μ_2 and ϕ_2 are defined in Table XXI. p. 132 hereinbefore.

[2] This number is made up as follows :—

		Living together but Wife's Age not stated.	Living together but Husband's Age not stated.	Wives Absent.	Total Husbands.
Husbands and wives completely specified as to age, and living together ..	616,738				
Both ages unspecified ..	506				
	617,244 +	4,108 +	2,368 +	112,129 =	735,849
Living together but wife's age not stated	4,108				‖
Living together but husage not stated	2,368				1.19313
Husbands absent	110,053				×
Total wives .. =	733,773	=	1.18976		616,738

TABLE LXVIII.—Number of Married Persons per 1,000,000 Married Couples, Living Together on the Night of the Census, 3rd April, 1911. In 5-year Age-groups.

Husbands' Ages.	Wives' Ages. 10 to 14	15 to 19	20 to 24	25 to 29	30 to 34	35 to 39	40 to 44	45 to 49	50 to 54	55 to 59	60 to 64	65 to 69	70 to 74	75 to 59	80 to 84	85 to 89	90 to 94	95 to 99	Total, 10 to 99
15–19	..	577	347	39	8	3	..	..	..	..	..	..	..	..	..	..	..	..	974
20–24	8	5,771	24,015	7,168	1,090	217	63	28	6	..	..	..	..	..	..	..	..	..	38,366
25–29	2	3,574	40,354	54,338	11.871	2,015	383	112	44	11	2	1	..	..	..	..	..	..	112,707
30–34	..	1,090	17,907	54,009	54,757	12,145	2,264	516	123	29	11	5	2	..	..	..	..	..	142,858
35–39	..	376	5.845	24,489	51,157	47,891	10,786	1,965	379	89	16	11	3	2	..	..	..	..	143,009
40–44	..	130	2,048	9,082	25,695	47,680	44,462	9,936	1,934	452	92	36	10	3	..	..	..	..	141,560
45–49	..	44	760	3,287	9,510	23,654	43,595	40,083	8,644	1,450	340	96	16	10	..	..	..	..	131,489
50–54	..	24	258	1,090	3.124	7,694	19,245	35,589	29,716	5.800	1,138	311	50	13	3	..	..	..	104,055
55–59	..	11	94	334	921	2,380	5,567	13,677	22,851	16,769	3,478	666	154	41	11	..	..	..	66,954
60–64	..	5	45	135	357	798	1,899	4,506	9,790	13,578	10,622	2,330	478	81	18	2	1	..	44,645
65–69	..	..	23	62	156	413	830	1,840	4,081	6,684	9,571	7,639	1,629	292	42	8	..	..	33,270
70–74	..	..	8	26	58	180	319	718	1,505	2,615	4,405	6,040	4,533	1,004	118	15	6	2	21,552
75–79	..	2	5	23	29	57	131	268	517	820	1,600	2,996	3,322	2,238	399	37	8	..	12,452
80–84	..	2	2	3	15	24	42	79	152	227	472	751	1,156	1,111	655	84	26	..	4,801
85–89	..	..	2	..	2	6	10	16	28	53	84	148	198	267	183	91	6	..	1,094
90–94	..	..	..	2	..	..	2	6	1	3	8	34	37	31	36	18	15	..	183
95–99	..	..	..	..	..	..	..	..	..	2	2	5	5	5	6	3	..	..	28
100–104	..	..	..	..	..	..	..	..	..	2	..	1	..	..	..	..	..	..	3
Totals 15–104	10	11,606	91,713	154,087	158,750	145,157	129,598	109,339	79,771	48.584	31,841	21,070	11,593	5,098	1,471	258	52	2	1,000.000

33. **Conjugal age-relationships.**—For certain estimations it is important to know, for given ages of husbands, the average difference of the age of the wives ; and also for given ages of wives the average differences of the ages of the husbands. These relationships as at marriage, *i.e.*, *initially*, may be ascertained from marriage records. They may be called the *protogamic age-relationships*. The instantaneous relationships at any moment, however, are disclosed only by a Census, and may be called the *gamic age-relationships*.

The age-groups, with the age of the husband as argument, and those with age of wife as argument, lead, it will be found, to different results, which have no obvious direct mutual relation. Hence this, in common with other analogous groupings of a non-homogeneous character, must be independently made, for a reason which we shall now more definitively indicate. In cases of the kind under consideration two formulae are needed ; in one the argument is the age of the husband (or bridegroom), in the other the age of the wife (or bride).

34. **Non-homogeneous groupings of data.**—If, associated with any group-range, viz., x_k to x_{k+1} say, of any class of elements (ages of husbands in the case under review), there is a class of related elements (ages of wives), viz., y_{k-a} to y_{k+b} say, where a and b, in general, have large values ; and if, reciprocally, a group-range, y_k to y_{k+1} say, is associated

with the group x_{k-A} to x_{k+B} say. A and B also having large values, the result obtained from the former will have no simple relation with that based on the latter. For a result based on the argument x, has not the same constitution as one based on the argument y. If the distribution about the mode in such cases be not symmetrical in each, in fact *if it be not similar in all respects*, no direct functional relationship subsists between results for groupings arranged according to the values of x, and those for groupings arranged according to the values of y. Groupings subject to this limitation may be called *non-homogeneous groupings*, and require special consideration.

35. **Average differences in age of husbands and wives, according to Census.** — In Chapter XIX., Vol. I., § 2, of the Report on the Australian Census of 1911, results are given for a series of age-groups of husbands and of wives. The results are also given in greater detail in Vol. III., Table I., pp. 1106-7. The difference for the central-age of the group, which is sensibly, though not exactly, the mean-age, of those included therein, is as shewn on Fig. 64,[1] the curve marked A, representing the excess of the age of husband over the average age of their wives, as determined from groupings according to the age of the husbands, and the curve marked B, representing the excess of the age of the wife over the average age of their husbands, as determined from groupings according to the age of the wives.

The differences are given in Table LXIX. hereunder. The tangent line to curve A is coincident with the curve for the ages 40 to 60 inclusive (beginning point of year): hence for this interval the relation is—

$$(438) \ldots\ldots D_w = 0 + 0.098\, x_h, \text{ for ages 40 to 60,}$$

D_w denoting the average excess in years of the age of the husband over the average age of the wives, and x_h being the age of the husband.

The tangent is coincident with curve B for the ages 30 to 67 inclusive, and the age of the wife is greater than the average age of the husbands by the amount D_h, where

$$(439) \ldots\ldots D_h = -6.275 + 0.058\, x_w, \text{ for ages 30 to 67.}$$

in which x_w denotes the age of the wife. It is obvious from the table that the assumption ordinarily made is invalid. The characteristics of a table of values of the differences will be evident from the table itself.

[1] See page 227.

TABLE LXIX.—Differences of the average Age of Wives for Husbands of various Ages, and of the Average Ages of Husbands for Wives of various Ages. Australia, 1911.

Age of Husband A; Wife B.	Calculated Result, Curve A.				Calculated Result, Curve B.			
	Position of Tangent	Ordinate to Curve.	Smoothed value of D_w.	Crude value from Data.	Position of Tangent	Ordinate to Curve.	Smoothed value of D_h.	Crude value from Data.
14½	+1.42	. .	. .	. .	—5.43	. .	. .	—8.4
15½	1.52	—6.52	—5.00	—5.0	5.38	—5.02	—10.40	10.4
16½	1.62	5.27	3.65	0.9	5.32	3.07	8.39	9.1
17½	1.72	4.52	2.80	2.8	5.26	2.25	7.51	7.5
18½	1.81	3.78	1.97	1.2	5.20	1.76	6.96	7.2
19½	1.91	3.48	1.57	1.1	5.14	1.44	6.58	6.6
20½	2.01	3.12	1.11	0.6	5.09	1.16	6.25	6.2
23	2.25	2.38	—0.13	—0.4	4.84	.80	5.70	5.7
27½	2.70	1.33	+1.37	+1.2	4.68	—.17	4.85	4.7
30 †	2.95	.90	2.05	. .	4.54	.0	4.54	. .
32½	3.19	.56	2.63	2.5	4.39	.0	4.39	4.4
37½	3.67	.10	3.57	3.6	4.10	.0	4.10	4.1
40 *	3.92	.0	3.92	. .	3.96	.0	3.96	. .
42½	4.16	.0	4.16	4.2	3.81	.0	3.81	3.8
47½	4.66	.0	4.66	4.7	3.52	.0	3.52	3.4
52½	5.15	.0	5.15	5.2	3.23	.0	3.23	3.1
57½	5.64	.0	5.64	5.8	2.94	.0	2.94	3.0
60 *	5.88	.0	5.88	. .	2.80	.0	2.80	. .
62½	6.13	.08	6.21	6.5	2.65	.0	2.65	2.9
67 †	6.55	.19	6.74	. .	2.50	.0	2.50	. .
67½	6.61	.20	6.81	7.3	2.36	.08	2.28	2.3
72½	7.11	0.66	7.73	8.1	2.07	.70	—1.37	—1.3
77½	7.60	1.58	9.18	9.2	1.78	1.96	+0.18	-0.4
82½	8.09	3.14	11.23	11.3	1.49	3.76	2.27	2.2
87½	8.58	5.70	14.28	14.4	1.20	6.70	5.50	4.2
92½	9.07	9.10	18.17	18.6	0.91	12.01	11.10	11.1
97½	9.56	14.90	24.46	22.3	0.62	25.62	25.00	25.0
102½	10.05	29.95	40.00	40.0	—0.33	. .	. .	. .

*† The asterisks and daggers denote the ages between which curves A and B, respectively, are straight lines.

In the figure the curves A and B are very approximately the smoothed values. The tangents are shewn by dotted lines; the data by the dots; it is instantly evident that the difference is not constant, but is a definite function of age. A and B are the curves of the gamic age-relationship.

36. **Average differences of age at marriage.**—A similar table to the preceding can be constructed for the ages at marriage. In order to eliminate the uncertainties due to paucity of data the results for the eight years 1907 to 1914 were combined. The combinations shewed the same tendency as was revealed by the Census, viz., for the numbers to be unduly large for the ages ending with the digits 0 and 5. The numbers for the purpose of the following table have, however, not been smoothed; the smoothing in the table itself making that unnecessary.

Differences between Ages of Husbands of any Age and the Average Ages of their Wives, and between the Ages of Wives and the Average Ages of their Husbands.

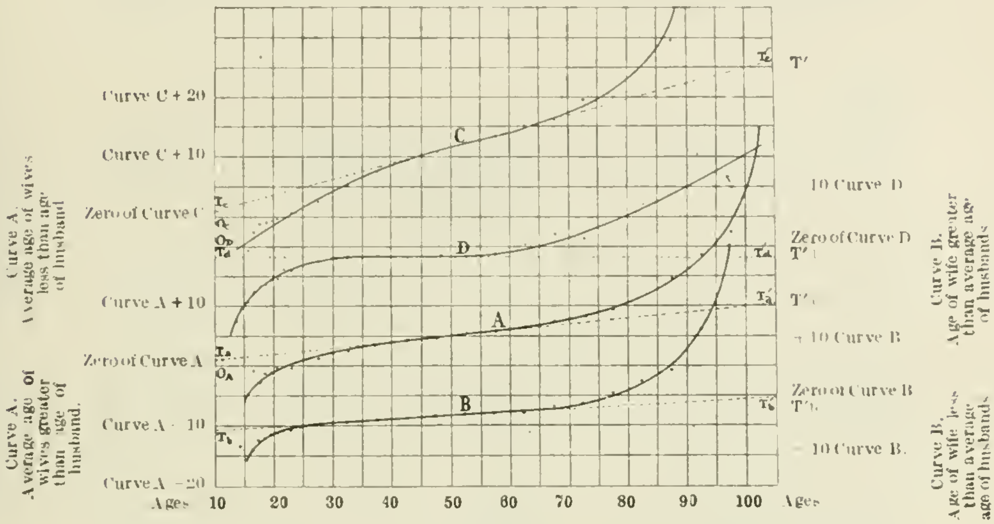

Fig. 64.

Curve A.—Excess of the husband's age over the average age of their wives, at the 1911 Census. See Table LXIX., p. 226. O_A is the zero for the curve.

Curve B.—Excess of the wife's age over the average age of their husbands, at the 1911 Census. See Table LXIX., p. 226. O_B is the zero for the curve.

Curve C.—Excess of the bridegroom's age over the average age of their brides, 1907-1914. See Table LXX., p. 228

Curve D.—Excess of the bride's age over the average age of their bridegrooms. See Table LXX., p. 228.

The results are shewn by curves C and D in Fig 64. The tangent to curve C, which is analogous to curve A, is identical with the results for ages 42½ to 67½ years; thus :—

(440)...... $D'_w = -1.745 + 0.266\, x_h$; for ages 42½ to 67½.

For curve D, the difference of ages is analogous to curve B. The tangent is parallel to the age-axis at the distance

(441)...... $D'_h = -1.76$; for ages 32½ to 60.

The table shews the differences outside these limits.

Towards the ends of the curves the results for all four curves are of course somewhat uncertain. C and D are the curves of the protogamic age-relationship.

TABLE LXX.—Difference of the Average Age of Brides for Bridegrooms of various Ages, and of the Average Age of Bridegrooms for Brides of various Ages.

Age of Bridegroom C; Bride D.	Calculated Result, Curve C.				Calculated Result, Curve D.			
	Position of Tangent	Ordinate to Curve.	Smoothed value of D'_w.	Crude value from D'_w.	Position of Tangent	Ordinate to Curve.	Smoothed value of D'_h.	Crude value from Data D'_h
13½	..	..	..	..	1.76	11.04	12.80	12.80
14½	..	..	..	..	1.76	8.45	10.21	10.21
15½	+2.38	—5.35	—2.97	—5.50	1.76	7.10	8.86	9.18
16½	2.64	5.08	2.44	2.36	1.76	6.10	7.86	7.86
17½	2.91	4.85	1.94	1.08	1.76	5.24	7.00	6.95
18½	3.18	4.56	1.38	0.81	1.76	4.50	6.26	6.25
19½	3.44	4.35	0.91	0.37	1.76	3.92	5.68	5.66
20½	3.71	4.08	—0.37	—0.18	1.76	3.42	5.18	5.26
23	4.37	3.49	0.88	+0.49	1.76	2.24	4.00	3.94
27½	5.57	2.52	3.05	2.72	1.76	.70	2.46	2.46
32½†	6.90	1.48	5.42	5.35	1.76	.00	1.76	1.76
37½	8.23	.56	7.67	7.67	1.76	.00	1.76	1.72
42½*	9.56	.00	9.56	9.45	1.76	.00	1.76	1.91
47½	10.89	.00	10.89	10.95	1.76	.00	1.76	1.66
52½*	12.22	.00	12.22	12.30	1.76	.00	1.76	1.75
57½	13.55	.00	13.55	13.42	1.76	.00	1.76	1.31
60 †	..	..	..	..	1.76	.00	1.76	..
62½	14.88	.00	14.88	15.03	1.76	.06	1.82	.30
67½	16.21	.00	16.21	16.16	1.76	.28	2.04	2.08
72½	17.54	.90	18.44	19.52	1.76	.73	2.49	1.31
77½	18.87	2.30	21.17	19.93	1.76	1.54	3.30	5.83
82½	20.20	4.50	24.70	37.05	1.76	3.00	4.76	7.14
87½	21.53	8.09	29.62	29.62	1.76	5.30	7.06	..
97½	22.86	..	..	..	1.76	..	..	10.00

*† The asterisks and daggers denote the ages between which the curves C and D, respectively, are straight lines.

37. **The gamic surface.**—The data furnished in Table LXVIII. may be used to construct the *gamic surface*, on the same principle as was followed in the construction of the protogamic surface, dealt with in § 25 hereinbefore. The results are shewn on Fig. 65, from which it will be seen that the isogams are more elliptical in form than isoprotogams, and are more regular; see Fig. 61. The principal meridians AB, AC and AD, AE are in much the same positions as on the protogamic surface, but the point of maximum frequency A, and the line of greatest slope are for higher ages than on that surface. The interpretation of the curves is, *mutatis mutandis*, the same as that for the isoprotogams; in the case of Fig. 65, however, everything applies to persons "living in the state of marriage," instead of to "persons at the moment of marrying."

Curves of Equal Conjugal Frequency.—The Gamic Surface, 1911.

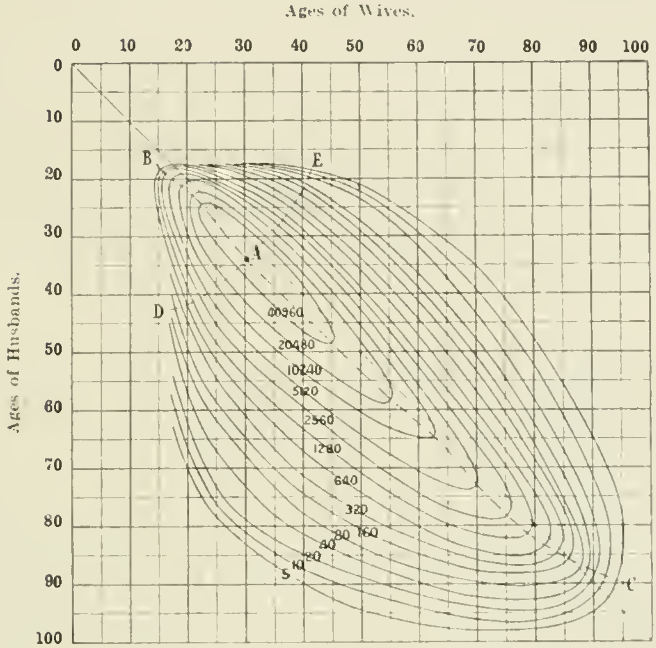

Fig. 65.

NOTE.—The pairs of ages for which an equal frequency of married couples existed at the Census of 1911 are found by following the course of any isogam. The remarks in the footnote to Fig. 61, p. 209, apply, *mutatis mutandis*, to the contours of the Gamic Surface.

38. **Smoothing of surfaces.**—Let it be supposed that the nature of statistical data is such that the most suitable representation is by means of the heights of series of parallelepipeds, as for example, in the case just considered, of the numbers of marriages of bridegrooms between given age limits and of brides between the same or other given age limits. For simplicity we may assume that the combination is according to age last birthday, and thus is in single year groups. Since the general equation of a surface of a second degree will involve nine constants, we can deduce the constants of a surface representing its integral between the limits $x = 0, 1, 2$, and 3, and $y = 0, 1, 2$, and 3, the deduced expression will give totals corresponding to those of the nine contiguous groups. By means of the corresponding surface equations, deduced from these, for lines parallel to the x-axis, or parallel to the y-axis, we can find the height to

this surface, along the four edges of the central parallelepiped. If this operation be then repeated, making each of the four adjoining parallelepipeds the central ones in a group, we shall obtain a second series of values for the distances along the four edges to the surface ; if these do not differ very greatly then the means of each pair of values may be taken, in general, as the smoothed result. In this way the greater part of the entire surface can be dealt with, and the series of verticals to the surface thus found will have reduced the original irregularities, and may be regarded as a first smoothing of the surface, conforming, however, as nearly as possible to the general series of group-heights. The results so obtained, however, are "instantaneous values," that is, they are the heights corresponding to the ranges x to $x + dx$, and y to $y + dy$.

If the numbers be very irregular the process above indicated is extremely tedious, and of little value. It may then be preferable to regard the group results as vertical ordinates with the central values of the group-ranges as the horizontal co-ordinates. The procedure then involves the independent smoothing of a double system of curves, and the taking throughout of the means of the pairs of verticals so found. The whole procedure is then repeated, with the means thus obtained, until the smoothing is satisfactory. The criterion of good smoothing is that the "accumulated deviations" in either of the two directions (at right angles to one another) do not attain to appreciable values, and that they alternate in size. It should be noted that smoothing in this way does *not* give "instantaneous values," that is where k is the extent of the range, the heights now denote values true for the ranges $\frac{1}{2}k$ on either side of the values x and y, these being the ordinates of the centre of the ranges.

There is another possible scheme of solution, *viz.*, to ascertain the constants of an equation, which will give at once the group values for groups of the same double-range, the arguments being the ordinates of the centres of the groups. The method is analogous to that treated for a surface in Part V., § 10, formulæ (211) to (216), pp. 72-73, and the solution by a process analogous to that indicated in the section immediately following, will give the group-height for any value of x and y, the range being $x \pm \frac{1}{2}k$, $y \pm \frac{1}{2}k$.

39. **Solution for the constants of a surface representing nine contiguous groups.**—The most general expression for a surface, every section of which parallel to the x-axis and parallel to the y-axis is a curve of the second degree is

$$(442)\ldots z = A + Bx + Cy + Dxy + Ex^2 + Fx^2y^2 + Gy^2 + Hx^2y + Ixy^2$$

Let the values of the groups be denoted by the letters l, $m \ldots\ldots t$, according to the following scheme :—

	$0\ y = 0$	$y = 1$	$y = 2$	$y = 3$
$x = 0$				
		l	o	r
$x = 1$				
		m	p	s
$x = 2$				
		n	q	t
$x = 3$				

The integral of the above, divided by xy, the area of the base, is :

$$(443)\ldots\ \frac{1}{xy}\iint F(x,y,k)\,dx\,dy = A + \frac{1}{2}Bx + \frac{1}{2}Cy + \frac{1}{4}Dxy$$
$$+ \frac{1}{3}Ex^2 + \frac{1}{3}Fy^2 + \frac{1}{6}Gx^2y + \frac{1}{6}Hxy^2 + \frac{1}{9}Ix^2y^2$$

from which we deduce, by putting x (or y) successively 1, 2, 3, and making y (or x) equal 1, 2, or 3, the following values of the constants A to I in terms of $l, m, \ldots\ldots t$. The results are :—

$$(444)\ldots\ A = -(q+o-2p) - (n+l-2m) + (l-p) - 3(s-p)$$
$$+ (t-q) + 3(r-o) + (p-o) - \frac{1}{2}(m-l)$$

$$(445)\ldots\ B = 3(q+o-2p) - 3(n+l-2m) + 8(s-p) - 3(t-q)$$
$$- 5(r-o) - 2(p-o) + (m-l)$$

$$(446)\ldots\ C = 2(q+o-2p) + 2p - 2(p-o) + 9(s-p) - 9(r-o) - 3(t-q)$$

$$(447)\ldots\ D = 4(p-o) - 6(q+o-2p) - 24(s-p) + 15(r-o) + 9(t-q)$$

$$(448)\ldots\ E = \frac{3}{2}(n+l-2m) - \frac{3}{2}(q+o-2p) - 3(s-p)$$
$$+ \frac{3}{2}(t-q) + \frac{3}{2}(r-o)$$

$$(449)\ldots F = \frac{9}{2}(r - o) - \frac{9}{2}(s - p) + \frac{3}{2}(t - q)$$

$$(450)\ldots G = -\frac{9}{2}(t - q) - \frac{9}{2}(r - o) + 9(s - p) + 3(q + o - 2p)$$

$$(451)\ldots H = 12(s - p) - \frac{15}{2}(r - o) - \frac{9}{2}(t - q)$$

$$(452)\ldots I = \frac{9}{4}(t - q) + \frac{9}{4}(r - o) - \frac{9}{2}(s - p)$$

It will be seen that the arithmetical labour of deducing the constants of a surface which will exactly reproduce any square system of 9 contiguous group-values, is very great, and ordinarily prohibitively so. In general, therefore, less rigorous methods have to be adopted, and are ordinarily quite satisfactory, particularly in view of the fact that in practical calculations values according to a given double-range are required.

40. **Nuptiality and conjugality norms.**—It would appear desirable to establish decennially, what may perhaps be called a *nuptiality or protogamic norm*, and also a *conjugality or gamic norm*, on the basis of an aggregation of the marriages of a large number of populations for the former; and of the Census results for the latter. The norms should preferably shew single-year results up to 24 years for brides, and 29 years for bridegrooms; and up to 34 years for wives, and 39 years for husbands, respectively.

The protogamic norm will reflect the trend in regard to the early institution of marriage, and the gamic norm the modification of this by change in longevity, the frequency of divorce, etc. These norms could include the curves of the totals according to the age of the males (bridegrooms and husbands), and according to the age of the females (brides and wives), and could include also the frequency of the group-pairs.

The norms of the conjugal state, "never married," "divorced," and "widowed," might, with advantage—as well as those of the "married"—also give the frequencies according to group-pairs.

41. **The marriage-ratios of the unmarried.**—It has already been shewn that the probability of marriage depends, among other things, upon the relative numbers among the unmarried of the sexes. So long, however, as a population does not greatly change its constitution according to sex and age, the crude probability of marriage according to sex and age may be regarded as varying approximately as the annual rate. This probability may be called the *peithogamic coefficient*[1] for the sex and age in question. It will be further discussed in Part XIII. in connection with fecundity.

[1] From πείθω to prevail upon, (Πειθώ the Goddess of Persuasion) and γαμικός, of or for marriage.

XIII.—FERTILITY AND FECUNDITY AND REPRODUCTIVE EFFICIENCY.

1. **General.**—The phenomena which directly concern the measure of the reproductive power of the human race will be dealt with in this part. These phenomena are in general complex, the variation of the reproductive power being in part of physiological origin, and in part of the result of the reaction of social traditions upon human conduct. This will appear in any attempt to determine the laws of what has been called bigenous[1] (better, digenous) natality, or natality as affected by the ages of both parents, as distinguished from those affecting merely monogenous natality, or natality as related to the producing sex. In deducing the most probable value for certain of the phenomena it will be necessary to minimise the effect of misstatement of age. This can probably be done more effectually than would at first sight appear probable. The final results, however, must be subject to some small degree of uncertainty.

The question of the reproductive efficiency of a population has in part been dealt with in Parts XI. and XII., dealing with Natality and Nuptiality; this, however, is derivative and depends in its turn upon the age-distribution and conjugal condition of the producing sex.

Many questions concerning the measurement of fertility and fecundity can be settled with sufficient precision without recourse to a differentiation depending on the age of the father, the better in Australia, perhaps inasmuch as the decay of virility with age is not well marked, and in this aspect the digenous fertility stands in marked contrast with that of Hungary.

2. **Definitions.**—It is desirable, initially, to define the sense in which several terms will be used hereinafter.

Monogenous fertility and monogenous fecundity will denote the fertility and fecundity of the female considered without regard to the age of the associated male.

Digenous fertility and digenous fecundity will denote the fertility and fecundity of the female, as modified by the age of the associated male, and therefore is considered in relation to the ages of both males and females. Consequently computations of monogenous fertility or fecundity will be based upon the age of the female. It follows from this, that two populations will be (i.) exactly, or (ii.) approximately, comparable, only when the conjugal age-relationships are (i.) sensibly identical, or (ii.) are similar.

[1] By Joseph Körösi, see Phil. Trans. Lond. B., 1895, p. 781.

Isogeny will denote either equal fertility or equal fecundity, the former to be called *initial isogeny or isoprotogeny*; the latter *general isogeny*, or *characteristic isogeny*, or simply *isogeny*.

A curve, passing through a series of pairs of ages plotted as co-ordinates, in such a manner that it will pass through all ages which give either equal initial or equal general fertility or fecundity, will be called an *isogen* as applied to either. The curves may therefore, in the cases considered, be called *isoprotogens*, and *isogens*.

The terms "fertility" and "fecundity"[1] though ordinarily sensibly identical in meaning, have sometimes been assigned different meanings by statisticians, one being employed to signify the qualitative, and the other the quantitative, aspect of reproductivity.

Owing to their phonic resemblance the words "sterility" and "fertility" are the more appropriate to employ in order to denote the difference between producing or non-producing; while "fecundity," which biologically is used without qualificative to imply *producing in great numbers* (a meaning which requires the qualification "*great*" when fertility is used), is obviously the more appropriate word to denote "multiple fertility."[1]

[1] In Latin, although "fertilitas" and "fecunditas" have no marked difference of meaning, the latter word seems to be the preferable one for denoting frequency of bearing offspring. The root of fecundus is "feo" (obsolete), or FE = Greek $\phi\upsilon$; c.f. Sanskrit bhu; Zend bû: see φύω Liddell and Scott's Greek-English Lexicon, 8 Edit., p. 1703.

The root of "fertilis" is "fero" = Greek root $\phi\epsilon\rho$: c.f., Sanskrit "bhar"; Zend "bar"; A.S., "bear-n"; the radical meaning being to bear or carry. See Liddell and Scott op. cit., p. 1662.

In regard to "sterilitas," c.f., Sanskrit "stari" (vacca sterilis).

In other languages the following correspondence might be suggested:—

English.	French.	Italian.	German.	Danish.	Swedish.
Fertility;	Fertilité;	Fertilità:	Fruchtbarkeit or Gebarfähigkeit	Frugtbarhed	Fruktsamhet;
Fecundity.	Fécondité.	Fecondità.	Ergiebigkeit or Fruchtbarkeit	Avledygtighed	Afvelsamhet.

Inasmuch "Fruchtbarkeit," "Frugtbarhed" and "fruktsamhet" ought, if possible, to be appropriated to the one meaning, the first suggestion as regards the German is to be preferred. That is, it is better to adopt "Fruchtbarkeit" for fertility and "Ergiebigkeit" for fecundity.

Körösi suggests "Ergiebigkeit der Ehen." "Mutterschaftsfrequenz" and "Maternitätsfrequenz" refer only to cases of maternity.

J. Matthews Duncan, in his "Fecundity, fertility, sterility and allied topics," 1866, 2nd Edit., 1871, has used "fecundity" to imply the quality of producing "without any superadded notion of quantity," and "fertility or productiveness" "the amount of births as distinguished from the capability to bear." For the reasons indicated in the text, it is better to adopt the terms "sterile" and "fertile" as contrasted, that is, as meaning "non-productive" and "productive" without reference to quantity, and the term "fecund" as conveying the idea of quantity. The matter seems of sufficient importance to abandon Duncan's usage.

Physiological or potential fecundity is, at present, not ascertainable; what is discoverable is only *actual fecundity*. Both rise to a maximum and fall away, the latter very early in life, while it is improbable that this is true of the former. The difference is theoretically (and of course practically) important. The following definitions make the matter clear:—

(i.) *Physiological fecundity at a given age* is the probability that a female of that age, subject to a *definite degree of physiological risk*, uniform for all ages, will reproduce.

(ii.) *Actual fecundity at a given age* is the probability that a female of that age, *subject to average actual risk* (as modified by social traditions, etc., and also by reproduction itself, and not necessarily uniform for all ages), will reproduce.

Inasmuch as physiological fecundity is probably not identical in populations of different races or nations, or even in populations of different localities and times, and is, moreover, dependent upon general health and mode and standard of living, the obtaining of its measure is in a high degree important, though at present impracticable.

Actual fecundity is, naturally enough, different for married and unmarried females. While it does not, even with married females, measure without correction the urgency of the reproductive impulse, or in unmarried females measure the force which this impulse opposes to restrictions created by social environment, it throws, as we shall later see, important light on this question.

3. **The measurement of reproductive efficiency.**—The determination of an unequivocal method of measuring the reproductive efficiency of a population is not without difficulty for the following reasons, viz., that—

(*a*) The life of women varies in duration;

(*b*) The reproductive period is only a limited portion of it;

(*c*) Fertility and fecundity are neither uniform for all ages, nor for all women;

(*d*) It appears to be qualified by the age of the associated males;

(*e*) Marriage and child-bearing initiate at different ages;

(*f*) Reproductive efficiency must take account of the duration of life of the children; and that

(*g*) The exercise of the reproductive function is subject to adventitious influences.

By way of enforcing the penultimate point, it may be noticed that generally a high birth-rate is associated with a high rate of infantile mortality, and the rate measured by taking account only of survivors at the end of one year or other prescribed period may give quite a different indication to that derived from births only. The following outline of various schemes of measurement, some of which have already been dealt with, will indicate the nature and limitations of each.

Rate Measured by— Numerator.	Denominator.	Deduced Result known as—	Remarks.
Total births, B	Total population, P	Crude birth-rate, B/P	Is dependent on age, sex, and conjugal constitution of total population, and therefore not strictly comparable as between different populations; it measures merely one element determining increase.
Total births, B	Total female population, F	Birth-rate referred to total number of women, B/F	Is dependent on female population only and is affected of course by the age and conjugal condition of that population.
Total births, B	Female population of reproductive age (viz., from about 10 to 60), F', say	Birth-rate referred to women of reproductive age only B/F'	Indicates reproductive efficiency of all women within the reproductive period. Owing, however, to the limits of this period being ill-defined at the initial and terminal ages, to the largeness of the number of women at those ages, and to the fact that it is dependent on the age-constitution within the group chosen to represent the reproductive age, the rate is not as definite as is desirable. The denominator, however, is a good crude measure of the potential of reproductive efficiency of the population.
Births in each age-group, B_x	The women in same groups, F_x	Birth-rate referred to women of each age-group in question, B_x/F_x	Is uncertain for comparison because the ratio of married to unmarried women may vary, and the relative frequency of maternity in each is not identical.
Nuptial births in each age-group, B'_x	Married women in same age-group, M_x	Nuptial maternity rate for each age-group, B'_x/M_x	Shews only the average frequency of maternity (average probability of maternity) for married women in each age-group.
Ex-nuptial births in each age-group of unmarried women, B''_x	Unmarried women in age-group, U_x	Ex-nuptial maternity rate for each age-group, B''_x/U_x	Shews only average frequency of maternity (average probability of maternity) for unmarried women in each age-group.
Appropriately weighted sum of birth-rates of the married and unmarried	Unity	Modified "Nuptial Index of Natality"	This attributes the reproductive facts of an existing population to a supposititious "standard" population, in which the relative number of married and unmarried females is the general average (norm) for the groups of populations to be compared. The comparison so attained may be regarded a suitable comparative measure of reproductive efficiency (natality).

4. **Natality tables.**—The preceding methods of measuring productive efficiency are all more or less defective. A more satisfactory scheme is to construct a monogenous age-group "natality table" for married, and one for unmarried, females. Such tables shew for each age the probability of the occurrence of a birth and the average number of children per confinement: see hereinafter. This, without doubt, is a more definite method, and stands in much the same relation to statistics of births, as a mortality table does in relation to statistics of deaths. It is, however, not perfectly satisfactory, because, as already indicated, it would appear that the age of the father as well as that of the mother affects the probability of maternity. This will be dealt with hereinafter. Tables of digenous natality, *i.e.*, double-entry tables, shewing the natality for every combination of age, are more complete and exact, and would be perfectly so, if the fertility at any age were *unaffected* by the number of previous confinements. This, however, is probably not the case. These matters will be dealt with in the various sections and tables hereinafter in this part.

5. **Norm of population for estimating reproductive efficiency and the genetic index.**—In order to eliminate the effect of variations in the constitution of populations, it is desirable to establish on as wide a basis as possible the norm of its female conjugal constitution, preferably for every 5 years of the reproductive period. This norm would shew for a total of 1,000, 10,000 or 100,000, etc., women of all reproductive ages, the number aged 10-14, 15-19, ... 55-59; that is from the 10th to the 59th year of age inclusive.* For each age-group there would be (at least) two classes, viz., the "unmarried" which might include widows and divorcées not remarried, and the "married." If, then, to these numbers in the age-groups of the "married" we attribute the nuptial birth-rates* and compute the births, and to the "unmarried" we similarly attribute the ex-nuptial birth-rates†, which are actually experienced by any population considered, we shall have comparable measures; and the aggregate (divided if desired by 1,000; 10,000 or 100,000, etc.) will be the "Index of Natality" based on the women of reproductive age. In short, the birth-rates actually experienced in the various age-groups of females of reproductive ages, for a series of populations to be compared as regards reproductive efficiency, are attributed to a common standard population (the norm). The sums in the various cases are the comparable measures of reproductive efficiency. Symbolically this may be described as follows:— Let p_1 and p'_1, p_2 and p'_2, etc., denote the ratio of the married and of the unmarried respectively in age-groups 1, 2, etc., to the total number of women married and unmarried of reproductive ages in the norm or standard population; that is, to the total of all the reproductive groups of that population. Then the sum $p_1+p_2+\ldots.p'_1+p'_2+\ldots.=1$. Hence the index of natality, ν, which measures reproductive efficiency, is simply—

$$(453)\ldots\ldots\ldots\nu=\Sigma_{10}^{59}(p\beta)+\Sigma_{10}^{59}(p'\beta')$$

where β denotes the nuptial, and β' the ex-nuptial, birth-rate based upon the numbers of the married and unmarried respectively, and not upon the total population of each group. In practice these results may of course for convenience be actually multiplied by 1,000, or any higher number.

This index of reproductive efficiency we shall call the *genetic index*. It is formed in a manner identical with that adopted to determine the index of mortality.

6. **The Natality Index.**—Following a procedure similar to that dealt with in last section, let q_1 and q'_1, q_2 and q'_2, etc., denote the ratio in the standard population of the married and unmarried‡ respectively to the

* By dividing the nuptial births in each age-group by the mean number of married women in that group, b_m/M.

† By dividing the ex-nuptial births in each age-group by the mean number of unmarried women in that group; b_u/U. When desirable to distinguish them "never married" may be used instead of "unmarried," the latter would include "widowed" and "divorced."

‡ See preceding note.

total of the standard population. Then these quantities will be smaller than p_1, p'_1, etc., in the ratio of the sum of all females of reproductive age in the standard population to the total standard population, male and female. Hence if we attribute to each age-group-ratio the birth-rate experienced in the population to be compared, we get a total also smaller in the same ratio. This then would give the natality-index ν'. That is—

$$(454)\ldots.\nu' = \Sigma(q\beta) + \Sigma(q'\beta') = \frac{P'}{P}\,\nu$$

where P' denotes the females of reproductive age in the norm, and P denotes the total population, male and female, in the norm.

7. **Age of beginning and of end of fertility.**—The determination of the age at which fertility begins and ends is of importance, and also the range of the reproductive period, which, of course, may not extend in individual cases from the initial age to the terminal age for a large population. What will be discussed here is the latter. The limits may best be determined from the usual statistical data by considering the nature of the frequency as the limits are approached. Keeping in view the fact that the numbers from which the experience is drawn do not vary appreciably, the absolute numbers may preferably be used for judging the age-terminals We get, therefore, for the old-age limit the following results for the period from 1st January, 1907, to 31st December, 1914, for Australia, the population being nearly 5 millions.

TABLE LXXI.—Probability of Birth in Old-age, Australia, 1907 to 1914.

Line No.	Age of Mothers Nuptial and Ex-nuptial.	48	49	50	51	52	53	54	55	56	57	58	59	60	Totals.
1	No. of births in 8 years ..	322	113	39	13	6	5	3	2	1	1	0	0	0	505
2	Decrease at the rate of e^x ..	319	117.3	43.2	15.9	5.8	2.1	0.8	0.3	0.1	0.04	0.014	.0053	.0020	504.613
3	Decrease at varying rate..	322	113.4	42.0	16.4	6.8	3.0	1.4	0.7	0.4	0.2	0.1	0.069	0.053	506.522
4	Ratio of decrease	2.84	2.70	2.56	2.42	2.28	2.14	2.00	1.86	1.72	1.58	1.44	1.30		
5	"Equivalent number" of married women	16938	16105	15113	13898	12759	11716	10819	9940	8989	8071	7269	6608	6033	
6	Probability per 100,000* ..	2,377	877	323	117	59	53	35	25	14	15	0	0	0	
7	Married women of same age per annum† ..	2,377	877	320	117	71	49	35	25	17	10	4	? 2	? 1	

* Crude result. † Smoothed result, see formula (454).

The above results indicate that towards the end of the child-bearing period the numbers decrease (above 48 years of age) roughly at about the rate e^x, where x is the number of years ; see line 2. This at least holds from 48 to 52, when it would appear that the decrease is much more slow. A closer correspondence can be had by forming the numbers according to a formula varying the rate of decrease such as—

$$(455)\ldots\ldots n_{x+1} = \{2.84 - 0.14\ (x-48)\}\ n_x$$

where n_x denotes the number of mothers of age x, last birthday.

The figures in line 1 in Table LXXI. are 8 years' experience of nuptial and ex-nuptial births with women of from 48 to 60 years of age in Australia. During this time there were 476 of the former to 26 of the latter, the number of married and unmarried females of the ages mentioned being respectively 136,781 and 21,615, giving one case of maternity in 287.3 and 831.3 women, respectively. The frequency of maternity with unmarried women between the age-limits in question is thus 0.346 that of married women (or that of married women is 2.89 times that of unmarried women). If, therefore, we add to the number of married women 0.346 times the number of unmarried that will be the total "equivalent number" of married women to whom the cases of maternity can be ascribed. These, divided into one-eighth[1] of the numbers on line 1, give the crude probabilities of maternity for married women of the ages in question. The values, as calculated from the data, are given in line 6: the smoothed values obtained from these are given in line 7. Although a probability is given for age 58, the actual fact is that in over 7,000 possible cases (see line 5) no birth occurred: 57 is the greatest age at which a birth actually occurred. The values are shewn as curve A and on a larger scale, as curve B on Fig. 66. It will be noted that the continuation of the curve for ages 49 to 51 (see a b) on the figure, suggests that 53 is the age at which the value approaches zero, point c, and the curve for ages, 51 to 60, b d in figure, seems to be quite a different curve. No simple exponential relation, however, will bring these two curves under a single formula.[2] See page 244 for Fig. 66.

For the lower limit we have the following data, viz.:—

TABLE LXXII.—Probability of a Birth in Early Age, Australia, 1907-1914.

Line	Age.	11	12	13	14	15	16	17	18
1	Nuptial births, 8 years	0	0	4	30	170	1,138	4,062	11,761
2	Ex-nuptial births, 8 years	0	5	21	126	537	1,500	2,950	4,504
3	Total births, 8 years	0	5	25	156	707	2,638	6,942	14,265
4	Ratio of ex-nuptial to nuptial births ..	—	∞	5.2	4.2	3.16	1.32	0.73	0.38
5	Married women	..	..	1	18	93	349	1,145	2,551
6	"Never married" women	42,222	42,001	42,071	42,484	43,273	43,915	43,813	42,854
7	Probability of nuptial maternity per annum per 1,000	0	0	? 500	? 208	228	408	443	576
8	Probability of ex-nuptial maternity per annum per 1,000,000 unmarried women	0	1.6	6.5	37.1	155.1	427	850	1,313

[1] Approximately, see § 8, p. 240.

[2] Results deduced from the initial value 2377 by means of the formula—

$$n_{x+1} = \{2.75 - 0.15\ (x - 48)\}\ n_x.$$

would be in substantial agreement with (455), and are as follows:—

2377 864 332 136 59 27 14 7 4 3 2 1 1

They are less probable, however, than those given on line 7 in the table.

The results on line 8 do not need smoothing. Those on line 7 for the ages 13 and 14 are, of course, very uncertain, the normal values would probably be much smaller than 200. It is evident from the above, that the cases of ex-nuptial maternity throw most light upon the question of the commencing age of fertility. These are shewn on line 2, and will be given very nearly by the equation.[1]

$$(456) \ldots\ldots n_{x-1} = \{1.50 + 0.50\ (18-x)\}\ n_x.$$

The results are shewn as curve D, and on a larger scale as curve E, on Fig. 66, on page 244.

The general result of the investigation as to the terminal conditions is that the null-points can be taken as say 11 and 60, the values being very small from ages 53 onward, and from 11 to 12.[2] The initial null-point is consistent with the curve of frequency of the first menstrual appearance, which would give a null-point of about 9 years[3] and a maximum just after 16 years of age are attained. The curve as shewn in Fig. 66, curve C, gives, according to Whitehead, the group-numbers of single year age-groups for a total of 4,000 cases under observation. These group-numbers are shewn by small circles, see p. 244.

8. **The maternity-frequency, nuptial and ex-nuptial, according to age, and the female and male nuptial-ratios.**—Let g, m, and u, denote respectively the number per annum (i.) of brides, (ii.) cases of nuptial maternity; and (iii.) cases of ex-nuptial maternity, and also let M and U denote the number of married and " never married " women among whom the latter occur. These numbers are given for each age from 12 inclusive onward, in Table LXXIII., see columns (ii.), (iii.), (iv.), (vi.), and (vii.), or g, m, u, M and U.

The numbers are for 8 years, and the mean population from which they are drawn is about 8.0406 times that of the moment of the Census, viz., 3rd April, 1911. Hence the epoch can be regarded as the date of the Census, and the numbers have been divided by 8.0406 to obtain the annual equivalent.

[1] If we take 4500 as the number of ex-nuptial births for the age 18, we shall obtain 4.0, 27.7, 145.6, 545.8, 1500.3, 3000.0, and 4500, instead of the numbers shewn on line 3 in Table LXXII.

[2] At Budapest, J. Körösi records two mothers at 54, one at 56, and one at 57 in 4 years; vide, Phil. Trans. 1895, B., p. 794. In Edinburgh and Glasgow Matthews Duncan records for the ages 51, 52 and 57, and for an aggregate of 16,301 married mothers, 2, 4 and 1 respectively, p. 9 of his "Fecundity, Sterility, &c." 1871 Edit. C. Ansell in 1874, *vide* his " Statistics of Families," regards an alleged case at 59 as needing verification. Tauffer, of Budapest, in 2083 cases, records one at 54. In handbooks of Forensic Medicine, Casper-Liman mentions one case at 54; one is mentioned by Hofmann at 55; see Phil. Trans. loc. cit. C. J. and J. N. Lewis' " Natality and Fecundity," published 1906, out of 84,971 cases of births in Scotland in 1855, give for the ages 15, 16, 17 and 50 and upwards to 58; the following results, viz.:—

Ages	15.	16.	17;	50.	51.	52.	53.	54.	55.	56.	57.	58.
Numbers	3.	23.	132;	16.	5.	7.	1.	3.	2.	1.	1.	2.

[3] See " Sterility and Abortion." Whitehead, p. 46, or M. Duncan, op. cit., p. 32.

The ratio (e) of ex-nuptial to nuptial cases of maternity is found by dividing the values in column (iv.) by those in column (iii.) in Table LXXIII. That is to say—

$$(457)\ldots\ldots e = u / m.$$

The ratio of "brides" to "unmarried" females, or to females "never married" given in column (viii.) of the table, may be called the "*female nuptial ratio*" ($\mathfrak{g}$) according to age, and is given by—

$$(458)\ldots\ldots \mathfrak{g} = g / U$$

the total number of brides being the same as the number of marriages J in (400), p. 176. Suffixes will denote the age to which the ratio refers. The values $\mathfrak{g}$ are the probabilities of marriage according to age of the unmarried. This probability corresponds to a mean of the marriage-rates of 0.008403, and to a marriage rate over all the eight years of 0.00842863.[1] For any particular year the distribution according to age will therefore approximately be in the ratio of the crude marriage rate for the year in question to that above; expressed ordinarily, say as—

$$(459)\ldots\ldots \mathfrak{g}' = \frac{g}{U} \cdot \frac{n}{0.00843}$$

n being calculated as indicated by (400), p. 176.

The greatest number of never married appears to be for the year between the ages 16.32 to 17.32, the number being about 43,950. Similarly the greatest number of brides appears to be for the ages 21.90 to 22.90, the number being about 27,955.

The curve shewing the number of brides of each age is curve F, Fig. 67, and that shewing the number of the females "never married" is curve G of the same figure; G′ and G′′ shew the terminal values on a larger scale. The circles with crosses denote the positions of the data when corrected for the error of statement of age at marriage; see pp. 193-6 hereinbefore. The crude results are shewn by circles on E′, G, G′ and G′′. It will be seen from these terminal values that there is considerable regularity in the curve even for advanced ages (see p. 244).

The "*male nuptial ratio*," according to age, is, similarly to (458) and (459)—

$$(460)\ldots\ldots \mathfrak{v} = v / V; \text{ or } (461)\ldots\ldots \mathfrak{v}' = \frac{v}{V} \cdot \frac{n}{0.00843}$$

The values are given in Table LXXIII., the crude results being shewn in column (xiv.). The curve shewing the number of bridegrooms of each age is curve W, Fig. 70, and that shewing the unmarried males is curve V of the same figure. V′ and V′′ shew the terminal values on a larger scale. The smoothed values of the probability $\mathfrak{g}'$, and $\mathfrak{v}'$ are given in columns (xviii.) and (xix.) of Table LXXIII.

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[1] Expressed per thousand, as is usual, 8.42863.

TABLE LXXIII.—Shewing the Numbers of Brides and Bridegrooms and the Cases of Nuptial and Ex-nuptial Maternity during the Yea 1907-1914, Australia, and the Numbers of Married and Never Married Males and Females, at the Census of 3rd Apri 1911. Shewing also the Probabilities of Marriage for Never Married Males and Females, and the Probability of Nupti and Ex-nuptial Maternity, and Ratios Dependent upon these.

Age at Maternity.	Brides during 1907-1914 (g).	Cases of Nuptial Maternity (m).	Cases of Ex-nuptial Maternity (u).	Ratio of Ex-nuptial to Nuptial Cases u/m.	Married Women Census, 1911 (M).	Women Never Married Census, 1911 (U).	Ratio of Brides per annum to the Number Unmarried (q = g/ U).	Crude Rate of Nuptial Maternity (per annum) (p = m/M).	Crude Rate of Ex-nuptial Maternity (per annum) (u = u/ U).§	Ratio of Crude Ex-nuptial to Crude Nuptial Maternity Rate c = u/U ÷ m/M.	Number of Bridegrooms 1907-1914 (v).	Men never Married Census, 1911 (V).	Ratio of Bridegrooms per Annum to the Number Unmarried v = (v/ V).	Probability of Nuptial Maternity, Smoothed p'.	Probability of Ex-nuptial Maternity (Smoothed) u'§	Ratio of Probability of Ex-nuptial Maternity to Nuptial Maternity c'§	Probability of Marriage of Women (Smoothed) q'.	Probability of Marriage of Men (Smoothed) v'.
(i.)	(ii.)	(iii.)	(iv.)	(v.)	(vi.)	(vii.)	(viii.)	(ix.)	(x.)	(xi.)	(xii.)	(xiii.)	(xiv.)	(xv.)	(xvi.)	(xvii.)	(xviii.)	(xix.)
12	1	0	5	∞	0	42,001	.00000024	..	.000015	..	..	42,748	..	..	.000015	..	.00000024	..
13	4	4	21	5.25	1	42,071	.0000095	.497	.000062	..	..	42,990	..	..	.000062	..	.0000095	..
14	73	30	126	4.20	18	42,284	.000215	.207	.000371	.0018	..	43,621	..	.2070	.00037	.0018	.000215	..
15	489	170	538	3.165	93	43,273	.001405	.227	.00155	.0068	4	44,598	.000011	.2270	.0016	.0071	.00141	.00001
16	2,412	1,140	1,502	1,317	349	43,915	.006831	.4062	.00425	.0105	51	45,482	.000140	.3010	.0042	.0109	.00683	.00014
17	6,918	3,974	2,991	.7526	1,145	43,813	.01964	.4316	.00849	.0197	239	46,212	.000643	.4580	.0085	.0187	.01964	.00064
18	13,246*	9,796	4,519	.4613	2,551	42,854	.03844	.4776	.01311	.0275	1,205	46,620	.00321	.4835†	.0131	.0271	.04064*	.00321
19	18,140*	18,165	5,339	.2939	4,499	41,020	.05500	.5022	.01619	.0322	3,353	46,470	.00897	.4790	.0157	.0325	.05937*	.00897
20	20,231*	25,307	5,302	.2095	6,933	38,370	.06557	.4540	.01710	.0370	6,438	45,466	.0176	.4640	.0172	.0368	.07674*	.01760
21	32,673*	35,529	5,040	.1419	10,100	34,634	.11733	.4375	.01810	.0414	17,374	43,790	.0493	.4425	.0180	.0407	.09706*	.04930
22	27,950	43,607	4,267	.0978	13,047	30,878	.11258	.4157	.01729	.0413	19,977	41,097	.0605	.4157	.01835‡	.0446	.11966	.06050
23	26,402	50,655	3,878	.0765	16,521	26,360	.12457	.3813	.01830	.0480	23,655	37,541	.0784	.3813	.0181	.0474	.12544	.07840
24	23,903	53,569	3,208	.0599	18,905	22,725	.13082	.3524	.01756	.0498	24,918	34,003	.0911	.3524	.0176	.0497	.12623	.09110
25	20,707	54,712	2,567	.0469	20,683	19,600	.13139	.3290	.01629	.0495	24,650	30,338	.1011	.3330	.0169	.0500	.12613	.10110
26	17,731	55,453	2,181	.0393	21,620	17,267	.12771	.3190	.01571	.0492	23,494	26,823	.1089	.3190	.0160	.0510	.12407	.10890
27	14,440	54,229	1,812	.0334	22,180	15,370	.11684	.3041	.01466	.0482	21,011	23,597	.1107	.3071	.0154	.0512	.11867	.11270
28	12,372	53,753	1,719	.0320	22,584	13,646	.11276	.2960	.01567	.0529	19,384	20,808	.1159	.2930	.0149	.0515	.11376	.11270
29	10,010	49,739	1,417	.0285	22,765	12,153	.10244	.2717	.01450	.0534	16,113	18,677	.1073	.2735	.0146	.0527	.10202	.10850
30	8,405	48,535	1,373	.0283	22,784	10,839	.09644	.2649	.01575	.0594	13,392	16,700	.0997	.2560	.0143	.0548	.09200	.10150
31	5,847	40,683	861	.0212	22,726	9,659	.07530	.2226	.01109	.0498	10,349	14,875	.0805	.2415	.0140	.0569	.08348	.09270
32	5,558	42,093	969	.0230	22,542	8,730	.07918	.2322	.01380	.0594	9,745	13,250	.0915	.2250	.0137	.0606	.07620	.08600
33	4,342	37,934	827	.0218	22,421	7,835	.06892	.2104	.01313	.0624	7,712	11,925	.0804	.2104	.0135	.0643	.07014	.08100
34	3,854	34,848	792	.0227	22,059	7,278	.06586	.1965	.01353	.0689	6,796	10,900	.0775	.1965	.01325	.0673	.06492	.07630
35	3,521	31,794	705	.0222	21,700	6,791	.06448	.1822	.01291	.0708	6,066	10,177	.0741	.1850	.0130	.0699	.06008	.07250
36	2,914	29,895	648	.0217	21,350	6,319	.05735	.1741	.01275	.0732	5,345	9,529	.0698	.1741	.0127	.0726	.05584	.06980
37	2,438	26,632	551	.0207	21,000	5,910	.05130	.1577	.01160	.0736	4,411	9,032	.0607	.1635	.0122	.0748	.05205	.06550
38	2,369	25,044	564	.0225	20,560	5,630	.05233	.1515	.01246	.0822	4,530	8,613	.0654	.1485	.0115	.0788	.04861	.06120
39	2,005	21,116	441	.0209	20,215	5,303	.04702	.1299	.01034	.0796	3,737	8,265	.0562	.1299	.01045	.0792	.04572	.05300
40	1,688	17,255	389	.0225	19,851	4,987	.0421	.1081	.00970	.0897	3,252	7,973	.0507	.1081	.0095	.0833	.04263	.04660
41	1,126	12,425	204	.0164	19,457	4,623	.0303	.0794	.00549	.0691	2,336	7,669	.0379	.0870	.0076	.0817	.03999	.04195
42	1,287	11,138	205	.0184	19,026	4,226	.0379	.0728	.00603	.0828	2,447	7,386	.0412	.0670	.0059	.0820	.03763	.03980
43	1,005	7,542	156	.0207	18,453	3,940	.0336	.0508	.00492	.0968	2,058	7,077	.0362	.0495	.0043	.0860	.03540	.03770
44	948	4,783	86	.0180	18,040	3,707	.0318	.0330	.00289	.0877	1,745	6,763	.0321	.0330	.0030	.0909	.03330	.03570
45	946	2,776	58	.0209	17,554	3,441	.0343	.01967	.00210	.107	1,847	6,453	.0356	.0197	.0020	.100	.03133	.03395
46	712	1,399	37	.0264	17,064	3,212	.0276	.01020	.00143	.140	1,575	6,185	.0317	.0102	.0012	.133	.02999	.03245
47	663	688	17	.0247	16,554	3,000	.0275	.00517	.00070	.135	1,350	5,957	.0282	.0052	.00085	.170	.02828	.03100
48	654	311	12	.0386	15,975	2,783	.0292	.09242	.00054	.223	1,363	5,731	.0296	.0025	.00055	.220	.02670	.02960
49	493	106	7	.066	14,216	2,570	.0239	.00087	.00034	.391	1,146	5,532	.0258	.0012	.00035	.283	.02525	.02825
50	481	34	5	.147	14,303	2,340	.0256	.00030	.00028	.933	1,113	5,232	.0265	.0006	.00022	.367	.02393	.02695
51	308	12	1	.083	13,162	2,127	.0180	.00011	.00006	.545	740	4,797	.0192	.0003	.00012	.400	.02274	.02565
52	351	7	..	..	12,088	1,938	.0225	.00007	..	..	579	4,332	.0228	.00015	.00004	.27	.02166	.02435
53	238	4	1	.250	11,100	1,780	.0166	.00004	.00007	1.750	647	3,942	.0204	.00007	.00002	.29	.02067	.02300
54	228	3	..	..	10,286	1,549	.0191	.00004	..	..	621	3,642	.0212	.000035	.00001	.29	.01975	.02175
55	179	2	..	..	9,468	1,363	.0163	.00003	..	..	545	3,342	.0203	.000017	..	..	.01888	.02067
56	189	1	..	..	8,557	1,248	.0188	.00001	..	..	489	3,042	.0200	.000008	..	..	.01810	.01967
57	165	1	..	..	7,675	1,145	.0179	.00002	..	..	400	2,762	.0180	.000004	..	..	.01735	.01876
58	159	..	..	..	6,912	1,032	.0192	..	..	..	419	2,512	.0207	.000002	..	..	.01659	.01787
59	98	..	..	..	6,293	910	.0134	..	..	..	286	2,302	.0155	.000001	..	..	.01582	.01700
60	136	..	..	..	5,746	830	.0204	..	..	..	347	2,152	.0201	..	..	..	.01504	.01615
61	58	..	..	..	5,277	760	.0095	..	..	..	236	2,032	.0144	..	..	..	.01425	.01532
62	93	..	..	..	4,871	715	.0162	..	..	..	227	1,922	.0147	..	..	..	.01345	.01451
63	81	..	..	..	4,505	690	.0146	..	..	..	229	1,822	.0156	..	..	..	.01270	.01372
64	68	..	..	..	4,161	650	.0130	..	..	..	203	1,732	.0146	..	..	..	.01200	.01295
65	93	..	..	..	3,829	600	.0193	..	..	..	219	1,622	.0168	..	..	..	.01135	.01220
66	60	..	..	..	3,502	550	.0136	..	..	..	170	1,490	.0142	..	..	..	.01075	.01147
67	54	..	..	..	3,194	500	.0134	..	..	..	163	1,396	.0145	..	..	..	.01019	.01078
68	51	..	..	..	2,880	450	.0141	..	..	..	154	1,328	.0144	..	..	..	.00965	.01013
69	43	..	..	..	2,621	405	.0132	..	..	..	122	1,272	.0119	..	..	..	.00912	.00952
70	36	..	..	..	2,365	360	.0124	..	..	..	134	1,230	.0135	..	..	..	.00860	.00895
71	21	..	..	..	2,099	320	.0082	..	..	..	70	1,190	.0073	..	..	..	.00809	.00842
72	21	..	..	..	1,867	280	.0093	..	..	..	70	1,140	.0076	..	..	..	.00759	.00794
73	14	..	..	..	1,652	240	.0073	..	..	..	77	1,080	.0089	..	..	..	.00712	.00749
74	11	..	..	..	1,444	205	.0067	..	..	..	64	1,010	.0079	..	..	..	.00667	.00710
75	10	..	..	..	1,224	180	.0069	..	..	..	53	920	.0072	..	..	..	.00023	.00675
76	4	..	..	..	1,004	155	.0032	..	..	..	33	810	.0051	..	..	..	.00580	.00644
77	6	..	..	..	818	135	.0055	..	..	..	33	690	.0059	..	..	..	.00535	.00614
78	6	..	..	..	650	120	.0062	..	..	..	22	560	.0049	..	..	..	.00496	.00585
70	4	..	..	..	510	105	.0047	..	..	..	23	450	.0064	..	..	..	.00456	.00557
80	2	..	..	..	307	92	.0027	..	..	..	28	370	.0094	..	..	..	.00418	.00530
81	1	..	..	..	317	80	.0016	..	..	..	10	312	.0040	..	..	..	.00380	.00504
82	3	..	..	..	241	69	.0054	..	..	..	9	206	.0042	..	..	..	.00344	.00479
83	1	..	..	..	184	58	.0021	..	..	..	8	225	.0044	..	..	..	.00308	.00455
84	0	..	..	..	140	48	..	..	..	..	7	189	.0046	..	..	..	.00265	.00432
85	0	..	..	..	105	38	..	..	..	..	5	158	.0039	..	..	..	.00243	.00410
86	0	..	..	..	80	28	..	..	..	..	3	121	.0031	..	..	..	.00213	.00389
87	0	..	..	..	56	20	..	..	..	..	3	90	.0041	..	..	..	.00185	.00369
88	0	..	..	..	35	14	..	..	..	..	1	65	.0019	..	..	..	.00159	.00350
89	0	..	..	..	24	10	..	..	..	..	1	46	.0027	..	..	..	.00135	.00332
90	0	..	..	..	20	8	..	..	..	..	0	33	..	..	..	..	.00113	.00315
91	0	..	..	..	16	6	..	..	..	..	0	26	..	..	..	..	.00093	.00299
92	0	..	..	..	12	4	..	..	..	..	0	20	..	..	..	..	.00075	.00284
93	0	..	..	..	9	3	..	..	..	..	0	15	..	..	..	..	.00059	.00270
94	0	..	..	..	7	2	..	..	..	..	0	11	..	..	..	..	.00045	.00257
95	1	..	..	..	6	2	.062	..	..	..	1	8	.0163	..	..	..	.00033	.00245
96	0	..	..	..	5	1	..	..	..	..	0	6	..	..	..	..	.00022	.00234
97	0	..	..	..	4	1	..	..	..	..	0	5	..	..	..	..	.00012	.00224
98	0	..	..	..	3	1	..	..	..	..	0	4	..	..	..	..	.00005	.00215
99	0	..	..	..	2	1	..	..	..	..	1	3	.0416	..	..	..	.00002	.00207
100	0	..	..	..	1	1	..	..	..	..	73	2	..	..	..	..	.00001	.00200

Unspecified, 111. Total including the Unspecified, 301,922. For notes see next page.

9. **Nuptial and ex-nuptial maternity and their frequency-relations.**— The crude rate, according to age, of nuptial and of ex-nuptial maternity is found by dividing the number of cases of maternity of each kind by the number of married or of "unmarried" or "never married" women. That is to say $\mathfrak{p}$, $\mathfrak{n}$, and $\mathfrak{P}$ denoting the probability of maternity, according to age, respectively of the married, the never married, or of both combined, we shall have :—

$$(462)\ldots\ldots\mathfrak{p} = m \;/\; M\,;$$

$$(463)\ldots\ldots\mathfrak{n} = u \;/\; U\,;$$

$$(464)\ldots\ldots\mathfrak{P} = (m + u) \;/\; (M + U).$$

The relation, according to age, between the ex-nuptial and nuptial rates, is—

$$(465)\ldots\mathfrak{e} = \mathfrak{n}/\mathfrak{p} = \frac{u}{U}\Big/\frac{m}{M} = \frac{M}{m}\cdot\frac{u}{U}$$

These crude rates and their ratio to each other are given in Table LXXIII. for the whole reproductive period in columns (ix.), (x.) and (xi.). The smoothed values are given in columns (xv.), (xvi.) and (xvii.).

The graphs of the numbers of cases of nuptial and of ex-nuptial maternity are shewn respectively by curves H and I, on Fig. 68, the dots in the former case, and the crosses in the latter, denoting the crude results. The ratio of the ex-nuptial cases to the nuptial cases are shewn by curve J, and on a larger scale by curve J,' Fig. 68. The nuptial and ex-nuptial maternity-rates are shewn on the same figure by curves K and L, the dots in the former, and the small circles in the latter indicating the crude results (see p. 244).

It should be noted that m and M in (462), etc., are not necessarily homogeneous, since each will contain, though in unequal proportions, primiparous and multiparous women, and these will have been subject to risk for unequal periods. Moreover the multiparæ may have given birth to very different numbers of children. If, therefore, the probability of maternity is affected by previous issue, the value of $\mathfrak{p}$ must be regarded as merely a *crude probability*. An exact probability would have to be defined in categories *according to the age, the number of previous issue, and the length of exposure to risk*. This will appear more clearly in the theory of fertility and sterility. For this reason the values given of $\mathfrak{p}$ and $\mathfrak{n}$ in Table LXXIII, are for the "average risk" of the "average married woman" or the "average never married woman" during twelve months, and takes no account of variation of the "risk" according to the age of the husband. In section 11 hereinafter it will be seen that the maxima vary.

Notes to Table LXXIII. on preceding page.

* If the corrections referred to in Part XII., § 15 and 16, pp. 193-6, be applied, these numbers become 14,004 ; 19,580 ; 23,678 ; 26,927 ; see formula (407). This will change the ratios in column (viii.) from .03844 to .04064 ; .05500 to .05937 ; .06557 to .07674 ; and .11733 to .09706.

† The maximum is for the *central age* 18.73, that is for the group of ages 18.23 to 19.23, and the amount is 0.4849.

‡ The maximum is for the central age 22.50 ; that is for the group of ages 22.00 to 23.00, and the amount is 0.01835.

§ The ex-nuptial births are attributed to the "never married" but may, perhaps, be equally well attributed to the "unmarried," that is the "never married" together with the "widowed" and "divorced"

Terminal Frequencies of Fertility; Frequency of Nuptial and Ex-nuptial Maternity; Probability of Marriage of both Sexes at each Age; etc.

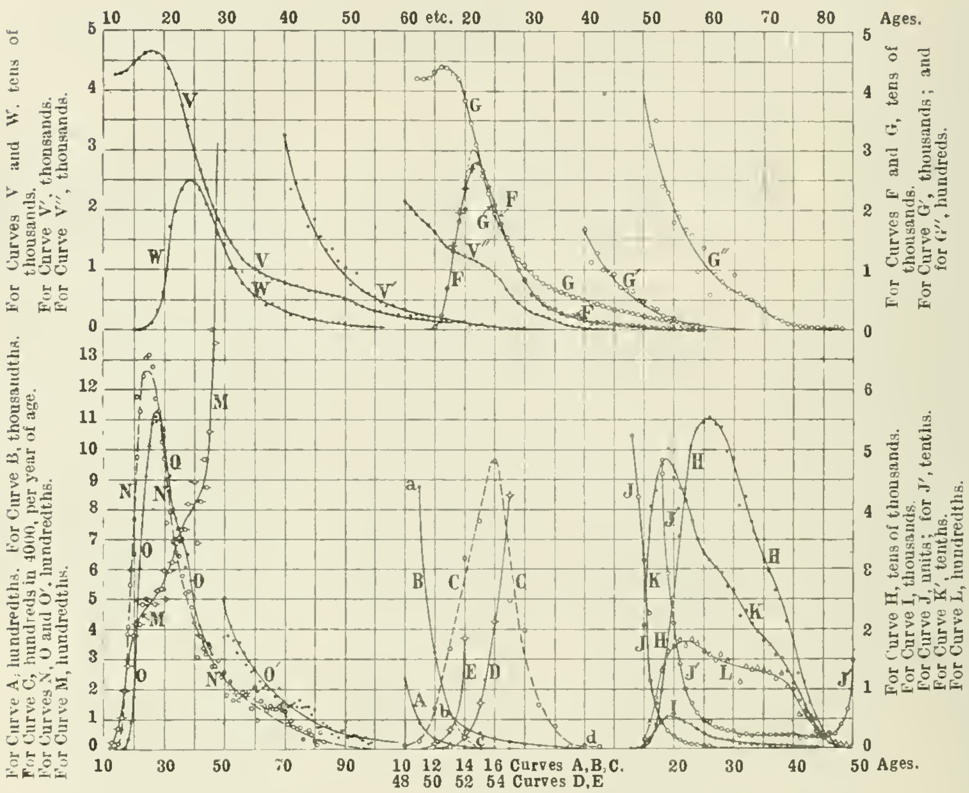


Fig. 66.—Curves A and B shew the terminal age of fertility. Curves D and E shew the initial age of fertility. Curve C shews the frequency of the appearance of menstruation according to age.

Fig. 67 —Curve F shews the numbers of brides at various ages. Curves G, G′ and G″ shew the numbers of the "never married" at various ages.

Fig. 68.—Curve H shews the number of cases of nuptial maternity, and Curve I those of ex-nuptial maternity at each age. Curves J and J′ shew the proportion of ex-nuptial to nuptial cases of maternity at each age. Curve K shews the nuptial and L the ex-nuptial rates of maternity at each age, the ex-nuptial rate being determined by attributing the births to the "never married."

Fig. 69.—Curve M shews the ratio of the ex-nuptial to the nuptial rates of maternity at each age. Curve N is the ratio of the brides at each age to the "never married females" of the same ages. Curves O and O′ are similarly the ratio of the bridegrooms at each age to the "never married males" of the same ages, curve O′ being displaced one division (10 years) to the right so as not to be confused with curve N.

Fig. 70.—Curve W shews the number of bridegrooms of each age, and V, V′ and V″ the number of "never married males" at each age.

In all the above cases the age is the "age last birthday."

10. Maximum probabilities of marriage and maternity, etc.—The position and amount of the maxima determined from the smoothed results in columns (xv.) to (xix.) of Table LXXIII. are as follow :—

Table LXXIV. Maximum Probabilities, Marriage and Maternity.

Maximum probability of—	Year-group from— Age.		Age.	Amount.
Nuptial maternity	18.45		19.45	.0486
Ex-nuptial maternity	22.00		23.00	0.01835
Ratio of ex-nuptial on nuptial maternity	Probably no maximum value point of inflexion at—			
	25	to	26	0.0510
Marriage of women	24.52	to	25.52	0.12632
Marriage of men	27.5	to	28.5	0.11320

The maxima are for the two heterogeneous groups "nuptial" and "ex-nuptial" aggregated according to age merely. In the next section it will be shewn that the maxima are dependent upon age at marriage.

The largest number of marriages of brides would appear to be for the ages 21.9 to 22.9, and to be about 28,000 in 8 years; and the largest number of marriages of bridegrooms, for the ages 24.8 to 25.8, the number being about 25,000 in 8 years, the total mean population aggregated for the years in question being 35,821,000 persons. The largest number of *cases of nuptial maternity* occurred for ages 26.12 to 27.12, the number being about 55,500 in 8 years, and the ratio at the crude maximum consequently 0.3182. The largest number of *cases of ex-nuptial maternity* occurred for the ages 19.5 to 20.5, the number being about 5,400 in 8 years, and the ratio at the crude maximum of cases, therefore, 0.01691.

The question of a more accurately defined maximum will be considered hereinafter.

11. Probability of a first-birth occurring within a series of years after marriage.—To determine the variation of initial fertility with age, the initial probability of maternity may be deduced by ascertaining primarily the number of women at different ages who were married during a given period. Then, tracing these through the first portion of their married life, the respective periods which elapsed after marriage before they gave birth to their *first living child* may be ascertained.

For this purpose the six-year period, 1909-14, was brought under observation, the experience being all cases in the Commonwealth of Australia within a series of years, viz., 6 after marriage. Owing to misstatements regarding age, however, the number of brides registered at each age during the several years under observation required correction. It was found that, if the actual numbers of brides registered at ages 18, 19, 20 and 21 years were accepted, without adjustment, anomalous results would be obtained. Evidently serious errors existed owing to brides of 18, 19, and 20 years overstating their age as 21, and therefore the numbers of brides upon which the rates of fertility should be founded

needed correction. A special type of smoothing of the number of brides of 18, 19, 20 and 21 years to remedy the misstatement of age had therefore to be adopted.[1]

A similar misstatement of age had evidently occurred in the case of *mothers* (registered as being 19, 20, and 21 years of age), who gave birth to a *first-born* child during the period 1909-14, and the numbers consequently had also to be smoothed, so as to eliminate the effect of misstatements in the age of mothers.[2]

Tables were compiled shewing the mean number of brides of each age in any year and in the year immediately preceding ; and for the same ages the number of first confinements in successive years of duration of marriage. Assuming then that the migration elements balanced each other, the table gave a series of results shewing for the years 1909 to 1914 inclusive the aggregate number of brides of each age at marriage to which the aggregate number of first confinements could be referred, hence the ratio of the latter to the former gave the probability required.[3]

[1] The justification for this smoothing is really that the probability of a misstatement of age is very great, and the probability of some physiological or other cause, for the anomaly, is relatively negligible.

[2] The following are the unadjusted and adjusted figures :—

Age.	Nuptial First Births, according to Successive Years of Duration after Marriage.							Number of Brides to whom the Births may be ascribed, according to Successive Years of Duration after Marriage.						
	Total	0–1	1–2	2–3	3–4	4–5	5–6	Total	0–1	1–2	2–3	3–4	4–5	5–6
18	7,568	5,899	1,291	262	81	29	6	10,159	10,159	8,331	6,513	4,735	3,039	1,484
	7,568	5,899	1,291	262	81	29	6	10,736	10,736	8,802	6,880	5,003	3,213	1,571
19	11,625	9,071	1,943	429	118	48	16	13,838	13,838	11,364	8,899	6,463	4,156	1,998
	11,228	8,761	1,877	414	114	46	16	14,902	14,902	12,227	9,557	6,917	4,457	2,177
20	13,596	10,141	2,618	556	202	56	23	15,496	15,496	12,737	9,978	7,244	4,657	2,241
	14,400	10,741	2,773	589	214	59	24	18,100	18,100	14,860	11,630	8,453	5,475	2,675
21	17,507	12,613	3,699	823	262	81	29	24,850	24,850	20,309	15,838	11,520	7,498	3,702
	17,100	12,320	3,613	804	256	79	28	20,600	20,600	16,848	13,158	9,588	6,264	3,002

The upper number is that furnished by the registration records, the lower is that which was obtained after adjustment. The only adjustment deemed essential as a preliminary is for these ages, viz., 18 to 21. For all other ages the results are as given by the unadjusted data.

[3] The following illustration of the method of compiling will suffice :—

Year.	Age at Marriage.	Mean No. of Brides for Year and preceding Year.	Number of First Confinements in successive Years of Duration of Marriage.						Duration of Marriage, 0-1.			
			0-1	1-2	2-3	3-4	4-5	5-6	Age.	Brides.	Confinement.	Ratio.
1909	26	1,864	1,002	443	160	71	54	15	26	13,637	7,279	0.5338
	27	1,563	835	417	107	66	39	22	27	11,054	5,721	0.5176
									Duration of Marriage, 1–2.			
1910	26	2,076	1,047	551	219	79	46	..	26	11,068	3,095	0.2795
	27	1,616	853	444	149	73	41	..	27	9,004	2,566	0.2850
									Duration of Marriage, 2–3.			
1911	26	2,268	1,171	645	212	101	..	..	26	8,571	800	0.0933
	27	1,781	967	527	178	88	..	..	27	6,971	619	0.0888
	(From these the totals on the right were formed.)											

The probabilities so ascertained are shewn on Table LXXV. up to 6 years. The crude results are shewn by the dots on Fig. 71, on which the curved lines give the smoothed results, the corresponding numerical values appearing on the right hand side of the table.

TABLE LXXV.—Probability of a Nuptial First Birth occurring within 6 Years of Marriage, Based on Australian Data, 1909 to 1914.

Age last Birthday.	CRUDE RESULTS. Probability of Giving Birth to a First Child for a Duration of Marriage of—							ADJUSTED RESULTS. Probability of Giving Birth to a First Child for a Duration of Marriage of—							Age last Birthday.
	less than 1 yr.	1–2 yrs.	2–3 yrs.	3–4 yrs.	4–5 yrs.	5–6 yrs.	less than 6 yrs.	less than 1 yr.	0–1 yrs.	1–2 yrs.	2–3 yrs.	3–4 yrs.	4–5 yrs.	5–6 yrs.	
11	..	..	..	..	..	..	..	.0000	.0000	.0000	.0000	.0000	.0000	.0000	11
12	..	..	..	..	..	..	..	.1308	.0963	.0217	.0066	.0030	.0020	.0012	12
13	..	..	..	..	..	..	..	.2568	.1881	.0433	.0131	.0060	.0039	.0024	13
14	..	..	..	..	..	..	..	.3781	.2755	.0647	.0195	.0091	.0058	.0035	14
15	.3324	.1233	.0470	.0353	.0278	..	..	.4946	.3585	.0860	.0258	.0121	.0076	.0046	15
16	.4352	.1042	.0424	.0177	.0149	.0075	.6219	.6063	.4370	.1073	.0321	.0150	.0093	.0056	16
17	.4979	.1271	.0413	.0128	.0141	.0053	.6985	.6975	.4985	.1263	.0377	.0176	.0108	.0066	17
18	.5495	.1467	.0381	.0162	.0090	.0038	.7633	.7770	.5485	.1455	.0432	.0199	.0123	.0076	18
19	.5879	.1535	.0433	.0165	.0103	.0073	.8188	.8414	.5800	.1664	.0497	.0229	.0138	.0086	19
20	.5934	.1866	.0506	.0253	.0108	.0090	.8757	.8856	.5950	.1854	.0551	.0252	.0153	.0096	20
21	.5981	.2144	.0611	.0267	.0127	.0093	.9223	.9176	.5958	.2051	.0614	.0280	.0168	.0105	21
22	.5919	.2301	.0675	.0299	.0151	.0122	.9467	.9429	.5908	.2247	.0673	.0306	.0182	.0113	22
23	.5800	.2425	.0783	.0314	.0173	.0094	.9589	.9619	.5819	.2423	.0730	.0331	.0195	.0121	23
24	.5545	.2466	.0827	.0344	.0231	.0130	.9543	.9730	.5688	.2569	.0785	.0354	.0206	.0128	24
25	.5314	.2636	.0815	.0375	.0235	.0158	.9533	.9771	.5533	.2679	.0831	.0378	.0216	.0134	25
26	.5338	.2795	.0933	.0404	.0254	.0081	.9805	.9750	.5357	.2754	.0872	.0402	.0225	.0140	26
27	.5176	.2850	.0888	.0458	.0252	.0141	.9765	.9667	.5168	.2795	.0903	.0423	.0233	.0145	27
28	.5037	.2677	.1013	.0465	.0260	.0126	.9578	.9530	.4967	.2813	.0922	.0439	.0240	.0149	28
29	.4548	.2774	.0836	.0359	.0198	.0107	.8822	.9330	.4766	.2792	.0929	.0446	.0245	.0152	29
30	.4686	.2421	.0898	.0498	.0224	.0107	.8834	.9075	.4545	.2751	.0930	.0448	.0247	.0154	30
31	.4602	.3084	.1003	.0447	.0238	.0178	.9552	.8745	.4310	.2668	.0923	.0416	.0245	.0153	31
32	.4191	.2464	.0873	.0368	.0220	.0132	.8248	.8381	.4073	.2571	.0907	.0440	.0240	.0150	32
33	.4057	.2422	.0825	.0428	.0217	.0194	.8143	.7938	.3789	.2463	.0883	.0426	.0231	.0146	33
34	.3310	.2526	.0928	.0353	.0232	.0204	.7553	.7411	.3487	.2319	.0843	.0407	.0217	.0138	34
35	.3036	.1950	.0771	.0387	.0113	.0155	.6412	.6748	.3123	.2135	.0784	.0382	.0198	.0126	35
36	.3024	.1820	.0724	.0395	.0236	.0061	.6260	.6063	.2768	.1935	.0718	.0354	.0178	.0110	36
37	.2241	.1910	.0741	.0341	.0173	.0000	.5406	.5367	.2423	.1730	.0650	.0315	.0157	.0092	37
38	.1919	.1576	.0634	.0252	.0105	.0144	.4630	.4662	.2088	.1520	.0573	.0276	.0134	.0071	38
39	.1844	.1391	.0406	.0275	.0087	.0000	.4003	.3946	.1755	.1303	.0490	.0237	.0110	.0051	39
40	.1436	.0986	.0520	.0131	.0049	.0000	.3122	.3245	.1426	.1082	.0415	.0198	.0088	.0036	40
41	.1323	.0870	.0336	.0194	.0076	.0000	.2799	.2558	.1111	.0863	.0333	.0158	.0070	.0023	41
42	.0756	.0627	.0211	.0135	.0000	.0073	.1802	.1951	.0855	.0656	.0254	.0119	.0053	.0014	42
43	.0669	.0665	.0131	.0051	.0000	.0083	.1599	.1411	.0634	.0174	.0178	.0080	.0037	.0008	43
44	.0384	.0462	.0064	.0030	.0000	.0000	.0940	.0937	.0441	.0321	.0116	.0041	.0014	.0004	44
45	.0258	.0066	.0086	.0000	.0000	.0000	.0410	.0622	.0296	.0220	.0070	.0022	.0012	.0002	45
46	..	..	..	..	..	..	..	.0400	.0199	.0147	.0035	.0012	.0006	.0001	46
47	..	..	..	..	..	..	..	.0252	.0131	.0094	.0019	.0005	.0003	.0000	47
48	..	..	..	..	..	..	..	.0159	.0093	.0056	.0007	.0002	.0001	..	48
49	..		..	..	..	..	..	.0095	.0062	.0029	.0003	.0001	.0000	..	49
50	.0026	.0031	.0000	.0000	.0000	.0000	.0057	.0053	.0040	.0012	.0001	.0000	..	..	50
51	..	..	..	..	..	..	..	.0028	.0023	.0005	.0000	..	..	..	51
52	..	..	..	..	..	..	..	.0013	.0011	.0002	..	..	..	..	52
53	..	..	..	..	..	..	..	.0006	.0005	.0001	..	..	..	..	53
54	..	..	..	..	..	..	..	.0002	.0002	.0000	..	..	..	..	54
55	..	..	..	..	..	..	..	.0001	.0001	..	..	..	..	..	55

The probabilities in tho table apply to the *total number of women married at the given ages, not to the survivors* after the series of years under observation. The probabilities are of course *cumulative*, that is to say

the probability, ${}_0p_n$, that a first birth will occur before the end of the n-th year after marriage, is the sum of the probabilities that it will occur during the first, during the second, etc., up to and including the n-th year. Or

$$(466) \ldots\ldots {}_0p_n = {}_0p_1 + {}_1p_2 + \cdots {}_{n-1}p_n$$

12. **Maximum probabilities of a first birth.**—From the smoothed results in the table, it will be seen that, as the interval to the first birth increases, the age of maximum increases. Thus the greatest probability of a first birth within the first year from marriage is for age at marriage 21.24, during the year succeeding that of marriage it is at age 28.47, and so on as shewn in the following table, viz., LXXVI.

TABLE LXXVI.—Shewing the Age of Maximum Probability of a First Birth. Australia 1909-1914.

Interval Years.	0–1	1–2	2–3	3–4	4–5	5–6	0–1	0–2	0–3	0–4	0–5	0–6	0–7*	0–8	0–9	0–10	0–11*
Vertex at (years)	20.74	27.97	29.62	29.75	30.03	30.14	20.74	23.75	24.52	24.91	25.06	25.16	25.22†	25.26	25.30	25.32	25.33†
Corresponding to Median Age at Marriage ..	21.24	28.47	30.12	30.25	30.53	30.64	21.24	24.25	25.02	25.41	25.56	25.66	25.72	25.76	25.80	25.82	25.83
Or to Median Age at Birth ..	21.74	29.97	32.62	33.75	35.03	36.14	21.74	25.25	26.52	27.41	28.06	28.66	29.22	29.76	30.30	30.82	31.33
Probability ..	.5962	.2813	.0931	.0448	.0247	.0154	.5962	.8259	.9050	.9421	.9637	.9772	.9859	.9916	.9953	.9978	.9998
By Formula (467)	..	..	..	..	..	..	..	.840	.920	.947	.960	.968	.973	.977			

* The ratios 9050/8259, 9421/9050, etc., are 1.0958, 1.0410, 1.0229, 1.0140, which continued, are 1.0089, 1.0058, 1.0038, 1.0025, 1.0020, the factor of the last two figures converging to 52/80. This, however, would give 1.0011 for 0-11. It is more probable, however, that the probability is of the type ${}_0p'_{n+1} = ({}_0p'_n + m)/(1 + m)$ where m may perhaps be taken even as unity, implying that the residual chance is reduced about one half each year. The matter requires special investigation.

† These correspond to the values of the vertices on Fig. 71.

It is worthy of note that the above results for 0-2 up to 0-8 are roughly given by the formula—

$$(467) \ldots\ldots {}_0p'_n = 1 - \frac{0.16}{n-1},$$

in which n is the total duration of marriage.

The figure (71) and table shew clearly that the maximum is a function of the duration of marriage as well as of age. To find the maximum value for any durations 0 to t the line of vertices C D on Fig. 71 must be followed, or during year-intervals t to $t + 1$, the line C E must be followed. Thus for age 20 last birthday, the duration is 0 to 0.93, the probability is about 0.555. The graphic solution may also follow the method indicated in Fig. 71A, which needs no comment when examined in connection with Table LXXVI.

Probabilities of a first-birth during first 6 years of marriage.

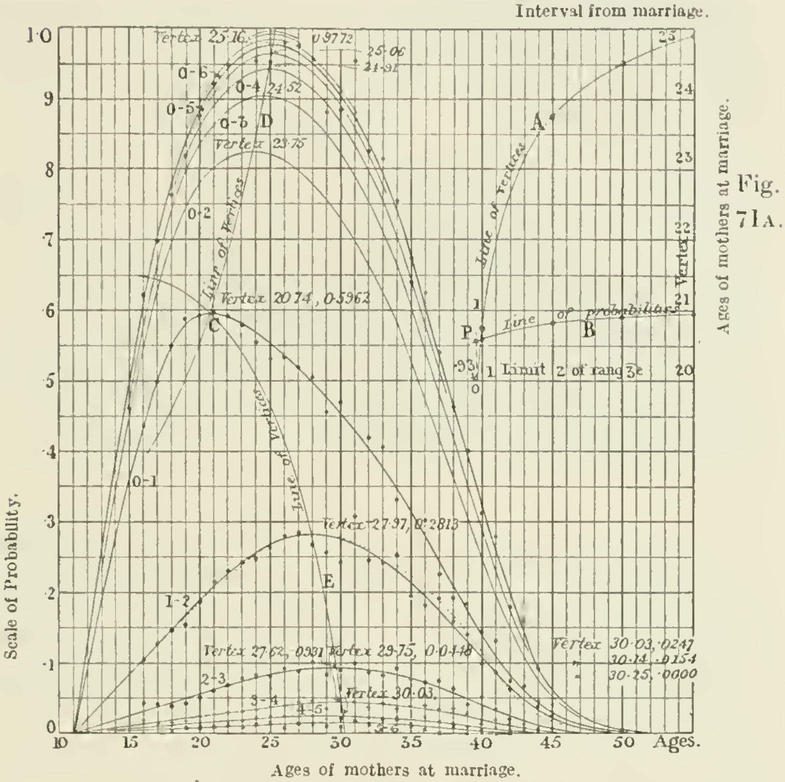


Fig. 71.

13. **Determination of the co-ordinates of the vertices.**—The representation of group-totals by means of integral functions of the values of the central abscissa of the group-base (central value of the interval) has been referred to in Part V., § 10, pp. 72, 73. In curves of the type which has just been considered, the results about the vertices may be closely represented by a curve of the second degree, and the curve itself may be regarded as defining the curve of group-totals for all values of the central abscissa (the abscissa of the middle ordinates of the group). In such instances the co-ordinates for the maximum-group may be very accurately ascertained from the tabular maximum together with the tabular values on either side of it. Let the maximum tabular value denote the point M on the curve, and the adjoining tabular values denote the points A, B, viz., the points on either side. Then, if the difference of the mean of the ordinates of the points A and B, and the ordinate of M be denoted by h, and the half difference of the ordinates of B and A be denoted by l, that is if—

(468)......$h = y_m - \frac{1}{2}(y_b + y_a)$; and $l = \frac{1}{2}(y_b - y_a)$;

then we shall have—

(469)......$y_{max} = y_m + \dfrac{l^2}{4h}$; $x_{max} = x_m + \dfrac{l}{2h}$

The proper maximum is greater than the tabular maximum by the amount $l^2/4h$, and its abscissa lies between that of the tabular maximum and the next highest tabular quantity distant from the former by the amount $l/2h$.

The positions of the vertices have been computed in this way. It remains to be noted, however, that when the value of the abscissa indicates merely the "age last birthday," it is necessary to add the amount $\frac{1}{2}$ to the value given by the formula in order to refer the co-ordinates to the middle values of the group-abscissæ. Thus, in Fig. 71, the curves are plotted with the argument "age," *i.e.*, last birthday, hence the vertex-value 20.74, see curve 0-1, and the maximum 0.5962, refer to the group of brides whose ages ranged between 20.74 years of age and 21.74 years of age. The middle value of the range is 21.24, but the average value is not that. The probability 0.5962 applies to the brides whose ages were between 20.74 and 21.74. Of 10,000 such, 5,962 would give birth to a first child within one year of marriage.

14. **Average age of a group.**—The error of adopting the middle value of any range has been considered in Part XII., § 20, pp. 200-201. It is sometimes preferable to relate the values of the dependent variable, not to the middle values but to the *average* values of the independent variable. In such a case formula (416), p. 201, may be used. Let A, M, and B be three group totals on equal bases k (equal intervals on the axis of abscissæ). The values of the co-efficients of a rational integral function of the second degree—the graph of which will represent, viz., the areas standing on the equal bases, the group-totals—may be found by the formulæ of Part V., §§ 1 to 9., pp. 64-72. The weighted mean abscissa of the middle group may be denoted by x'_m. If then we make the origin at 0, so that A is the integral of the equation $a + bx + cx^2$ between the limits 0 and k, M the integral between k and $2k$, and B between $2k$ and $3k$, then we shall have—

$$(470) \ldots\ldots x'_m = \frac{\frac{1}{12}bk^2 + \frac{1}{4}ck^3}{a + \frac{3}{2}bk + \frac{7}{3}ck^2}$$

which may be put in the very simple form—

$$(471) \ldots\ldots x_m + \epsilon = x_m + \tfrac{1}{24}k(B - A)/M.$$

This is the same formula as (416). In general, therefore, it is sufficient to find the value of the abscissa to which a group may be referred by using the value of the group and of these on either side: see the results as to average interval in §§ 21, 24, etc., hereinafter.

15. **Curves of probability for different intervals derived by projection.**—Reverting to Fig. 71, it may be noted that the probabilities of a first birth between 1 and 2 years, 2 and 3 years, etc., after marriage

may be derived for each age approximately by projection if the ratio of the aggregates and the position of the maximum are known. For example the faintly-dotted curve is the curve for the interval between 1 and 2 years after marriage derived by projection from that up to 1 year (0-1). The difference between the two curves is nearly negligible. The following are the relations between the curves :—

Let x, y be the co-ordinates of any point P on a curve, and let x', y' denote the co-ordinates of what may be called the *corresponding point* P′, on a curve derived therefrom by drawing the line P P′ Q to cut the axis (OX) of abscissæ in the point Q, so as to make the angle of intersection therewith, XQP, equal to θ, and also the ratio QP′/QP equal to ρ. Then, if θ and ρ be constant, the derived curve will belong to a family of curves of the type of the original, but differing therefrom in "skewness" if θ be not 90°. The co-ordinates of any point P′, viz., of the "corresponding point" on the derived curve are simply related to those of the point P on the original curve from which it was derived, being given by the equations—

$$(472)\ldots\ldots y' = \rho y\,;\;\; x' = x - y(1 - \rho)\cot.\theta.$$

Hence if the equation of the original curve be $f(y) = F(x)$, that of the derived curve will be—

$$(473)\ldots\ldots f\left(\frac{y'}{\rho}\right) = F\,(x' - ky')\,;$$

in which $k = \cot.\,\theta\,(1 - \rho)/\rho$.

To determine whether the succession of probabilities for 0-1, 0-2, 0-3, etc., and 0-1, 1-2, 2-3, etc., are rigorously derivable by projection would involve data embracing larger numbers and free from all uncertainty as to the effect of migration thereupon.

16. **Numbers of first-births according to age and duration of marriage.**—There were in Australia during the years 1907-14 inclusive, 220,021 cases of nuptial first births. The records of these were compiled according to "age last birthday," and duration of marriage." Multiplying the numbers as compiled by a factor, that would make the total 1,000,000, the results are as shewn in Table LXXVII., compiled for single months of duration of marriage from 1 to 12 months, and for single years of duration of from 1 to 26. The table thus furnishes the distribution of 1,000,000 nuptial first births according to age and duration of marriage. The figures for the months are of course only one-twelfth of the figures which would be comparable to the yearly values. This distribution may be called the *nuptial protogenesic distribution*.

TABLE LXXVII.—Shewing the Number in 1,000,000 Nuptial First-births of Births occurring for 1
Births occurring in Australia during the Ye

Age of Mothers.	Interval after Marriage during which a Birth Occurs.																	
	0–1 mths.	1–2 mths.	2–3 mths.	3–4 mths.	4–5 mths.	5–6 mths.	6–7 mths.	7–8 mths.	8–9 mths.	9–10 mths.	10–11 mths.	11–12 mths.	0–1 year.	1–2 years.	2–3 years.	3–4 years.	4–5 years.	5–6 yrs.
12 ..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..
13 ..	0	5	0	5	0	0	0	0	0	0	5	0	15	4	..	..	..	..
14 ..	32	14	23	18	9	9	5	0	0	5	9	0	124	0	..	..	..	..
15 ..	45	73	109	68	91	77	68	36	9	27	23	0	626	18	..	..	..	..
16 ..	382	423	514	532	568	541	486	295	91	95	73	91	4,091	209	18	9	..	..
17 ..	959	1,073	1,336	1,663	1,532	1,836	1,859	1,250	532	704	450	304	13,498	1,054	73	5	9	..
18 ..	1,523	1,773	2,754	3,163	3,613	3,891	4,468	2,950	1,636	2,118	1,359	1,082	30,330	3,417	377	50	0	5
19 ..	**1,886**	2,227	**3,513**	4,272	5,127	6,054	6,790	4,609	2,850	3,909	2,972	2,345	46,554	7,794	1,054	155	32	0
20 ..	1,754	2,236	3,172	4,104	5,086	5,995	7,068	5,590	3,254	6,154	4,777	3,318	52,508	11,921	1,968	382	64	27
21 ..	1,877	**2,309**	3,454	**4,640**	**5,704**	**6,981**	**8,331**	**6,613**	4,613	9,122	6,845	4,772	65,261	16,125	2,950	682	227	36
22 ..	1,532	1,827	2,682	3,445	4,281	5,740	7,254	6,159	5,077	**11,981**	8,935	**6,708**	**65,621**	22,225	3,995	1,118	395	132
23 ..	1,113	1,523	1,941	2,909	3,854	4,740	6,263	5,413	**5,177**	11,953	**9,226**	6,331	60,443	**24,316**	5,672	1,859	677	236
24 ..	986	1,104	1,586	2,086	2,886	3,441	4,959	4,231	4,045	11,467	8,726	6,313	51,830	24,261	6,413	2,268	904	377
25 ..	768	818	1,273	1,573	1,873	2,850	3,754	3,250	3,895	10,549	7,986	5,645	44,234	21,988	**6,954**	2,550	1,040	550
26 ..	691	677	1,027	1,168	1,500	2,154	2,836	2,732	3,272	9,031	7,222	4,877	37,187	20,670	6,613	2,482	1,268	782
27 ..	382	432	718	895	1,136	1,463	2,254	2,145	2,373	7,649	5,909	4,177	29,533	18,419	6,009	**2,650**	1,373	**800**
28 ..	491	345	691	736	964	1,232	1,595	1,677	2,091	6,372	5,140	3,450	24,784	15,315	5,508	2,454	**1,432**	695
29 ..	305	282	414	600	568	845	1,177	1,245	1,613	4,536	3,659	2,795	18,039	12,281	4,436	2,263	1,218	727
30 ..	227	255	282	418	432	677	877	1,054	1,404	4,191	3,336	2,304	15,457	10,221	4,113	2,077	1,182	732
31 ..	209	159	241	395	359	450	641	691	945	2,600	2,245	1,782	10,717	7,549	2,936	1,618	868	532
• 32 ..	177	182	227	264	373	405	523	627	786	2,145	2,000	1,354	9,063	6,680	2,532	1,300	823	577
33 ..	173	168	195	150	273	345	441	436	564	1,654	1,468	1,082	6,949	5,086	2,032	1,154	650	441
34 ..	73	105	191	141	218	256	355	327	382	1,264	1,027	773	5,112	3,918	1,712	800	586	373
35 ..	105	100	132	127	227	177	268	318	314	1,014	863	559	4,204	3,530	1,314	714	423	282
36 ..	95	55	150	132	118	150	223	255	264	750	777	450	3,419	2,654	1,232	577	423	268
37 ..	55	91	36	105	64	114	141	155	182	609	527	345	2,424	2,054	1,041	459	282	259
38 ..	41	59	59	45	68	86	105	150	136	400	364	332	1,845	1,786	786	477	314	232
39 ..	73	32	32	68	82	86	123	109	82	309	259	232	1,487	1,427	641	377	200	114
40 ..	41	27	64	59	27	45	64	59	64	168	164	136	918	1,114	523	300	195	109
41 ..	14	27	36	27	41	23	59	32	45	109	105	68	586	600	286	173	114	68
42 ..	9	14	27	32	45	41	14	27	18	55	50	73	405	386	318	159	91	68
43 ..	5	14	18	27	14	18	23	18	14	41	36	36	264	295	132	105	36	64
44 ..	14	0	14	4	14	0	14	14	14	27	9	14	138	209	82	45	50	18
45 ..	0	18	9	0	9	..	5	5	..	27	18	18	109	91	55	50	14	14
46 ..	0	0	5	0	..	..	..	..	..	9	0	5	19	27	18	23	9	0
47 ..	4	0	4	5	..	..	..	..	..	0	0	9	22	14	18	14	5	5
48 ..	..	0	..	0	..	..	..	..	..	0	0	5	5	9	5	0	4	5
49 ..	..	5	..	0	..	..	..	..	..	5	5	0	15	5	0	5	..	4
50 ..	..	..	..	4	..	..	..	..	..	0	..	4	8	0	..	..	..	9
51 ..	..	..	..	..	..	..	..	..	..	0	..	..	0	4	..	..	..	..
52 ..	..	..	..	..	..	..	..	..	..	4	..	..	4	..	..	..	..	..
*Age at Maximum	20.0	20.1	20.1	21.0	20.2	20.6	20.9	21.2	22.4	22.4	23.0	22.80	..	23.6	25.1	26.0	27.6	27.4
Mean Interval	0.0	0.1	0.2	0.3	0.4	0.5	0.5	0.6	0.7	0.8	0.9	1.0	..	1.5	2.5	3.5	4.5	5.5
Age at Marriage	20.0	20.0	19.9	20.7	19.8	20.1	20.4	20.6	21.7	21.6	22.1	21.8	..	22.1	22.6	22.5	23.1	21.9
do.,smoothed	20.0	20.0	20.0	20.1	20.2	20.3	20.4	20.8	21.6	21.7	21.8	21.9	..	22.0	22.3	22.7	22.7	22.3
Frequency at Max.†(crude)	1,890	2,291	3,437	4,527	5,800	6,980	7,933	6,565	5,065	12,100	9,226	6,700	..	24,600	6,960	2,620	1,395	803
do.,(smooth'd	1,890	2,290	3,440	4,530	5,800	6,980	7,930	6,560	5,070	12,100	9,230	6,700	72,520	24,600	6,960	2,620	1,400	800
Totals ..	16,041	18,452	26,929	33,880	41,156	50,722	63,043	52,472	45,742	111053	86,569	61,789	607848	247676	71,816	29,354	14,908	8,541
Smoothed ..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..
Ratio Max. to Total ..	.1178	.1241	.1277	.1337	.1411	.1377	.1258	.1250	.1108	.1089	.1066	.1084	..	.0993	.0969	.0893	.0939	.0936

* Age at beginning of year of maximum. Add 0.5 year for the median age of

...rations of Marriage up to 26 Years with Women of Ages 13 to 52 inclusive. Based upon 220,021

...907-1914 inclusive. Unadjusted Numbers.

	Interval after Marriage during which a Birth Occurs.																		Total Number of First Births.		Age of Mothers.	
	8–9 yrs.	9–10 yrs.	10–11 yrs.	11–12 yrs.	12–13 yrs.	13–14 yrs.	14–15 yrs.	15–16 yrs.	16–17 yrs.	17–18 yrs.	18–19 yrs.	19–20 yrs.	20–21 yrs.	21–22 yrs.	22–23 yrs.	23–24 yrs.	24–25 yrs.	25–26 yrs.	1–26 years.	0–26 years.		
																			..	..	..	12
	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	4	19	..	13
	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	0	124	..	14
	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	18	644	..	15
	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	236	4,327	..	16
	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	1,141	14,639	..	17
	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	3,849	34,179	..	18
	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	9,035	55,589	..	19
0	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	14,371	66,879	..	20
4	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	20,057	85,318	..	21
9	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	27,906	**93,527**	..	22
36	18	..	9	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	32,918	93,361	..	23
32	59	18	9	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	**34,582**	86,412	..	24
36	55	14	18	14	0	4	..	..	..	..	..	..	..	..	..	..	..	..	33,600	77,834	..	25
00	118	55	18	18	5	0	..	..	..	..	..	..	..	..	..	..	..	..	32,688	69,875	..	26
32	145	77	64	5	14	0	..	..	..	..	..	..	..	..	..	..	..	..	30,215	59,748	..	27
45	200	127	36	27	18	5	..	..	..	..	..	..	..	..	..	..	..	..	26,703	51,487	..	28
05	255	127	82	23	41	9	..	..	..	..	..	..	..	..	..	..	..	..	22,222	40,261	..	29
91	**264**	**227**	114	86	55	14	9	9	0	0	4	..	..	..	..	..	..	..	19,975	35,432	..	30
27	227	177	118	45	32	41	9	4	0	0	0	..	..	..	..	..	..	..	14,860	25,577	..	31
80	191	191	**205**	100	91	27	41	5	4	14	0	4	..	..	..	..	..	..	13,526	22,589	..	32
23	191	177	123	77	**109**	68	36	9	9	5	0	0	..	..	..	..	..	..	10,885	17,834	..	33
50	205	136	145	**132**	100	82	27	23	5	9	5	5	..	..	..	..	..	..	8,836	13,948	..	34
32	177	150	105	73	105	**114**	**41**	36	23	9	14	0	4	..	..	..	..	..	7,614	11,818	..	35
95	164	95	145	100	50	45	32	**68**	18	14	9	14	0	..	..	..	..	..	6,362	9,781	..	36
36	86	86	77	77	77	59	41	41	36	32	9	9	5	..	..	..	..	..	5,034	7,458	..	37
23	132	123	50	82	73	18	32	36	**45**	**64**	**27**	9	**18**	5	4	..	..	..	4,595	6,440	..	38
77	59	64	64	59	91	45	27	23	9	18	23	5	14	9	4	..	5	..	3,455	4,942	..	39
14	45	32	32	32	27	59	14	23	27	18	14	14	9	14	4	**9**	0	..	2,710	3,628	..	40
36	36	41	27	23	18	9	0	14	23	18	23	9	5	**14**	5	0	4	..	1,601	2,187	..	41
36	23	36	27	9	14	9	5	9	9	18	5	**27**	4	0	5	5	0	..	1,285	1,690	..	42
32	18	14	5	9	0	9	0	18	5	9	4	5	0	0	5	5	0	..	802	1,066	..	43
18	5	19	9	5	0	0	18	14	9	0	..	0	0	9	0	0	0	**9**	537	675	..	44
5	14	5	0	5	..	9	..	..	..	0	..	0	..	0	**14**	0	..	0	281	390	..	45
9	4	..	0	0	..	5	..	..	..	4	..	4	..	5	0	4	..	0	117	136	..	46
5	..	..	0	4	..	..	..	..	..	..	..	..	..	..	..	..	..	5	74	96	..	47
0	..	..	4	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	27	32	..	48
4	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	18	33	..	49
	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	9	17	..	50
.	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	4	4	..	51
.	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	4	..	52
36	30	31	32	34	33	35	35	36	38	38	38	40	39	41	45	?	?	?	..	..	*Age at Maximum	
7.5	8.5	9.5	10.5	11.5	12.5	13.5	14.5	15.5	16.5	17.5	18.5	19.5	20.5	21.5	22.5	..	..	..	..	..	Mean Int'rv'l	
2.5	21.5	21.5	21.5	22.5	20.5	21.5	20.5	20.5	21.5	20.5	19.5	20.5	18.5	19.5	22.5	..	..	..	..	..	Age at Marriage	
2.0	21.9	21.8	21.7	21.6	21.5	21.4	21.3	21.2	21.1	21.0	20.9	20.8	20.7	20.6	20.5	20.4	20.3	20.2	..	..	do., sm'thed	
391	264	227	205	132	109	114	41	68	45	64	27	14	14	14	14	9	5	9	..	..	Frequency at Max†(crude)	
390	285	225	175	140	115	93	75	60	47	36	27	20	15	11	8	6	5	4	..	..	do., (sm'thed)	
472	2,691	1,991	1,486	1,005	920	631	332	332	221	232	137	105	59	56	41	23	9	14	392,152	1,000,000	Totals	
..	..	..	..	..	820	620	442	322	264	193	140	101	72	51	37	25	16	9	1,000,000	..	Smoothed	
101	.106	.112	.117	.139	.140	.150	.170	.186	.178	.187	.193	.198	.208	.216	.216	.240	.313	.444	..	..	Ratio Max. to Totals	

...maximum 12 months. † The frequency at the maximum is for the age.

The detailed results for the successive years shew considerable regularity in the frequency of first births even for individual ages, as for example the births, for ages 23 and 25 during the tenth month and first year after marriage, were respectively as follows :—

Interval.	Year.	1908.	1909.	1910.	1911.	1912.	1913.	1914.	1908–1914.
Months 10-11	Number (23)	239	232	261	302	328	314	354	290.0*
	Number (25)	195	184	237	249	288	296	308	251.0*
	Corresponding Marriages ..	32,480	32,704	34,127	36,953	39,815	42,078	41,808	37,138*
Years 1–2	Number (23)	622	685	688	698	860	888	909	764.3*
	Number (25)	559	631	604	654	757	813	820	691.1*
	Corresponding Marriages	31,440	32,510	33,163	35,183	38,037	40,814	41,870	36,145*

* Average for the period 1908–1914.

The significance of these figures, which are taken at random, is seen, when the "corresponding marriages" (*i.e.*, the marriages earlier, by the proper interval, than the year indicated) are taken into account. The interval in question is about $10\frac{1}{2}$ months in the one case, and 18 months in the other. Thus for the two upper numbers the figures adopted for 1908 are those for 1907, plus one-eighth of the difference between them and those for 1908, and so on ; and for the lower numbers the mean of the figures for 1906 and 1907 ; and similarly throughout. The ratio of each number to the seventh of the total shews the degree of correspondence since the whole of these ratios are relative, and the vertical columns should be identical for exact correspondence. The ratios corresponding to the six lines above are :—

Interval.	Year.	1908.	1909.	1910.	1911.	1912.	1913.	1914.
Months 10–11	Number (23)	.82	.80	.90	1.04	1.13	1.08	1.22
	Number (25)	.78	.73	.94	.99	1.15	1.18	1.23
	Corresponding Marriages	.87	.88	.92	.99	1.07	1.13	1.13
Years 1–2	Number (23)	.81	.89	.90	.91	1.13	1.16	1.19
	Number (25)	.81	.91	.87	.95	1.09	1.18	1.19
	Corresponding Marriages	.87	.90	.92	.97	1.05	1.13	1.16

Seeing that the original numbers are very limited, the agreement is remarkably good, and confirms the utility of Table LXXVII., and the utility of the graphs of the protogenesic surface, to which surface reference will now be made.

17. **The nuptial protogenesic boundary and agenesic surface.**—If the relative numbers of first-births, after different durations of marriage and for different ages of women, given on Table LXXVII., are regarded as vertical (z) ordinates, with the ages of women and duration of marriage as the other two ordinates (x and y), the surface so defined may be called the *nuptial protogenesic surface* or *surface of nuptial primiparity*. In the graph of such a surface the area for which the ordinates are zero may be called the *agenesic region*, or the surface of absolute sterility; and the boundary between the two may be called the *agenesic boundary*.

The values of x and y for all points on the boundary between the agenesic region and the protogenesic surface are the ages and corresponding durations of marriage which define the existence of perfect sterility. Thus with a duration of marriage of say $6\frac{1}{2}$ years there were no cases of first-births among women of $19\frac{1}{2}$ years of age in the records extending from 1908 to 1914; see Table LXXVII. or Fig. 72.

The Protogenesic Surface. Australia, 1908-1914.

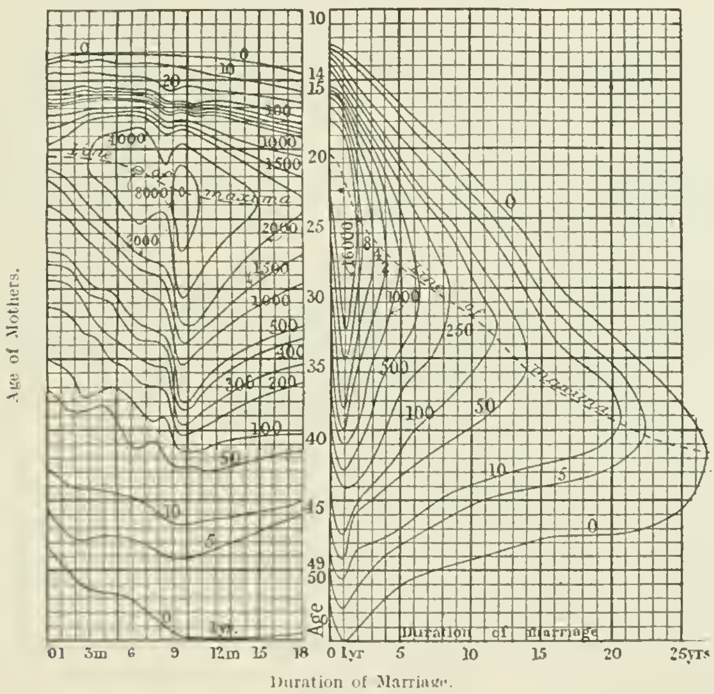


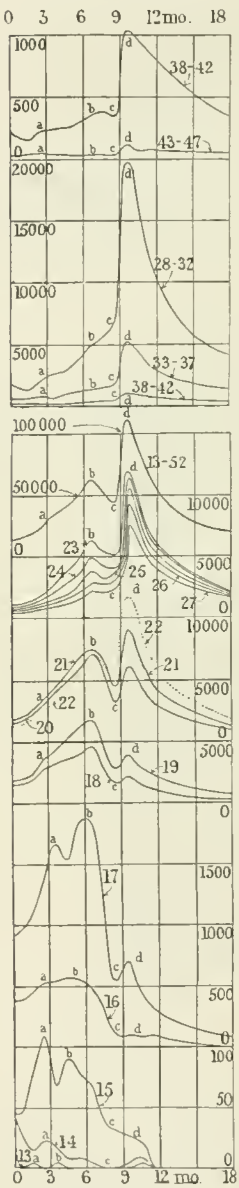
Fig. 72.

The contours represent equal frequency of first-births with varying age and duration of marriage. The area outside the contour 0 is the agenesic region. The figures on the contours are per million first births, for all women of age x last birthday, and for durations of marriage t to $t+1$, where t is expressed in months on the left hand part of the figure and in years on the right hand part.

The characteristic features of the protogenesic surface are shewn in Figs. 72 and 73. On Fig. 72 this surface is defined, by contours, on extended lateral scale for 0 to 18 months after marriage, and on a smaller

lateral scale from 0 to 27 years after marriage, and in both cases for the whole nuptial reproductive period, say 13 to 52 first-births. A vertical frequency of 1 on the right hand side of Fig. 72 corresponds to the frequency of 12 on the left hand side. The line of maxima is shewn by a broken line on the figure. In Fig. 73 the vertical sections of the protogenesic surface are shewn for each age from 13 to 27 years ("age last birthday"), and for the 5-year groups 28-32, 33-37, 38-42, 43-47, and for the group for all ages from 13 to 52.

Profiles of the Protogenesic Surface.



Duration of Marriage.

Fig. 73.

The frequencies of first-births, which are identical on any contour, are indicated by figures. These are per million total first births for intervals of a month of duration of marriage on the left-hand side of the figure, and for intervals of a year's duration of marriage on the right hand side. The "age" indicated is always to be taken as the "age last birthday," or what is the same thing, and more general, for the age x to the age $x+1$.

It will be seen that these contours constitute a family of curves for which there is no simple mathematical specification. The unique maximum shewn by a small contour like an "O" on the left hand side of the figure and by an asterisk on the right hand side.

The profiles of the protogenesic surface, shewn on Fig. 73, from 0 to 18 months, are the curves shewing the frequency at various ages, for a total of a million first-births at all ages, and for the first 18 months after marriage. These curves have characteristic similarities, indicated by the points letters a, b, c, d, on the figure. The similarities are important since they shew that there is a remarkable regularity in the interval between marriage and first-birth in women of different ages. The curves drawn are not for instantaneous group-values, viz., for the groups x to $x+dx$, but are the values for mensual groups, the abscissæ for which are referred to the middle of the month.

18. **Curve of nuptial protogenesic maxima.**—The curved broken line on Fig. 72, shewing the ordinates for greatest frequency of first-birth, can be replaced by a regular curve, which will give the actual values of these ordinates with sufficient pre cision. Adopting as the argument the "age last

birthday," that is the initial value of the age where the range is from x to $x+1$, and for the corresponding initial value of the duration y, where the duration meant is from y to $y+1$, we have—

$$(474)\ldots\ldots y = 1.45\ \xi^{\frac{5}{3}} = 1.45\ (x - 20)^{\frac{5}{3}}$$

ξ being the "age last birthday" less 20. This gives the values on the upper line, while those on the lower are scaled from Fig. 72 :—

Age ..	20	25	30	35	40	
Calc. Value	0.0	2.12	6.73	13.23	21.37	(Initial value of the duration.)
Graph Value	0.0	2.2	7.0	13.0	21.5	(.. ,, ,,

The maximum frequency per million total births, where the age is "age last birthday," and the duration is from y to $y+1$, cannot be expressed by any simple mathematical formula. The values, however, are given at the bottom of Table LXXVII.

19. **Ex-nuptial protogenesis.**—The previous issue is not ascertained in the case of ex-nuptial births, and the point of time to which the interval corresponding to duration of marriage should be referred cannot be defined. Hence no comparison can be made with nuptial protogenesis.

20. **Average age for quinquennial age-groups of primiparæ.**—The following table gives the average age of mothers of first-births in quinquennial groups :—

TABLE LXXVIII.—Average Age of Mothers, First-births, for Quinquennial Groups.

Age-group ..	14	15–19	19	20–24	25–29	30–34	35–39	40–44	45–49	50–52
Average Age ..	14.36	18.78	18.77	22.61	27.19	32.06	37.08	41.74	46.31	41.84
Middle Age	14.0	17.50	16.51	22.50	27.50	32.50	37.50	42.50	47.50	46.50
Difference ..	+0.36	+1.28	+2.27	+0.11	−0.31	−0.44	−0.42	−0.76	−1.19	−4.66

The differences between the middle and average ages are obviously too large to be neglected, and therefore it is always necessary to decide whether the average value or the middle value of the ranges of the argument (age-group ranges) shall be used. In general the middle value is the more convenient.

21. **Average interval between marriage and a first-birth, a function of age.**—The data furnished in Table LXXVII. shew that the average interval between marriage and first-births is a definite function of age.[1]

[1] T. A. Coghlan, in his "Child-birth in New South Wales, a study in statistics," has given results (see his Table VIII., on p. 26) for the average period from "marriage to birth of first-child" for "post-nuptial conceptions only." He introduces an adjustment for the non-stationary character of the population from which they are derived, see p. 26. His main result, however, is wholly erroneous, and the true result is inconsistent with his conclusion, viz., that for married women between the ages of 17 and 39 the average period between marriage and a first-birth is only 19.4 months, and the range between 18.3 and 21.5 months. The matter will be referred to more fully later, see pp. 271-2, and particularly the note on the latter page.

If age-groups of primiparæ be formed, it is found that the mean ages of the groups and the average intervals between marriage and first-births are as shewn in the third column of the Table hereunder, viz., LXXIX., see also Figs. 74 and 75. The average values of the ages and of the corresponding intervals are as follows :—

TABLE LXXIX.—Average Ages and Average Interval between Marriage and First-births.

(i.) Age of Married Women ..	Under 20	20–24	25–29	30–34	35–39	40–44	45–49
(ii.) Average Age	18.77	22.61	27.19	32.06	37.08	41.74	46.31
(iii.) Average interval between Marriage and First-birth (crude data)	0.623	0.994	1.483	2.026	2.862	3.501	4.048
(iv.) Average interval by formula (smoothed data)	0.604	0.991	1.502	2.100	2.766	3.420	4.209
(v.) Difference (data–calc.) ..	+0.019	+0.003	—0.019	—0.074	+0.096	+0.080	—0.161

The values on line (iii.) are fairly well given by the simple formula :—

$$(475)\ldots\ldots i = 0.0437\,\chi + 0.01221\,\chi^{1\cdot 5}$$

where i is the *average interval* between marriage and the first-births, and χ is 11 years less than the average age, a 5-year group, that is to say, the age 11 is taken as the zero of χ. This age has not been arbitarily adopted, but, as is shewn by the line OP on Fig. 74, is indicated as the minimum age to which reproductive facts should be referred. (See Table LXXII., p. 239 and p. 268).

The small crosses in Fig. 75 are the results for individual years of age last birthday, computed by means of the formula (475) ; see p. 268.

There is a fairly definite indication that the continuation of the curve should be as shewn by the broken line in Figs. 74 and 75, terminating therefore at about age 55. There are, however, so few births at ages greater than 45, that this part of the curve cannot be regarded as yet well determined or determinable : see p. 268.

The following Table, LXXX., gives the results in greater detail, and furnishes also smoothed values of the *approximate* average interval[1] between marriage and first-births for all first-births within a year of marriage, and for all ages during the reproductive period. Since formula (475) refers to the average age, it will not give the quantities in the Table LXXX :—

[1] The intervals are only approximate. They have been calculated by assuming that the births in each month during the first 12 months may be referred to the middle of the months, and those during the intervals of from 1–2 years onwards may be referred to the middle of the year. The change in rapidity of births is so great during the year after that of marriage that a correction is necessary for rigorous accuracy. That the difference is appreciable is obvious from the following results :—

	1st Births to 9 mths	1st Births to 12 mths	All First-births.
Approximate average interval, age 22 ..	5.52	7.53	11.70 months
Average interval more rigorously calculated . .	5.53	7.54	10.88 months

The intervals are found more rigorously hereinafter for births occurring not earlier than nine months after marriage.

TABLE LXXX.—The Protogenesic Indices, according to Age. (Approximate Average Intervals between Marriage and First-births)* Australia, 1908-1914.

Age of Mother at Birth.	Average Interval. Births Occurring within 12 months after Marriage. Crude Result.	Births Occurring within 12 months after Marriage. Smo'th'd Result.	Births Occurring after 12 mths. Marriage. Crude Result.	All First-births. Crude Result.	All First-births. Smoothed Result.	All First-births. Smoothed Result.	Interval for Age-Group. Crude Result.
Years.	Months.	Months.	Months.	Months.	Months.	Years.	Months.
10 ..	..	..	..	..	0.00	0.00	
11 ..	..	..	..	..	0.85	0.071	
12 ..	..	..	..	..	1.72	0.143	
13 ..	5.17	2.88	18.00	8.38?	2.61	0.217	4.03
14 ..	3.30	3.49	..	3.39	3.53	0.294	
15 ..	4.41	4.06	18.00	4.80	4.47	0.373	(7.48)
16 ..	4.49	4.60	19.85	5.33	5.44	0.453	
17 ..	5.12	5.11	19.15	6.21	6.44	0.537	7.55
18 ..	5.64	5.58	19.54	7.20	7.47	0.623	
19 ..	6.07	6.03	19.94	8.32	8.53	0.711	
20 ..	6.56	6.44	20.57	9.57	9.62	0.802	
21 ..	6.89	6.82	21.19	10.25	10.73	0.896	
22 ..	7.53	7.11	21.51	11.70	11.91	0.992	11.93
23 ..	7.77	7.48	22.84	13.08	13.10	1.092	
24 ..	8.04	7.77	23.99	14.42	14.33	1.195	
25 ..	8.26	8.02	25.29	15.61	15.60	1.300	
26 ..	8.36	8.24	26.69	16.94	16.91	1.409	
27 ..	8.55	8.43	27.75	18.20	18.26	1.522	17.80
28 ..	8.52	8.58	29.46	19.38	19.65	1.638	
29 ..	8.61	8.71	30.95	20.94	21.08	1.757	
30 ..	8.75	8.80	33.68	22.81	22.56	1.880	
31 ..	8.64	8.86	34.39	23.60	24.08	2.007	25.15
32 ..	8.55	8.89	37.44	25.85	25.65	2.137	
33 ..	8.49	8.88	39.41	27.36	27.26	2.272	
34 ..	8.46	8.85	42.38	29.96	28.93	2.411	
35 ..	8.30	8.78	43.75	31.11	30.64	2.553	
36 ..	8.30	8.68	45.85	32.73	32.41	2.700	
37 ..	8.44	8.55	47.51	34.82	34.22	2.852	34.34
38 ..	8.45	8.38	52.24	39.70	36.09	3.008	
39 ..	7.99	8.19	50.15	37.47	38.02	3.168	
40 ..	7.71	7.96	49.74	39.09	40.00	3.333	
41 ..	7.62	7.70	55.74	42.82	42.03	3.503	
42 ..	7.32	7.41	56.15	44.50	44.13	3.677	42.01
43 ..	7.26	7.08	53.32	41.90	46.28	3.856	
44 ..	6.87	6.73	59.69	48.99	48.50	4.041	
45 ..	7.46	6.34	55.77	42.13	50.77	4.231	
46 ..	8.25	5.92	78.92	69.50	53.11† 51.50‡	4.426† 4.292‡	
47 ..	5.90	5.17	61.50	50.55	55.51 50.55	4.626 4.212	48.58
48 ..	11.50	4.98	52.00	46.21	57.98 46.22	4.832 3.851	
49 ..	7.17	4.47	54.00	33.93	60.52 40.50	5.043 3.375	
50 ..	7.50	3.92	66.00	36.75	63.12 30.50	5.260 2.542	
51 ..	..	3.34	18.00	18.00	65.79 18.00	5.483 1.500	28.54
52 ..	9.50	2.73	..	9.50	68.53 9.50	5.711 0.792	
53 ..	..	..	..	..	71.34 3.00	5.945 0.250	
54 ..	..	..	..	..	74.23 1.20	6.185 0.100	
55 ..	..	..	..	..	77.18 0.00	6.432 0.000	
	7.49 months 24.20 years		29.06 m. 27.34 yrs.		15.05 months. 25.43 years.		

* Based on a total of 220,021 births. † These values from ages 46 to 55 are merely extensions of the curve. ‡ These values are probably fairly reliable.

The yearly groups may with advantage be referred to the "age last birthday," instead of the middle-age value, which is approximately the "age last birthday plus $\frac{1}{2}$." Let then ξ denote the "age last birthday," less 10; the intervals are found to be very accurately given in months and in years respectively by the following formulæ, viz.:—

(476)....$i' = 0.8364\xi + 0.01062\xi^2 + 0.000198\xi^3$, and for months;[1]

(476a)....$i'' = 0.0697\xi + 0.000885\xi^2 + 0.0000165\xi^3$, for years:[1]

ξ is of course expressed in years in either case. The values may be readily computed by taking the interval for age 10 as zero, and the smoothed results for 20, 30 and 40, and applying formulæ (199) to (199c), see Part V., § 7, p. 69, and remembering that the coefficients b, c, d vary inversely as the variable, and as the square, and the cube of the variable, respectively. To develope the table we may calculate the values for 11, 12 and 13 (*i.e.*, for $\xi = 1, 2$ and 3), or calling the leading differences for 10 years as D_1, D_2, and D_3, the leading differences d_1, d_2 and d_3 can be found by the formulæ[2]—

(477)......$d_1 = 0.1\,D_1 - 0.045\,D_2 + 0.0285\,D_3$

(477a)....$d_2 = \ldots\quad 0.01\,D_2 - 0.009\,D_3$

(477b)....$d_3 = \ldots\quad \ldots\quad 0.001\,D_3$

We have also, for the coefficients of the equations above:—

(478)......$b = d_1 - \frac{1}{2}d_2 + \frac{1}{3}d_3$

(478a)....$c = \ldots\quad \frac{1}{2}d_2 - \frac{1}{2}d_3$

(478b)....$d = \ldots\quad \ldots\quad \frac{1}{6}d_3$

The agreement between the crude values and the values by formula (476) for the average interval between marriage and first-birth is remarkably close throughout, the curve applying as far as age 45. Beyond this age the values for the extrapolated curve are given as well as those of the probable value of the interval.

22. **The protogenesic indices.**—The average interval, calculated as shewn in the preceding section (viz., by formula (484) in the section next following, § 23, but omitting the correction term ϵ) is not rigorously

[1] These formulæ are for the "approximate" average interval; see the preceding note.

[2] See Text Book, Institute of Actuaries, Part II., Chap. XXIII., Art. 22, p. 443, Edit. 1902.

exact, but is sufficiently approximate to be used as an index of the frequency distribution throughout the interval. We shall call the interval so calculated the *protogenesic index* for married women of the age in question, and for all ages, the *general protogenesic index*.[1]

Table LXXX. is thus a table of protogenesic indexes rather than a table of average intervals, though the intervals are approximately correct.

We shall now consider methods of correctly estimating the interval.

23. Exact evaluation of the average interval from a limited series of group-values.—The average interval may be determined with a higher degree of approximation from the series of group-values *for equal ranges* themselves by formulæ developed as follows:—Since the group-values can often[2] be reproduced with sufficient accuracy by a rational integral function we have, in such cases, for the value (x_m) of the interval (the distance to the centroid vertical):—

$$(479)\ldots\quad x_m = \frac{\int_0^x xy\,dx}{\int_0^x y\,dx} \equiv \frac{\frac{1}{2}ax^2+\frac{1}{3}bx^3+\text{etc.}}{ax+\frac{1}{2}bx^2+\text{etc.}} = \frac{1}{2}x + \frac{bx^2+cx^3+\frac{9}{10}dx^4+\frac{4}{5}ex^5+\ldots}{12a+6bx+4cx^2+3dx^3+\frac{12}{5}ex^4+\ldots}$$

in which last expression we may substitute, by means of formulæ (195) to (197), see Part V., § 5, pp. 67, 68, the values of the groups themselves for a, b, c, etc. This will give a series of formulæ according to the number of groups taken simultaneously into account. We may take the common value and the ranges as unity: if it be k the value deduced will then be multiplied finally by k.

It will be convenient to call the group values A, B, C, etc., hence if n of these are included, n will be the value of x. That is to say, in formulæ in which D appears, x will be 4. From (479) we thus obtain the following series of formulæ, viz:—

$$(480)\ldots\ldots x_m = 1 + \frac{2}{3} \cdot \frac{-A+B}{A+B}\,;\ \text{for } x = 2;$$

$$(481)\ldots\ldots x_m = 1\tfrac{1}{2} + \frac{9}{8} \cdot \frac{-A+C}{A+B+C}\,;\ \text{for } x = 3;$$

$$(482)\ldots\ldots x_m = 2 + \frac{4}{45} \cdot \frac{-19A-3B+3C+19D}{A+B+C+D}\,;$$

for $x = 4$; and

$$(483)\ldots\ldots x_m = 2\tfrac{1}{2} + \frac{125}{288} \cdot \frac{-5A-2B+2D+5E}{A+B+C+D+E}\,;\ \text{for } x = 5$$

[1] To fully define the term it should be preceded by the term "nuptial"; but for obvious reasons this may be always understood.

[2] But not invariably: see latter part of Section 24.

If the common range be k, these expressions should of course be multiplied by that quantity.

From these formulæ multiply-infinite series of formulæ may be developed, and such development can be effected by processes similar to those indicated in Part VI., § 2, and Table VI., pp. 75 to 77.

A practical way of applying the formulæ is to calculate by an approximate method and make the necessary correction, if it be sensible. Thus :—

$$(484)\ldots\ldots x_m = \frac{(A+3B+5C+7D+\text{etc.})}{2(A+B+C+D+\text{etc.})} + \epsilon$$

where ϵ is a small quantity. For the value of ϵ, we have, from (480) to (483) :—

$$(485)\ldots\epsilon_2 = \frac{1}{6}\cdot\frac{-A+B}{A+B};\ \text{when there are two quantities only.}$$

$$(486)\ldots\epsilon_3 = \frac{1}{8}\cdot\frac{-A+C}{A+B+C};\ \text{when there are three quantities only.}$$

$$(487)\ldots\epsilon_4 = \frac{1}{90}\cdot\frac{-17A+21B-21C+17D}{A+B+C+D};\ \text{when}$$

there are four quantities only.

$$(488)\ldots\epsilon_5 = \frac{1}{288}\cdot\frac{-49A+38B-38D+49E}{A+B+C+D+E};\ \text{when there}$$

are five quantities only.

Σ_n denoting the sum of n successive groups, A, B, etc., these expressions may be put in the arithmetically more convenient form hereunder, viz. :—

$$(489)\ldots\epsilon_2 = 0.1\dot{6}(B-A)/\Sigma_2;\ \epsilon_3 = 0.125(C-A)/\Sigma_3;$$

$$\epsilon_4 = \{0.1\dot{8}(D-A)-0.2\dot{3}(C-B)\}/\Sigma_4$$

$$(490)\ldots\epsilon_5 = \{0.17014(E-A)-0.13194(D-B)\}/\Sigma_5$$

Whenever each group-value in a series is not greater than say 2 to 2½ times an adjoining group-value, the preceding formulæ give fairly good results, and may be used for a succession of three, four, or five groups in a way which will now be indicated.

24. **Evaluation of group-intervals for an extended number of groups.**—To apply the preceding formulæ to a large number of groups it will be convenient to adopt the following notation. Let A, B, C, etc., be denoted by A_1, A_2, A_3, etc., and let also $x' = x_m$ for A_1 to A_k reckoned from the beginning of A_1, $x'' = x_m$ for say A_{k+1} to A_m, etc., reckoned

not from the beginning of A_{k+1}, but from the beginning of A_1; and so on. Let also A', A'', etc., denote the totals of the various series of groups in question; that is, let $A' = A_1 + A_2 +$ etc.; $A'' = A_{k+1} + A_{k+2} +$ etc.; and so on.[1] Then the value for the entire series is :—

$$(491)\ldots\ldots {}_0x_m = \frac{A'x' + A''x'' + A'''x''' + \text{etc.}}{A' + A'' + A''' + \text{etc.}} = \frac{\Sigma(A'x')}{\Sigma A'}.$$

Consequently, if $x' = w' + \epsilon'$, where w' is an approximate value of x' and ϵ' is the *correction* to make it exact, we shall have for the true value of ${}_0x_m$:—

$$(492)\ldots\ldots {}_0x_m = \frac{\Sigma(A'w')}{\Sigma A'} + \frac{\Sigma(A'\epsilon')}{\Sigma A'}$$

in which $\Sigma(A'\epsilon') = A'\epsilon' + A''\epsilon'' +$ etc. Let the factors 1/6, 1/8, 1007/90, 3051/90, 49/288, 38/288, on formulæ (485) to (488) be denoted by a_1, a_2, etc., and generically by a', a'', etc. Then, since when ϵ', ϵ'', etc., are multiplied respectively by A', A'', etc., their denominators disappear, we have, for the total correction ϵ_0 say, the sum of the numerators divided by the sum of all the groups. Thus $a A'$, $a'' A''$, etc., denoting the numerators, we have :—

$$(493)\ldots\ldots {}_0x_m = \frac{\Sigma(A'w')}{\Sigma A'} + \frac{\Sigma(a'A')}{\Sigma A'}$$

that is to say, the approximate value of the average interval, found by multiplying each group by the middle value of its interval, and dividing the sum of all the products by the sum of all the groups, merely requires the correction found by multiplying each group by its correction coefficient (a), and dividing by the sum of the whole of the groups. Hence formulæ may be developed to embrace the corrections by multiplying the individual groups by factors, and these factors are readily found by summations. Thus we obtain the following, viz. :—

$$(494)\ldots {}_0x_m = (0.375\,A_1 + 1.5\,A_2 + 2.625\,A_3 + 3.375\,A_4 + 4{\cdot}5\,A_5 + 6.625\,A_6 + \text{etc.}) / \Sigma A.$$

the series of coefficients being in threes; thus the coefficient for the third term from any term of the series is 3 greater than that of the term from which it is reckoned. Further, :—

$$(495)\ldots {}_0x_m = (0.3\dot{1}\,A_1 + 1.7\dot{3}\,A_2 + 2.2\dot{6}\,A_3 + 3.6\dot{8}\,A_4 + \text{etc.}) / \Sigma A; \text{ and}$$

$$(495a)\ldots {}_0x_m = (0.32986\,A_1 + 1.63194\,A_2 + 2.5\,A_3 + 3.36806\,A_4 + 4.57014\,A_5 + \text{etc.}) / \Sigma A;$$

the series of coefficients being respectively in fours and in fives: thus the coefficient of the fourth term in the one case, and of the fifth term in the other, from any term in the series, is 4 greater in the former case and 5 greater in the latter, than the coefficient of the term from which it is reckoned.

[1] It is of course immaterial what number of groups are combined.

25. **Average interval for curves of the exponential type.**—In cases where A_2 is very small (or very large) compared with A_1, the preceding formulæ are not very accurate.[1] In general, if the curve giving the groups be approximately of the type $e^{\pm mx}$, and the groups be also very different in magnitude, it is preferable to proceed as follows :—[2]

Let A_1, A_2 be two adjoining groups ; these can be satisfied by the equation :—

$$(496)\ldots\ldots y = Be^{bx}, \text{ or } y = e^{a+bx} ;$$

in the former of which, therefore, $B = e^a$. Similarly three adjoining groups, A_1, A_2 and A_3 may be satisfied by the equation :—

$$(497)\ldots\ldots y = A + Be^{bx}$$

Putting A_1 the group for the range 0 to 1 ; A_2 the group with the range 1 to 2 ; A_3 the group with the range 2 to 3 ; we have from these equations the following, viz. :—From (496) :—

$$(498)\ldots\ldots \frac{A_2}{A_1} = \frac{A_3}{A_2} = e^b ; \text{ or } b = 2.3025851 \; log_{10} \frac{A_2}{A_1}$$

and this applies to a whole series of groups if the ratio A_{n+1} / A_n be constant. Also :—

$$(499)\ldots\ldots B = \frac{b\,A_1}{e^b - 1} = \frac{b\,A_2}{(e^b - 1)e^b} = \frac{b\,A_3}{(e^b - 1)\,e^{2b}} = \text{etc.}$$

the final equation in (499) being true only if $A_3/A_2 = A_2/A_1 = n$, say.

From (497) we have, similarly to (499) :—

$$(500)\ldots A_1 = A + B\,(e^b - 1)/b \; ; \; A_2 = A + B\,(e^b - 1)\,e^b/b \; ;$$
$$A_3 = A + B\,(e^b - 1)\;e^{2b}\,/\,b \; ;$$

and consequently :—

$$(501)\ldots.(A_3 - A_2)\,/\,(A_2 - A_1) = e^b,$$
$$\text{or } b = 2.3025851 \; log._{10}\{(A_3 - A_2)/(A_2 - A_1)\} ; \text{ etc.}$$

---

[1] For example, if there be two groups, on equal bases 0-x, x-$2x$, one of which is three times greater than the other, the straight line (which in such a case would be the assumed curve, giving areas equal to the groups), would start at the terminal 0 (or $2x$) of one of the groups. If one is greater than 3 times the other, it will fall within one of the rectangles. The question has been exhaustively considered by Prof. Karl Pearson, see Biometrika, Vo. I., pp. 265-303, Vol. II., pp. 1-23.

[2] As the formulæ of this section are of general application x has been used for the independent, and y for the dependent variable.

Writing n for e^b, we have also :—

$$(502)\ldots B=b\,(A_2-A_1)/(n-1)^2 = b\,(A_3-2A_2+A_1)/(n-1)^3, \text{ etc. };$$

and

$$(503)\ldots\ldots A=A_1-\frac{B}{b}(n-1)=A_1-\frac{A_2-A_1}{n-1}=\frac{A_1n-A_2}{n-1}, \text{ etc.}$$

Thus the constants b, B and A in (497) are determined.

In applying these formulæ to ascertain the average interval, four cases will require specially to be considered, viz., when the factor b is positive, and when it is negative, the range being either 0 to 1, or 1 to 2 in both cases. For the ascending and descending branches respectively, these cases correspond to the curves Be^{bx} and Be^{-bx}. For the purpose in view (496) is suitable, and the results, to be tabulated for various ratios of A_1/A_2 or A_2/A_1, will be the groups $B\int_0^1 e^{bx}$, $B\int_1^2 e^{bx}$, $B\int_0^1 e^{-bx}$, and $B\int_1^2 e^{-bx}$. The mean interval lies between the centre of the group-range and the side on which the groups have higher values. For the more general case, that is when three values are satisfied, we should have to determine

$$(504)\ldots x_m=\frac{\int xy\,dx}{\int y\,dx}\equiv\frac{\int x(A+Be^{bx})\,dx}{\int (A+Be^{bx})\,dx}\equiv\frac{\frac{1}{2}Ax^2+B\left\{(bx-1)e^{bx}+1\right\}/b^2}{Ax+B\,(e^{bx}-1)\,/\,b}$$

If A, however, be taken as zero, this last will become

$$(505)\ldots\ldots {}_0x_m \equiv \frac{(bx-1)\,e^{bx}+1}{b\,(e^{bx}-1)}=\frac{xe^{bx}}{e^{bx}-1}-\frac{1}{b}$$

which function is the basis of the tabulation hereunder for ratios of A_2 to A_1 and for ranges of $x=1$ and 2, by applying (498). It may be noted that the value of (505) $=0$ for $x=0$. In the table hereunder, LXXXI., the four cases above referred to are as follows :—

Case I. ;	$A_2 > A_1$;	Origin 0 ;	Range $0-k$;	Tabular Interval computed from	0.
„ II. ;	„ ;	„ ;	„ $k-2k$;	„ „ „ „	k.
„ III. ;	$A_2 < A_1$;	„ ;	„ $0-k$;	„ „ „ „	0.
„ IV. ;	„ ;	„ ;	„ $k-2k$;	„ „ „ „	k.

These four cases are illustrated by Fig. 78, hereinafter.

The necessary formulæ for calculating the required values are simple if we put $A_2 = n\,A_1$, viz. :—

$$(506)\ldots {}_0x'_1=1+\frac{1}{n-1}-\frac{1}{b}\,;\ {}_1x'_2=2+\frac{1}{n-1}-\frac{1}{b};\ \ldots {}_{p-1}x'_p=p+\frac{1}{n-1}-\frac{1}{b}\,;$$

formulæ which are convenient for computing tabular values.

For negative values of b, in which case A_2 is less than A_1, it is arithmetically convenient to use the ratio $A_1/A_2=m$, so that $m=1/n$, and put $\beta=-b$, then the preceding formulæ become :—

$$(507)\ldots {}_0x''_1=1-\frac{m}{m-1}+\frac{1}{\beta}\,;\ {}_1x''_2=2-\frac{m}{m-1}+\frac{1}{\beta}\,;\ {}_{p-1}x''_p=p-\frac{m}{m-1}+\frac{1}{\beta}$$

It may be easily verified that ${}_{p-1}x'_p + {}_{p-1}x''_p = 1$.

By means of the preceding formulæ Table LXXXI. has been computed: it will serve for readily estimating the position of the centroid vertical for any group by means of the relative magnitudes of the adjoining groups. The determination of that vertical from the relative magnitudes of the groups on either side of any group in question gives results of a fair degree of precision.

To satisfy three groups by means of (497) we have for the value of A in terms of A_1 to A_3, :—

$$(508)\ldots\ldots A = \frac{A_1 A_3 - A_2^2}{A_1 + A_3 - 2A_2}$$

instead of (503): hence we can subtract this quantity from the groups and we then obtain :—

$$(509)\ldots\ldots A'_1 = A_1 - A\ ;\quad A'_2 = A_2 - A\ ;\ \text{etc.};\ \text{etc.}$$

these reduced groups, denoted by accents, conforming to the relation $A'_3/A'_2 = A'_2/A'_1$.

The value of the average interval is therefore :—

$$(510)\ldots {}_0x_p = \frac{\tfrac{1}{2}(1+3+\ldots 2p-1)A + A'_1\,{}_0x'_1 + A'_2\,{}_1x'_2 + \ldots\text{to } p \text{ terms}}{A_1 + A_2 + \ldots\ldots\text{to } p \text{ terms.}}$$

Results so computed have a high order of precision. If A, and A'_1, etc., be expressed in ratios to $A_1 +$ etc., as unity, the denominator of course disappears.

TABLE LXXXI.—Abscissæ of the Centroid Verticals of Groups Bounded by the Curve Be^{bx} and Be^{-bx}. For the Computation of Average Intervals, etc.

Ratio A_2/A_1 or A_1/A_2	Case I.	Case III.	Ratio A_2/A_1 or A_1/A_2	Case I.	Case III.	Ratio A_2/A_1 or A_1/A_2	Case I.	Case III.
1.0	.50000	.50000	4.0	.61199	.38801	9	.66988	.33012
1.25	.51857	.48143	4.5	.62085	.37915	10	.67682	.32318
1.5	.53370	.46630	5.0	.62867	.37133	11	.68297	.31703
1.75	.54639	.45361	5.5	.63563	.36437	12	.68848	.31152
2.0	.55731	.44269	6.0	.64189	.35811	13	.69346	.30654
2.25	.56685	.43315	6.5	.64757	.35243	14	.69800	.30200
2.5	.57531	.42469	7.0	.65277	.34723	15	.70216	.29784
2.75	.58290	.41710	7.5	.65754	.34246	20	.71672	.28328
3.0	.58976	.41024	8.0	.66196	.33804	25	.73100	.26900
3.5	.60177	.39823	8.5	.66606	.33394	50	.76479	23521

For case II. add unity to the value for case I., and for case IV. add unity to the value for case III.

Applying the various formulæ to the results given on the penultimate line on Table LXXVII. for all first-births, 12 months or more after marriage, the following results are obtained :—

By formula (484), neglecting the correction ϵ, 29.06 months (Index).
,, (494), applied through same range, 28.18 ,, (Interval)
,, (495) ,, ,, ,, 28.00 ,, ,,
,, (506) ,, ,, ,, 27.72[1] ,, ,,
By graduating and using monthly values for the groups up to 48 months 27.70[1] ,, ,,

26. **Positions of average intervals for groups of all first-births.**—The positions of the average intervals (abscissæ of the centroid verticals), computed on the basis of the results shewn on the penultimate line of Table LXXVII., will probably be found approximately true for any population. By means of Table LXXXI., they may be readily found.

TABLE LXXXII.—Average Intervals* in Months for First-births, for Various Ranges of Interval. Australia, 1908-1914.

Range of Int'rval	Average Value.	Range of Int'rval	Average Value.	Range of Int'rval	Average Value.	Range of Int'rval	Average Value.	Range of Int'rval	Average Value.	Range of Interval	Average Value.
months.	months.	years.	months.	years.	months.	months.	months.	months.	months.	years.	months.
0– 1	.051	0– 1	7.51	12–13	149.76	0–3	1.70	0– 6	3.68	¾– 1	10.34
1– 2	1.52	1– 2	16.35	13–14	161.69	3– 6	4.65	0– 9	5.41	¾– 5	17.35
2– 3	2.53	2– 3	28.95	14–15	173.67	6– 9	7.40	0–12	7.51	¾–10	19.60
3– 4	3.52	3– 4	39.22	15–16	185.73	9–12	10.34	years		¾–15	20.43
4– 5	4.52	4– 5	53.39	16–17	197.74	years		0– 5	13.07	¾–20	20.72
5– 6	5.52	5– 6	65.53	17–18	209.69	0– 1	7.51	0–10	14.63	¾–25	20.79
6– 7	6.50	6– 7	77.61	18–19	221.68	1– 5	22.36	0–15	15.19	¾–26	20.80
7– 8	7.49	7– 8	89.63	19–20	233.67	5–10	81.12	0–20	15.38	1– 5	22.36
8– 9	8.55	8– 9	101.67	20–21	245.66	10–15	142.95	0–25	15.43	1–10	25.82
9–10	9.53	9–10	113.71	21–22	257.67	15–20	203.05	0–26	15.43	1–15	27.13
10–11	10.52	10–11	125.73	22–23	269.65	20–25	261.40			1–20	27.59
11–12	10.53	11–12	137.77	23–24	281.58	25–26	305.43			1–25	27.71
				24–25	293.49					1–26	27.72

* These will be sensibly true for any distribution at all similar to that shewn in Table LXXVII. and in Table LXXXIII. hereinafter.

The above results have been computed by using graphic graduation[2] where necessary, by means of the values given in Table LXXXI., and by formula (416), p. 201. In general the computed values proved to be sensibly identical. A result intermediate between the extreme values has always been taken, regard being had to the general probabilities of each case.

[1] These last results are the most accurate; the value for the month 11-12 is taken into account in the graduating; in applying formulæ (494) and (495) and (506) it is not considered.

[2] It is impossible in the absence of monthly data to determine the position of the centroid vertical with great precision. By graphic graduation conforming to the 11 to 12 months group and to the 1-2, and 2-3 years groups, the result, 16.46 was obtained. By extrapolating the 10-11, 11-12 months group-results, adopting this extrapolation for the year-group 0-1, and conforming to this fictitious year-group and the actual year-groups 1-2 and 2-3, the result is 16.25 by formula (510). Adopting the extrapolated result and the group 1-2 only, gives 15.91; while the exponential curve conforming to the group 1-2 and 2-3 only, gives the result 16.79. The groups 1-2, 2-3 and 3-4, treated by formulæ (508) and (510) give 16.63. After consideration of all the circumstances I have adopted 16.35 as the result which I believe to be nearest the correct value. Similarly the results 28.95, 28.93 and 29.11 were obtained for the group 2-3; of these the first was adopted.

Average Issue and the Protogenesic Indices.

Fig. 75.

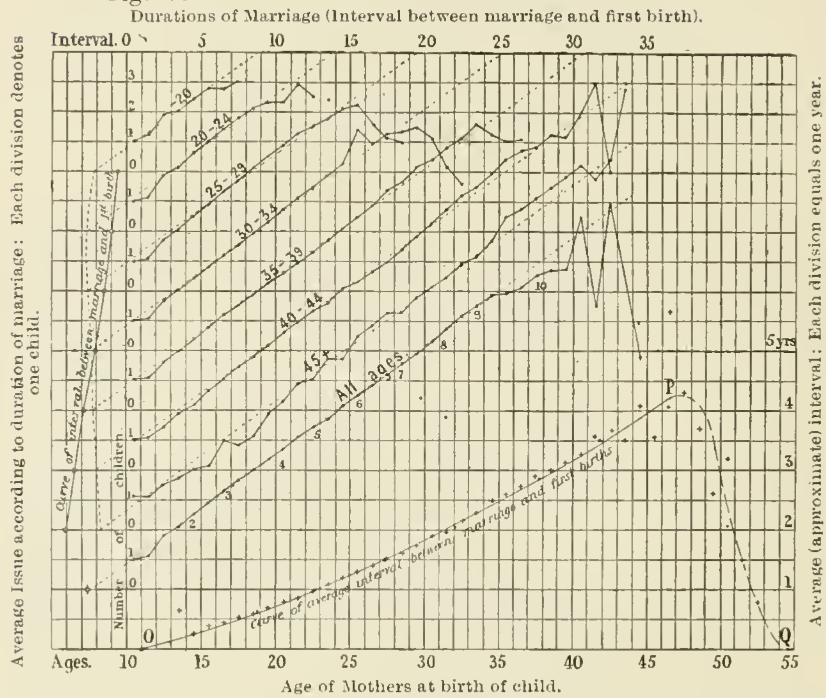


Fig. 74.

FIG. 74.—The lower curve OPQ is the curve of the protogenesic indices (or approximately computed average intervals between marriages and the first-births) according to the age of mother at the birth of the first child.

FIG. 75.—The upper series of lines are graphs of the average number of children born to all mothers under 20 years of age, to mothers of from 20 to 24, 25 to 29, etc., and to mothers of all ages—who come under observation—according to duration of marriage. The fine dots give the crude results. The parallel broken lines indicate that the average *rate of increase* is nearly independent of the age of the mothers, and is dependent on the duration of the marriage.

27. **The unprejudiced protogenesic interval.**—The protogenesic interval gives unequivocally a measure of what may be called the *modified-fertility* of married women, that is fertility as modified by physiological and social conditions, by Malthusianism, etc. It is evident that *first* births are likely to give the best available indication of the physiological element in fertility; that is to say, the ratio of cases of nuptial-maternity at any age to the total number of nulliparous women, is a better indication of variations with age of physiological fertility, than would be the indication given by later births. But what have been called "prejudiced cases" should obviously be excluded, viz., cases where maternity, being expected, leads to marriage. For this reason the interval obtained by excluding such cases is not only appreciably longer, but also gives a truer idea of the normal probability of maternity, other things being equal. Results were published in New South Wales in 1899, purporting to shew

that, *when prejudiced cases were excluded*, the "average period from marriage to the birth of a first child" was, for unprejudiced mothers of from 17 to 39 years of age, about 19½ months, individual cases ranging between 18.3 and 21.9 months.[1] In order to definitely ascertain whether there was any justification for the statement, the New South Wales statistics, upon which they were based, were examined and recompiled; the data are given in Table LXXXIII. hereunder.

Table LXXXIII.—Interval between Marriage and First Births occurring later than 9 Months after Marriage.—New South Wales, 1893–98.

Age.	Interval (mths.)			Interval Years.)																				
	9 to 10	10 to 11	11 to 12	1 to 2	2 to 3	3 to 4	4 to 5	5 to 6	6 to 7	7 to 8	8 to 9	9 to 10	10 to 11	11 to 12	12 to 13	13 to 14	14 to 15	15 to 16	16 to 17	17 to 18	18 to 19	20 to 21	22 to 23	
13	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	
14	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	
15	3	..	1	1	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	5
16	15	11	7	20	1	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	54
17	52	51	42	132	9	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	286
18	134	137	108	380	37	3	1	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	800
19	273	256	159	641	73	17	..	1	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	1,420
20	320	337	209	772	129	36	9	6	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	1,818
21	470	425	292	1,026	172	41	11	1	1	..	..	..	..	..	..	..	..	..	..	..	..	..	..	2,439
22	544	521	365	1,181	210	61	22	4	6	1	..	..	..	..	..	..	..	..	..	..	..	..	..	2,915
23	483	498	357	1,205	249	79	37	17	7	1	1	..	..	..	..	..	..	..	..	..	..	..	..	2,934
24	453	431	265	1,031	245	96	40	13	9	5	1	..	..	..	..	..	..	..	..	..	..	..	..	2,589
25	419	382	246	925	205	85	41	17	10	6	5	2	1	..	..	..	..	..	..	..	..	..	..	2,344
26	342	294	240	801	205	83	41	17	19	4	5	1	1	..	1	..	..	..	..	..	..	..	..	2,054
27	243	264	185	650	176	74	38	34	15	10	2	7	..	2	..	..	..	..	..	..	..	..	..	1,700
28	232	185	153	549	142	86	32	27	15	18	4	5	3	1	1	..	..	..	..	..	..	..	..	1,453
29	141	145	103	417	122	52	43	13	12	9	6	8	3	2	2	..	..	..	..	..	..	..	..	1,078
30	133	131	83	343	124	46	37	32	16	14	6	6	3	5	3	1	..	..	..	..	..	..	..	983
31	68	83	62	248	76	32	20	10	13	7	4	7	6	5	1	..	..	1	..	..	..	..	..	643
32	52	53	48	209	78	27	12	19	9	9	14	7	8	3	3	4	1	..	..	..	..	..	..	556
33	48	47	41	142	66	25	13	8	8	8	6	6	4	3	3	1	2	..	..	..	..	..	..	431
34	33	45	31	117	43	16	13	5	6	1	2	2	4	1	4	4	4	1	..	1	..	..	..	333
35	33	29	25	95	34	16	11	6	2	4	8	4	5	..	3	..	4	1	..	1	..	..	..	281
36	29	25	12	90	32	7	12	6	4	1	4	7	3	4	4	..	1	2	..	1	1	1	..	246
37	12	22	9	59	20	8	8	6	2	2	1	2	..	3	..	2	1	1	1	..	..	..	..	159
38	10	12	8	58	13	14	3	7	..	3	1	3	1	2	1	..	..	..	1	1	..	..	..	138
39	11	8	13	47	15	9	5	1	1	1	2	2	..	..	1	1	..	..	..	1	..	..	..	118
40	..	3	5	37	15	6	2	3	..	..	1	1	1	2	1	..	2	..	3	..	1	..	..	83
41	2	2	..	10	7	3	..	1	1	..	1	..	..	1	..	..	..	..	1	..	..	..	2	31
42	3	6	3	17	12	3	..	1	1	2	1	..	..	..	..	..	..	..	..	..	..	..	..	49
43	1	3	2	8	3	2	1	1	..	1	..	..	1	..	1	..	..	..	..	..	..	..	..	24
44	2	..	..	6	1	1	..	..	1	..	..	..	..	..	1	..	..	..	..	..	..	..	..	12
45	..	1	1	9	1	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	12
46	..	..	..	3	..	..	1	..	..	..	1	..	..	..	..	..	..	..	..	..	..	..	..	5
Σ	4,561	4,407	3,075	11229	2,515	928	453	256	158	107	76	70	44	34	30	13	15	6	6	5	2	1	2	27,993

Further, to ascertain whether any material difference existed between the results for New South Wales for the period 1893-8, and for the whole of Australia for the period 1908-14, the latter were also computed, and are shewn in the same table. On Fig. 79 the intervals for successive ages are shewn by a light zig-zag line, and for the Commonwealth by a heavy zig-zag line. The two are evidently substantially identical, as the figures in Table LXXXIV. also show.

[1] See note on page 257, hereinbefore.

Table LXXXIV.—Protogenesic Interval or Average Interval elapsing between Marriage and First-birth, for all First-births occurring not Earlier than 9 months after Marriage. New South Wales, 1893-8; and Australia, 1908-14.

Age of Mother last Birth-day.	Interval.		Age of Mother last Birth-day.	Interval.		Age of Mother last Birth-day.	Interval.		Age of Mother last Birth-day.	Interval.	
	N.S.W.	Aust.		N.S.W.	Aust.		N.S.W.	Aust.		N.S.W.	Aust.
years.	months.	months.	years.	months.	months.	years.	months.	months.	years.	months.	months
13	—	13.83*	23	16.10	16.25	33	28.87	30.70	43	—	49.38
14	—	10.14*	24	16.88	17.20	34	30.10	33.64	44	—	57.35
15	—	12.09*	25	17.30	18.23	35	32.21	35.04	45	—	47.49
16	—	13.65	26	28.28	19.49	36	35.91	36.92	46	—	58.70
17	13.48	12.72	27	19.71	20.54	37	32.34	38.55	47	—	—
18	13.93	13.46	28	20.91	21.92	38	33.00	43.62	48	—	—
19	14.40	14.05	29	22.07	23.39	39	28.47	42.14	49	—	—
20	15.03	14.46	30	24.65	25.42	40	43.88	43.77	50	—	—
21	14.71	14.76	31	25.15	26.28	41	—	48.57	51	—	
22	15.04	15.02	32	30.09	28.94	42	—	49.83	52	—	—

* Depend upon 9, 14, and 68 cases only.

The above table and Fig. 79 indicate that there has been no material change in the interval between marriage and first-birth during the elapsed 15 years, and also that the average period is not constant but is a function of the age when tabulated according to "age of mothers," that is, according to age at maternity. It will be shewn later that when the

TABLE LXXXV.—Approximate Protogenesic Index for

(These results are only approximate, the table being constructed from the data in Table

Ages of Mothers at Marriage.	Number of each Duration of Marriage, the total being 1,000,000,															
	0–9 months.	9–12 months.	1–2 yrs.	2–3 yrs.	3–4 yrs.	4–5 yrs.	5–6 yrs.	6–7 yrs.	7–8 yrs.	8–9 yrs.	9–10 yrs.	10–11 yrs.	11–12 yrs.	12–13 yrs.	13–14 yrs.	14–15 yrs.
12	..	0	4	0	0	0	0	0	0	0	0	0	0	0	4	0
13	10	5	0	0	9	9	5	0	0	0	0	9	0	0	0	0
14	110	14	18	18	5	0	0	9	14	0	0	9	14	5	0	0
15	576	50	209	73	50	32	27	23	9	18	18	18	18	14	5	0
16	3,832	259	1,054	377	155	64	36	32	36	59	14	18	5	18	9	9
17	12,040	1,458	3,417	1,054	382	227	132	95	82	55	55	64	27	41	14	9
18	25,771	4,559	7,794	1,968	682	395	236	191	136	118	77	36	23	55	41	41
19	37,328	9,226	11,921	2,950	1,118	677	377	277	200	145	127	82	86	32	27	36
20	38,259	14,249	16,125	3,995	1,859	904	550	459	232	200	127	114	45	91	68	27
21	**44,522**	20,739	22,225	5,672	2,268	1,040	782	427	345	255	**227**	118	100	**109**	82	41
22	37,997	**27,624**	**24,316**	6,413	2,550	1,268	**800**	**541**	305	**264**	177	**205**	77	100	**114**	32
23	32,933	27,510	24,261	**6,954**	2,482	1,373	695	455	**391**	227	191	123	**132**	105	45	41
24	25,336	26,506	21,988	6,613	**2,650**	**1,432**	727	477	327	191	177	145	73	50	59	32
25	20,054	24,180	20,670	6,009	2,454	1,218	732	377	300	191	136	105	100	77	18	27
26	16,057	21,130	18,419	5,508	2,263	1,182	532	441	323	205	150	145	77	73	45	14
27	11,798	17,735	15,315	4,436	2,077	868	577	395	250	177	95	77	82	91	59	0
28	9,822	14,962	12,281	4,113	1,618	823	441	323	232	164	86	50	59	27	9	5
29	7,049	10,990	10,221	2,936	1,300	650	373	268	195	86	123	64	32	18	9	0
30	5,614	9,831	7,549	2,532	1,154	586	282	259	136	132	64	32	23	14	9	18
31	4,090	6,627	6,680	2,032	800	423	268	168	123	59	32	27	9	0	0	0
32	3,564	5,499	5,086	1,712	714	423	259	159	77	45	41	27	9	0	9	0
33	2,745	4,204	3,918	1,314	577	282	232	100	14	36	36	5	5	0	5	0
34	2,048	3,064	3,530	1,232	459	314	114	82	36	23	14	9	5	0	0	..
35	1,768	2,436	2,654	1,041	477	200	109	55	36	18	19	0	0	0	..	..
36	1,442	1,977	2,054	786	377	195	68	23	32	5	5	0	4	0	..	..
37	943	1,481	1,786	641	300	114	68	32	18	14	0	0	0	..	..	..
38	749	1,096	1,427	523	173	91	64	18	5	4	0	4	0	..	..	..
39	687	800	1,114	286	159	36	18	5	9	0	0	0	..	..	..	..
40	450	468	600	318	105	50	14	5	5	0	..	..	..	..	..	..
41	304	282	386	132	45	14	0	4	0	0	..	..	..	..	..	..
42	227	178	295	82	50	9	5	0	4	0	..	..	..	..	..	..
43	151	113	209	55	23	5	5	0	0	..	..	..	..	..	..	..
44	88	50	91	18	14	4	4	0	..	..	..	..	..	..	..	..
45	46	63	27	18	0	0	9	..	..	..	..	..	..	..	..	..
46	5	14	14	5	5	0	0	..	..	..	..	..	..	..	..	..
47	13	9	9	0	0	..	..	..	..	..	..	..	..	..	..	..
48	0	5	5	0	..	..	..	..	..	..	..	..	..	..	..	..
49	5	10	0	0	..	..	..	..	..	..	..	..	..	..	..	..
50	4	4	4	0	..	..	..	..	..	..	..	..	..	..	..	..
51	0	0	0	..	..	..	..	..	..	..	..	..	..	..	..	..
52	0	4	0	..	..	..	..	..	..	..	..	..	..	..	..	..
Totals	348,437	259,411	247,676	71816	29354	14908	8,541	5,700	3,872	2,691	1,991	1,486	1,005	920	631	332

tabulation is according to "age at marriage," there is a great approach to constancy of the interval, though the distribution according to interval is very different for different ages.

28. **Protogenesic Index based on age at and duration of marriage.**— The protogenesic indexes as determined in the preceding sections, viz., §§ 21, 22, 26 and 27, are based upon the *ages at maternity*. For certain purposes, however, they might with advantage be based upon the *ages at marriage*, and for exact results the evaluation of the index would of course require a compilation according to those ages, and cannot be quite satisfactorily deduced from the results given in Table LXXVII. A very fair approximation, however, can be obtained by reconstructing that table (see pp. 252-3), and the simplest form which this reconstruction can take is to treat the results in columns 1-2, 2-3, etc., years as respectively applicable to "ages at marriage, 1 year, 2 years, etc., earlier than that in the age-column. Such a compilation will be sufficiently accurate to disclose the general characteristics of the protogenesic indices for ages at marriage. This has been done in Table LXXXV. hereunder, which is self-explanatory when compared with Table LXXVII.

Australia, 1908 to 1914 based on Age at Marriage.

LXXVII. by moving the successive columns upwards, 1, 2, 3, etc., places respectively).

including those Born within 9 Months of Marriage.												Protogenesic Index, or Approximate Average Interval.		Protogenesic Quadratic Index. (Crude).	Ages of Moth'r at Marriage.
15–16 yrs.	16–17 yrs.	17–18 yrs.	18–19 yrs.	19–20 yrs.	20–21 yrs.	21–22 yrs.	22–23 yrs.	23–24 yrs.	24–25 yrs.	25–26 yrs.	9 mths. to 26 yrs.	Crude.	Smooth'd		
0	0	0	4	0	..	..	..	..	..	..	12	134.0	134.0	159.0	12
0	0	0	0	4	0	..	..	..	..	..	41	80.9	88.0	101.9	13
0	0	0	0	0	0	0	0	0	0	..	106	66.0	67.0	82.5	14
9	0	14	0	5	4	0	0	0	5	0	601	57.4	58.0	81.8	15
4	4	5	5	0	0	0	4	0	0	0	2,167	33.3	34.8	47.8	16
5	9	9	14	11	5	5	4	9	1	0	7,190	29.5	28.5	41.3	17
9	5	9	9	9	18	9	1	0	0	0	16,424	24.8	25.3	35.8	18
23	23	14	9	9	14	14	5	5	0	9	27,406	23.5	23.6	34.0	19
36	18	32	27	5	9	14	5	5	0	0	39,196	22.9	22.4	32.6	20
68	36	64	23	14	5	0	5	0	0	0	54,645	22.3	21.6	31.6	21
41	45	18	14	9	4	0	0	0	0	5	64,922	21.1	21.2	29.4	22
36	9	18	23	27	0	9	14	4	0	..	65,125	21.0	21.0	29.2	23
23	27	18	5	5	0	0	0	0	..	..	61,525	20.8	20.9	28.0	24
23	23	18	4	0	0	5	0	..	..	..	56,667	20.7	20.8	27.8	25
14	9	9	0	0	0	..	..	..	..	..	50,539	21.2	20.9	28.4	26
9	5	0	0	4	0	..	..	..	..	..	42,252	21.2	21.0	28.5	27
18	9	0	0	0	..	..	..	..	..	..	35,220	21.1	21.1	27.7	28
14	0	4	0	..	..	..	..	..	..	..	27,283	21.4	21.2	28.1	29
0	0	0	..	..	..	..	..	..	..	..	22,621	21.3	21.3	27.9	30
..	..	..	..	..	..	..	..	..	..	..	17,248	21.1	21.4	26.6	31
..	..	..	..	..	..	..	..	..	..	..	14,060	21.7	21.4	27.8	32
..	..	..	..	..	..	..	..	..	..	..	10,728	21.2	21.5	26.7	33
..	..	..	..	..	..	..	..	..	..	..	8,882	21.6	21.5	26.5	34
..	..	..	..	..	..	..	..	..	..	..	7,045	21.9	21.6	26.7	35
..	..	..	..	..	..	..	..	..	..	..	5,526	21.4	21.6	25.9	36
..	..	..	..	..	..	..	..	..	..	..	4,454	21.5	21.5	25.8	37
..	..	..	..	..	..	..	..	..	..	..	3,405	21.2	21.4	25.2	38
..	..	..	..	..	..	..	..	..	..	..	2,427	19.8	21.2	22.9	39
..	..	..	..	..	..	..	..	..	..	..	1,565	21.8	21.0	25.2	40
..	..	..	..	..	..	..	..	..	..	..	863	19.5	20.7	22.0	41
..	..	..	..	..	..	..	..	..	..	..	623	20.7	20.5	23.9	42
..	..	..	..	..	..	..	..	..	..	..	410	19.9	20.4	22.4	43
..	..	..	..	..	..	..	..	..	..	..	181	20.8	20.0	24.1	44
..	..	..	..	..	..	..	..	..	..	..	117	19.5	19.3	24.6	45
..	..	..	..	..	..	..	..	..	..	..	38	20.0	18.3	22.9	46
..	..	..	..	..	..	..	..	..	..	..	18	14.3	17.0	14.7	47
..	..	..	..	..	..	..	..	..	..	..	10	14.3	15.1	14.7	48
..	..	..	..	..	..	..	..	..	..	..	10	10.5	13.5	10.5	49
..	..	..	..	..	..	..	..	..	..	..	8	14.3	11.3	14.7	50
..	..	..	..	..	..	..	..	..	..	..	..	..	8.8	10.5	51
..	..	..	..	..	..	..	..	..	..	..	1	10.5	6.0	..	52
332	221	232	137	105	59	56	11	23	9	11	651,563				Totals

Much more accurate results would be secured by that reconstitution of the data, which would be possible if monthly or quarterly graduations for at least the first 3 years after marriage were used. Such graduations would have to be both for the horizontal and vertical values, and when effected, the sub-divided numbers would admit of a new table being compiled, giving with considerable exactitude the required numbers of births occurring after various durations of marriage, borne by women of various ages at marriage (instead of ages at maternity). The general characteristics of the values determined from such a table will, however, not differ materially from those in the table pp. 270, 271.

In the final columns of Table LXXXV. are given the crude and smoothed protogenesic indexes or approximate of protogenesic intervals according to age, with the argument "age at marriage." These are quite different in form from those deduced with the argument "ages at maternity." The values exhibit considerable regularity and require relatively little smoothing. As might be expected *a priori*, the interval decreases rapidly as the age at marriage increases, until the age 20 is reached, when it is 21 months. It remains sensibly constant till age 46, and then rapidly diminishes. It is evident that it must necessarily have a small value at the end of the child-bearing period.

The protogenesic index, or the protogenesic interval, determined according to "age at marriage," is perhaps to be preferred to one or the other based on the "age of mothers" (*i.e.*, age at maternity). The average "period elapsing between marriage and the birth of the first child of post-nuptial conception" is evidently not the same for all women marrying at ages below 40 years, as had been stated,[1] but is a function of age, and is very nearly constant for a long period, viz., from about 22 to 45 years of age. The maximum frequency is about age 23.4 or 23.5, but cannot be very accurately ascertained without a special compilation.

29. **Protogenesic quadratic indices and quadratic intervals.**—The fact that the protogenesic indexes or the protogenesic intervals are sensibly identical through a wide range of ages, notwithstanding the "scatter" of the distributions varies enormously, necessitates the adoption of a second and different index, or of a second and different type of "interval." This will of course be of the nature of a higher moment since the higher the power the greater the influence of the distribution on the product. It will in most cases be sufficient to employ the second power of the "duration of marriage," and to use the quadratic index, viz., that

[1] T. A. Coghlan, "Childbirth in New South Wales," 1899, p. 26, says: ". . . . but where a marriage proves fertile, as the following table shews, the period elapsing from marriage to the birth of the first child of post-nuptial conception averages the same for all women marrying at ages below 40 years. This average period is 19.4 months, ranging between 18.3 and 21.5 months." In the table referred to the results are grouped under "age of mother," not under "age at marriage," but the text might suggest that what is implied is "age of mother at marriage" (age of brides). The table shews that from age at marriage 21 to 45 the average interval is sensibly constant, and only slightly larger than that deduced by Coghlan if in his Table VIII. "age at marriage" be substituted for "age of mothers."

analogous to the *radius of gyration* in mechanics. That is, we shall require the value of G where its square is given by :—

$$(511)\ldots\ldots G^2 = \frac{\int x^2 f(x)\,dx}{\int f(x)\,dx}$$

When $f(x)$ is a rational integral function $(a + bx + \text{etc.})$, this gives—

$$(512)\ldots{}_0G_x^2 = \tfrac{1}{3}x^2 + \frac{bx^3 + \frac{16}{15}cx^4 + dx^5 + \frac{96}{105}ex^6 + \ldots}{12a + 6bx + 4cx^2 + 3dx^3 + \frac{12}{5}ex^4 + \ldots}$$

a formula which is appropriate when the graphed areas extend from the origin. The values of b, c, etc., can be ascertained from the group-totals, see, for example, by formulæ (195) to (197d), etc., pp. 67, 68. When the graph-totals are not continuous to the origin, the solution is a matter of integrating between the same limits in both numerator and denominator. If the limits be $x-\frac{1}{2}k$ to $x+\frac{1}{2}k$, that is, if the middle of the group-range be taken as the value x in the formula, then it is easy to shew that

$$(513)\ldots G_m^2 = \frac{a\,(x^2 + \frac{1}{12}k^2) + b\,(x^3 + \frac{1}{4}k^2x) + c\,(x^4 + \frac{1}{2}k^2x^2 + \frac{1}{80}k^4) + \text{etc.}}{a + bx + c\,(x^2 + \frac{1}{12}k^2) + d\,(x^3 + \frac{1}{4}k^2x) + \text{etc.}}$$

G_m being the radius of gyration of the figure standing on the range referred to, viz., $x \pm \frac{1}{2}k$. This formula can be readily recast into arithmetically convenient forms.

When the function is a simple exponential one (Be^{bx}), we have :—

$$(514)\ldots{}_0G_1^2 = \frac{n}{n-1}\left(1 - \frac{2}{b}\right) + \frac{2}{b^2}\text{ ; or generally}$$

$$(515)\ldots{}_0G_p^2 = \frac{n}{n^p-1}\left(p - \frac{2}{b}\right) + \frac{2}{b^2}$$

in which $n = e^b$. These are also suitable only for the figure starting from the origin. When the limits of the integral are p and q, we shall have

$$(516)\ldots{}_pG_q^2 = \frac{n^q q\left(q - \frac{2}{b}\right) - n^p p\left(p - \frac{2}{b}\right)}{n^q - n^p} + \frac{2}{b^2} = \frac{n^t t^2 - n^s s^2}{n^t - n^s} + \frac{1}{b^2}\,*$$

in which last expression $s = p - 1/b$ and $t = q - 1/b$. When the values of the squares of the several "radii of gyration" have been obtained, the radius of gyration of the whole series of groups is given by :—

$$(517)\ldots G_0^2 = (A_1G_1 + A_2G_2 + \text{etc.})/(A_1 + A_2 + \text{etc.}) = \Sigma(AG)/\Sigma A$$

A_1 denoting the number in group 1, A_2 in group 2, and so on.

The *protogenesic quadratic index* is computed in a manner analogous to that for computing the simple protogenesic index : that is by multiplying the square of the middle value of the successive yearly ranges of

* This may be seen by adding $1/b^2$ to the first term, thus making the terms in brackets perfect squares when multiplied by q and p respectively ; and then multiplying both numerator and denominator by $e^{-\frac{1}{b}}$.

duration by the number in the group: that is in formula (491) x'^2, x''^2, etc., is written instead of x', x'', etc., x', etc., here denoting the durations of marriage.

30. **Correction of the protogenesic interval for a population whose characters are not constant.**—When a population is increasing, all other facts remaining the same, the first-births, after a given duration of marriage (i), are drawn from a smaller population than are those for any lesser duration and presumably also from a smaller number of marriages. For comparative purposes, therefore, they need to be "corrected" so as to agree with what would be shewn by a constant population. Thus, were the ratio of first-births to marriages constant, it might very properly be assumed that the number of first-births to be expected would vary roughly as the ratio of the total marriages (marriages at all ages) for the period i years earlier, to the total number for the period being compared. Thus, if J_{-i} be the total number in the former case, and J the total number in the latter, the correction to be applied would be[1] :—

$$(518)\ldots\ldots 1 + c_i = J\ /J_{-i}$$

a quantity ordinarily greater than unity, *i.e.*, c_i is ordinarily a positive factor since populations generally are increasing.

We may, however, envisage the problem more rigorously as follows:

Let M, with suffixes showing the age, denote the number of mothers of first-born children, and J the number of women marrying, from which they were derived. Then in the case of a "constant population," in which also the relative frequencies of nuptial first-births were constant, the former number would bear a constant ratio to the latter, for any age in question; that is to say, for any age and at any time we should have $M/J = \mu$, a constant. Actually this ratio, however, is not quite constant, hence, rigorously, the number of nuptial primiparæ must be taken as :—

$$(519)\ldots\ldots M_x = {}_x\mu_t J_x = J_x\ .\ f(x, t)$$

In short we cannot take the marriages as the basis of the correction, but we should take what may be called their *Malthusian equivalent*; that is the number of marriages so reduced (or increased) as to be of equal productive efficiency: thus, μJ must replace J, and μ is not a constant. The character of μ may not be simple; it is probably a function also of the interval elapsing before birth, *i.e.*,

$$(520)\ldots\ldots \mu = f\ (x,\ i,\ t).$$

The form and constants of this function can be ascertained only by computing μ for differing ages with different intervals and at different times. Thus, instead of (518) we should write :—

$$(521)\ldots\ldots 1 + c_i = \mu' J/(\mu'_{-i}\ .\ J_{-i})$$

[1] This was pointed out by Sir (then Mr.) T. A. Coghlan, Childbirth in New South Wales, 1899, p. 26. He used this correction, which, however, would not be completely satisfactory if the "Malthusian coefficient" were increasing.

in which μ' denotes :— (a) the value of μ for a given age and interval, (when J and c are to be ascertained for a given age and interval), or (b) :— its value for the total for all ages and for a given interval, (when J and c are required for the total of all marriages). For Australia the ratio M/J is known only since 1893. During the period 1893-1914 it ranged between .790 in 1903, and .901 in 1912, for first-births and women of all ages (see hereinafter). As this average 0.0156 per annum for the 9 years interval between the years mentioned, it is of the same order as the yearly increase of population, and in the case cited would increase the correction. It may fall or rise 0.03 in one year. This term may be neglected, however, because its effect is relatively negligible when the correction is large, so that it has very little influence on the result computed by ignoring it. This is shewn by the results in the following table :—

TABLE LXXXVI.—Correction to the Computed Average Interval between Marriage and First-birth when Population is Increasing.

Factors to be multiplied into the computed average interval between marriage and first-birth when the correction for increase is ignored See (511) to (514).	When the increase per unit per annum is,				
	0.010	0.015	0.020	0.025	0.030
	Multiply the computed interval by the factor :—				
(a) When the first-births after 12 months are taken into account	1.0195	1.0294	1.0395	1.0500	1.0604
(b) When the first-births after 9 months are taken into account	1.0132	1.0199	1.0267	1.0338	1.0408
(c) When all first-births are taken into account	1.0083	1.0125	1.0168	1.0213	1.0257

It is to be remembered that the epoch to which the results refer is (sensibly) the middle of the year of observation, and that the intervals are 0, 1, 2, etc., years.

Since the *relative* numbers for different intervals will probably differ from those of Australia but slightly for most countries, we obtain the following very simple rules :—(i) If the ratio of first-births to marriages increase continually at the rates indicated in Table LXXXVI., or (ii.) if that ratio be constant, and the number of marrjages increase continually at the rates in the table, or (iii.) if the sum of the ratios in question be as indicated in the table, then—

The correction to the interval for all first-births occurring more than twelve months after marriage is	Twice the rate of increase. $1 + 2r$
For all first-births occurring more than nine months after marriage the correction is	The rate of increase plus one-third. $1 + 1\frac{1}{3}r$
For all first-births occurring after marriage, the correction is	The rate of increase less one-sixth. $1 + \frac{5}{6}r$

in which r denotes the rate of increase.

31. **Proportion of births occurring up to any point of time after marriage.**—The rate of occurrence of first-births, for different intervals after marriage, is well shewn by giving the proportion of the whole which have occurred up to any given time. The following table furnishes the proportions in question :—

TABLE LXXXVII.—Shewing Proportion of Nuptial First-births occurring up to any point of time after Marriage.

Age of Mothers.	Up to End of Month.											
	1	2	3	4	5	6	7	8	9	10	11	12
15 ..	.0699	.1832	.3525	.4581	.5994	.7189	.8245	.8804	.8944	.9363	.9720	.9798
20 ..	.0262	.0597	.1071	.1685	.2445	.3341	.4398	.5234	.5721	.6641	.7355	.7851
25 ..	.0099	.0204	.0367	.0569	.0810	.1176	.1659	.2076	.2577	.3932	.4958	.5683
30 ..	.0070	.0140	.0229	.0362	.0493	.0685	.0945	.1240	.1631	.2761	.3677	.4346
35 ..	.0080	.0156	.0284	.0305	.0562	.0723	.0958	.1216	.1485	.2338	.3083	.3577
40 ..	.0120	.0201	.0324	.0466	.0606	.0748	.0977	.1161	.1340	.1884	.2376	.2781
45 ..	.0128	.0281	.0510	.0536	.0714	.0714	.0867	.1020	.1122	.1658	.1888	.2194
13–52 ..	.0160	.0345	.0614	.0953	.1365	.1872	.2502	.3027	.3484	.4595	.5461	.6078
13–52 Proportion of first year's births during month	.0264	.0304	.0443	.0557	.0677	.0834	.1037	.0863	.0753	.1827	.1424	.1017
Proportion of first year's births up to end of months ..	.0264	.0568	.1011	.1568	.2245	.3079	.4116	.4979	.5732	.7559	.8983	1.0000

Age of Mothers.	Up to End of Year.								
	2	3	4	5	6	10	15	20	26
15 ..	1.0000	..	..	..	..	(7)	..	..	..
20 ..	.9634	.9928	.9985	.9995	.9999	1.0000	(13)	..	..
25 ..	.8508	.9402	.9729	.9863	.9933	.9996	1.0000	(18)	..
30 ..	.7278	.8458	.9043	.9367	.9565	.9928	.9998	1.0000	(22)
35 ..	.6455	.7631	.8223	.8614	.8869	.9562	.9932	.9997	1.0000
									(25)
40 ..	.5704	.7052	.7842	.8314	.8585	.9202	.9676	.9916	1.0000
45 ..	.4974	.6276	.7270	.7908	.8316	.9107	.9541	.9745	1.0000
13–52 ..	.8555	.9273	.9567	.9716	.9801	.9943	.9988	.9998	1.0000
13–52 Proportion of first year's births during month	..	..	..	..	..	..	..	..	..
Proportion of first year's births up to end of month	1.4075	1.5254	1.5739	1.5984	1.6125	1.6359	1.6431	1.6448	1.6451

This table is interpreted as follows :—Taking the upper line, 13-52, 0.0160 of all nuptial first-births occur within one month of marriage, 0.3484 occur before the end of the ninth month after marriage, and 0.6078 before the end of the twelfth month. Again, of the nuptial first-births occurring, with women of all ages, during the year of marriage, 0.5732 are born before the end of nine months, and all births exceed those born during the first twelve months by only 0.6451. This is shewn on the last line of the table.

32. **Range of the gestation period.**—In order to accurately estimate the cases of first-births properly attributable to pre-nuptial insemination, the range of the normal gestation-period must be taken into account as

well as the frequency of premature live births. Contrary to popular opinion this gestation-period has a considerable range.[1] The following data represent the best available results :

TABLE LXXXVIII.—Relative Frequency of Births after Different Periods, between the last Menstruation and Parturition.

Duration Days.	Reid.* 500 Cases.	Hannes.† 561 Cases.	Hannes.‡ 314 Cases.	Hannes. 875 Cases.	Various§ 51 Cases.	Reid, with Hannes.
			AUTHORITIES.			
241–250	56	36	16	28	?	41
251–160	59	37	13	29	20	44
251–170	150	141	111	130	210	140
·71–280	317	325	366	340	510	329
Maximum (days)	(277.77)	(277.73)	(277.02)	(277.42)	(274.64)	(277.58)
281–290	269	271	258	267	160	268
291–200	97	121	118	120	100	109
301–310	24	50	76	59	?	41
311–320	18	14	22	17	?	18
321–330	10	5	19	10	?	10
Total	1,000	1,000	1,000	1,000	1,000	1,000
Average Duration	277.2	279.2	281.9	280.3	276.5	278.8

NOTE.—The cases for 241 to 251, 316-330, have been obtained by extrapolating Reid's curve.

* See Hart, Edinburgh Medical Journal, 1914, New Ser. XII., p. 401 ; also Journ. Edin. Obstetr. Soc., XXXVIII., pp. 107-134 ; 1912-3. Biometric analysis of some insemination-labour and menstrual-labour curves in certain mammalia. The distribution of Reid's results according to the normal curve of probability for a table of frequency is unquestionably unsatisfactory, as an examination of the original data will show. The distribution does not conform to the normal curve. The average is given as 278.3 ; it should be 278.84 ; there is an arithmetical mistake in the original calculation.

† Zeit. f. Geburt und Gynäk. LXXI., 1912, p. 524. Die körperliche Entwicklung der Frucht in ihrer Beziehung zur berechneten Schwangerschaftsdauer. Walther Hannes. Children 3000 to 4000 grammes weight.

‡ Same authority, children above 5000 grammes weight.

§ Interval reckoned from coitus, certain. These i clude 51 cases reported by Desormeaux, Girdwood, Montgomery, Rigby, Lockwood, Lee, Dewers, Beatty Skey, McIlvain, Ashwell, Clay and Reid.

The average durations indicated are not exactly identical with the maximum frequency, since the frequency curves are very slightly asymmetric.

If Hannes' cases are combined with Reid's, a total of nearly 1400 is obtained. If the result be "smoothed," so as to agree with the final column of Table LXXXVIII., the result shown in Table LXXXIX. on next page is obtained.[1]

[1] Other values are as follows :—Hippocrates, περὶ ὀκταμήνου, generally within 280 days ; Hensen, 128 cases, 272.5 days after coitus ; see Handbuch der Physiologie by Hermann, VI., 2., p. 73, 1881 ; M. Zöllner, after menstr., first-births 279.1, second births 282.0 ; see Zur Kenntniss und Berechnung der Schwangerschaftsdauer, Jenenser Dissertation, 1885, p. 6. Hasler, 195 cases, 281.0 ; after coitus 665 cases, 272 days ; Glusing, after menstr., 279.6 ; Würzburger Dissertation, 1888, p. 15 ; Voituriez, 274-8 after menstr. Thèse de Paris (Lille), 1885, p. 62 ; Winckel, 274.8, Lehrbuch d. Geburtshülfe, p. 78, 1889 ; Ahlfeld, 270.4 after coitus, Monatschr. f. Geburtskr u. Frauenkr., XXXIV., p. 304. 1869.

TABLE LXXXIX.—Shewing the Frequency per diem per 100,000 Births occurring between the 240th and 332nd day after the Termination of the Menstrual Period.

Day	No.	Ratio of Aggregate.	Day	No.	Ratio of Aggregate.	Day	No.	Ratio of Aggregate.	Day	No.	Ratio of Aggregate.	Day	No.	Ratio of Aggregate.
240	297	.00297	260	675	.09012	280	3,429	.56930	300	657	.93538	320	111	.99424
241	303	.00600	261	733	.09745	281	3,318	.60248	301	597	.94135	321	101	.99525
242	310	.00910	262	807	.10552	282	3,196	.63444	302	546	.94681	322	91	.99616
243	318	.01228	263	911	.11463	283	3,014	.66458	303	496	.95177	323	81	.99697
244	327	.01555	264	1,052	.12515	284	2,847	.69305	304	455	.95632	324	71	.99768
245	338	.01893	265	1,305	.13820	285	2,676	.71981	305	420	.96052	325	61	.99829
246	349	.02242	266	1,548	.15368	286	2,504	.74485	306	389	.96441	326	51	.99880
247	361	.02603	267	1,784	.17152	287	2,332	.76817	307	361	.96802	327	40	.99920
248	374	.02977	268	2,015	.19167	288	2,160	.78977	308	334	.97136	328	30	.99950
249	388	.03365	269	2,246	.21413	289	1,988	.80965	309	304	.97440	329	20	.99970
250	404	.03769	270	2,470	.23883	290	1,816	.82781	310	277	.97717	330	15	.99985
251	420	.04189	271	2,689	.26572	291	1,644	.84425	311	252	.97969	331	10	.99995
252	437	.04626	272	2,913	.29485	292	1,477	.85902	312	227	.98196	332	5	1.00000
253	455	.05081	273	3,132	.32617	293	1,320	.87222	313	207	.98403	333	0	..
254	474	.05555	274	3,420	.36037	294	1,189	.88411	314	188	.98591	240	..	..
255	496	.06051	275	3,455	.39492	295	1,077	.89488	315	171	.98762	to	100,000	..
256	521	.06572	276	3,501	.42993	296	976	.90464	316	156	.98918	333		
257	551	.07123	277	3,511	.40564	297	885	.91349	317	143	.99061			
258	587	.07710	278	3,506	.50010	298	804	.92153	318	131	.99192			
259	627	.08337	279	3,491	.53501	299	728	.92881	319	121	.99313			

Maximum frequency occurs on the 277.67th day. Average (240 to 332 days) = 279.28 days.*

* If the average date be found in the usual way (*i.e.*, from the weighted mean), it will prove to be 278.78. But the births occurring on the *n*th day range between n and $n + 1$, hence the average is about $n + \frac{1}{2}$, consequently the 278.78th day is from 278.78 to 279.78; hence the average interval is 279.28 about.

It would appear from these results that the *most frequent interval* between the termination of menstruation and parturition, and *the average interval*, may be regarded for practical purposes as identical, and may be taken as 278 days on the average for births of children of ordinary weight, and that only 2 or 3 days need to be added in the case of the birth of heavier children. For first-births the interval is about 3 days shorter. From insemination to parturition the interval is slightly shorter, perhaps 5 or 6 days on the average. In view of social custom, however, the interval for first-births may be taken as say about 14 days longer than the 278, or about 292 days in all. Making allowance for live births occurring after 210 days from insemination, and for the fact that 40 per cent. of births occur between the 261st and 278th day from the last menstruation, (see Table LXXXIX.), we may take 274 days, or 9 months, as the period to be rejected as uncertain as regards post-nuptial conception.

The frequency-curve for the interval between the termination of menstruation and parturition is curve E on Fig. 76, see later, page 284.

33. **Proportion of births attributable to pre-nuptial insemination.**—It is evident, from the preceding table, that there is a certain period during which it is not possible to ascertain what proportion of births should be regarded as attributable to pre-nuptial insemination.[1] The numbers

[1] T. A. Coghlan in 1899 based his computations on the assumption of a 9-months interval, see Childbirth in New South Wales. He points out that in the years 1893-8, the nuptial first-births registered were 41,384, of which 13,366, or 32.3 per cent., were "due to pre-nuptial conception." It may be observed that pre-nuptial insemination may have characterised some cases where birth occurred in the tenth or even eleventh month after marriage, and a small number of births may be attributable to cases of post-nuptial insemination from 200 to 240 days after marriage, and a considerable number from 240 to 270 days. However, the percentage he deduced for New South Wales in 1893-8 seems, on the whole, to be confirmed by the present investigation for Australia, 1908-14.

per million nuptial first-births for women of all ages born during various intervals after marriage are shown on the penultimate line of Table LXXVII.

By plotting the groups of first-births occurring monthly from 1 to 12 months, and drawing a continuous curve giving the same totals, results are obtained analogous to those shewn on Figs. 76 and 77. On the former figure the part of curve A, marked f, g, g', h, denotes the boundary of the groups, which may be attributed to pre-nuptial insemination. The curve i, i', j shews the boundary of the groups which may be attributed to post-nuptial insemination. On Fig. 77 the curve k, l, l', m, denotes the pre-nuptial insemination quota, and the curve n, n', o, p, the post-nuptial quota; see page 284.

By fixing the position of that part of the curve shewn by the dotted lines in the figures referred to, it would appear that about 0.634 of the births occurring during the 9th month after marriage are to be attributed to pre-nuptial insemination. Thus, about 0.952 of the first-births occurring within 9 months of marriage are due to pre-nuptial insemination. This is equal to 0.546 of all first-births occurring during the year of marriage, and 0.332 of all first-births, in every case for women of all ages. These ratios, it will be seen from Fig. 73, are a fairly definite function of the age of the mothers ; and this function could be ascertained by treating the group-results given in Table LXXVII. in the manner above described.[1]

34. **Issue according to age and duration of marriage.**—The recording of the number of children borne by married women of various ages, and after various durations of marriage, furnish data of value in any attempt to ascertain the law of increase "according to age and duration of marriage." But it is to be kept in view that the immediate results from such data apply only to those who thus, through maternity, come under observation, and does not apply to married women generally. That is to say, if averages be formed these averages are *not* averages for all married women of the given ages and durations of marriage. During the seven years, 1908-1914, 805,015 mothers came under observation in Australia, their total issue being 2,675,291, or an average of 3.3233 each. The results are shewn in Table XC. hereunder, the averages being found as follows :—

Let ${}_im''_x$ denote the mothers of age-group $x - k/2$ to $x + k/2$, and of duration of marriage $i - 1$ to i, and let the total issue of these be ${}_iC_x$; then the average, ${}_ic_x$, is given by :—

$$(522) \ldots\ldots c_x = {}_iC_x \;/\; {}_im''_x$$

and these are the averages which have been tabulated.[2]

[1] The attributing of the whole of the births occurring during the 9 months after marriage to pre-nuptial insemination, gives a result somewhat too great. Nevertheless it is clear that for practical purposes it is a satisfactory rule for eliminating the so-called "prejudiced" from the "unprejudiced" cases, to assume that, on the average, births occurring less than 9 months after marriage are "prejudiced."

[2] The original data will be found in the Population and Vital Statistics of Australia for the years 1908-1914, Bulletins 14, 20, 25, 29, 30, 31 and 32.

TABLE XC.—Shewing the Average Number of Children Born to those who Bear during Varying Intervals after Marriage, based upon the Experience of Australia during the Years 1908-1914.

Duration of Marriage.	Age-groups. (Age at Birth of Last Child.)								Totals, All Ages.	
	-19.	20-24.	25-29.	30-34.	35-39.	40-44.	45-	All Ages.	Mothers.	Issue.
Years.	AVERAGE NUMBER OF CHILDREN.									
0–1	1.006	1.010	1.016	1.030	1.051	1.029	1.142	1.013	134,171	135,996
1–2	1.250	1.157	1.085	1.087	1.089	1.113	1.151	1.125	61,213	68,906
2–3	1.925	1.882	1.747	1.700	1.627	1.454	1.545	1.802	64,229	115,759
3–4	2.145	2.171	2.087	2.039	1.997	1.923	1.786	2.107	70,317	148,160
4–5	2.466	2.622	2.520	2.441	2.401	2.207	2.041	2.525	59,407	150,009
5–6	2.791	3.020	2.919	2.825	2.803	2.670	2.153	2.906	53,275	154,836
6–7	*2.750	3.401	3.339	3.194	3.216	3.038	3.000	3.290	47,250	155,476
7–8	3.000	3.776	3.731	3.576	3.544	3.447	2.846	3.655	41,713	152,461
8–9	..	4.105	4.126	3.954	3.883	3.820	3.142	4.018	37,115	149,129
9–10	..	4.292	4.514	4.330	4.271	4.149	3.940	4.374	32,170	140,725
10–11	..	*4.347	4.910	4.705	4.600	4.619	4.318	4.726	29,607	139,942
11–12	..	4.950	5.256	5.122	4.965	4.954	4.931	5.091	25,887	131,795
12–13	..	4.571	5.541	5.513	5.329	5.319	5.037	5.443	23.372	127,226
13–14	..	..	5.790	5.868	5.725	5.608	5.761	5.718	20,339	117,691
14–15	..	..	6.131	6.269	6.091	6.056	5.721	6.156	17,572	108,160
15–16	..	..	*6.24	7.434	6.453	6.324	6.493	6.494	15.217	98,827
16–17	..	..	5.59	6.967	6.859	6.688	6.844	6.844	13,271	90,836
17–18	..	..	5.16	7.239	7.401	6.985	7.282	7.193	11,617	83,539
18–19	..	..	5.00	*7.371	7.679	7.431	7.291	7.575	10,073	76,308
19–20	..	..	..	7.480	8.018	7.865	7.775	7.926	8,520	67,530
20–21	..	..	..	7.111	8.418	8.282	8.168	8.329	7,424	61,839
21–22	..	..	..	6.192	8.824	8.750	8.449	8.751	5,988	52,403
22–23	..	..	..	5.60	9.154	9.230	8.962	9.191	4,726	43,437
23–24	..	..	..	..	9.609	9.503	9.171	9.483	3,561	33,770
24–25	..	..	..	16.00	9.265	9.973	9.700	9.884	2,664	26,330
25–26	..	..	..	..	*9.053	10.450	10.500	*9.932	1,809	17,967
26–27	..	..	..	..	9.105	10.730	10.773	10.16	1,146	11,637
27–28	..	..	..	..	7.000	10.860	11.150	10.54	643	6,781
28–29	..	..	..	..	..	11.260	11.480	10.71	383	4,102
29–30	..	..	..	..	..	11.210	11,840	10.75	192	2,064
30–31	..	..	..	..	..	12.00*	*12.220	12.51	77	963
31–32	..	..	..	..	..	13.00	11.770	9.51	45	428
32–33	..	..	..	..	..	10.00	12.460	12.94	17	220
33–34	..	..	..	..	..	..	14.80	7.80	5	39
All Durations	1.202	1.760	2.643	3.837	5.341	6.997	8.565	3.3233	..	..
Totals all dur'tions										
Mothers	29,371	185,694	239,066	181,191	118,310	46,705	4,678	805,015	805,015	..
Issue ..	35,292	326,868	631,954	695,220	626,641	326,095	40,181	2,675,291	..	2,675,291

* Owing to the limited data, the values are not reliable for the age-group 45, nor for the values shewn by the asterisks and those for greater durations of marriage.

The table shews that, for all ages; the average total issue of married women, with various durations of marriage, who each year appear in the Australian maternity records, increases approximately at the rate of one child in 2.745 years, or 0.3643 of a child per annum. The results are graphed in Fig. 75, p. 268. The parallel dotted lines in the figure shew that the *rate of increase of the total issue according to the duration of marriage is identical for all ages,* at least for the greater part of the range of duration. That the graphs approximate so closely to straight lines, and, moreover, to parallel straight lines, is remarkable.[1] These lines may be defined by equations :—

$$(523)\ldots\ldots c''_x = a_x + bi = 0.6667 + 0.3643i, \text{ approximately };$$

[1] This characteristic can no doubt be deduced, but no explanation of an elementary nature can be offered.

in which only ax is dependent on the age of the mothers, being about $\frac{2}{3}$, and b is constant for all ages. The more exact values of a are given in Table XCI. hereinafter.

The results shewn in Fig. 75, p. 268, and detailed in the table referred to, can be referred in a general way also to the age-groups, that is to say, if γ_x denote the average issue for mothers of a given age-group for all durations of marriage, then the number is as shewn in Table XCI. The average ages for these age-groups, as shewn in the table, are found on the supposition that the distribution of the cases of nuptial maternity occurring during the period 1907-1914, in Australia, apply. This distribution is given in Table LXXIII., p. 242, and the average ages of each age-group have been calculated strictly[1] : these are as given hereunder.

TABLE XCI.—Shewing the Total Issue for Mothers in various Age-groups, for All Durations of Marriage : the Constants of Formulæ for Computing this Number, and the Differences between the Observed and Computed Numbers. Australia, 1908-1914.

		–19	20–24	25–29	30–34	35–39	40–44	45–	13–52
Average age		18.92	22.87	27.46	32.35	37.29	41.91	46.29	..
Average number of children, all durations of marriage		1.202	1.760	2.643	3.837	5.341	6.997	8.565	3.3233
Smoothed result† ..		1.242	1.751	2.636	3.895	5.413	6.994	8.764	..
The above crude and smoothed results are equivalent to durations for all ages of:	(Crude)	1.37	2.90	5.33	8.60	12.73	17.28	21.58	Crude
	(Smoothed)	1.48	2.88	5.31	8.76	12.93	17.27	22.13	Smoothed
Values of A_x for age-group		.6515	.7909	.7778	.6921	.6646	.5977	.4939	.7029
Value of b		.3643	.3643	.3643	.3643	.3643	.3643	.3643	.3643
	Year of duration.	ϵ	ϵ	ϵ	ϵ	ϵ	ϵ	ϵ	ϵ
Calculated Values of $A_x + bu$ when u and the value of ϵ‡	1	1.016—.010	1.155—.145	1.052—.036	1.056—.026	1.029+.021	0.962+.067	0.858+.284	1.067—.054
	2	1.380 .130	1.520—.363	1.416—.331	1.421—.334	1.393—.304	1.326—.213	1.223—.072	1.432—.307
	3	1.744+.181	1.884—.002	1.781—.034	1.795—.095	1.758—.131	1.691—.237	1.587—.042	1.796+.006
	4	2.109+.036	2.258—.087	2.155—.068	2.159—.120	2.122—.125	2.055—.132	1.951—.165	2.160—.053
	5	2.473—.007	2.622—.000	2.519+.001	2.524—.083	2.486—.085	2.419—.212	2.315—.274	2.524+.001

† The smoothed result conforms to a rational integral equation of the fourth degree.
‡ ϵ is the quantity which, added to the tabular value (calculated), makes it identical with the data.

The smoothed results for the average number of children, according to age, for all durations of marriage, are given by :—

$$(524) \ldots\ldots \gamma_x = 1 + b\chi + c\chi^2 + d\chi^3 + e\chi^4 ;$$

in which $\chi = x - 13$, and the values of which for $2\frac{1}{2}$ years' intervals are as follow :—

TABLE XCII.—Shewing the Effect of "Age of Mothers" upon the Total Issue for All Durations of Marriage. Australia, 1908-1914.

Ages at birth of last child, in years ..	13	15.5	18	20.5	23	25.5	28	30.5	33	35.5	38	40.5	43	45.5	48
Children*	1.000	1.019	1.160	1.413	1.770	2.221	2.760	3.378	4.070	4.829	5.650	6.528	7.460	8.441	9.470
Difference for $2\frac{1}{2}$ yrs		**0.019**	**0.141**	**0.253**	**0.357**	**0.451**	**0.539**	**0.618**	**0.692**	**0.759**	**0.821**	**0.878**	**0.932**	**0.981**	**1.029**

* That these are given by a curve of the fourth degree, can be readily seen by taking the values for 13, 18, 23, etc.

[1] That is, the numbers are referred to the exact average for the year of age, not merely to the age for the middle point.

In the above table the differences for $2\frac{1}{2}$ years shew that for all durations of marriage, *differences of age have much less influence than differences in duration*. To obtain this relationship exactly, it is necessary to compile for each age, and for given durations of marriage the total issue. For all age-groups the general result is 0.3643 a child per year, that is 0.9107 for $2\frac{1}{2}$ years. From the above table, however, it would appear that this value is not attained for "all durations of marriage" until, almost exactly, age 40.

Such results as are referred to, are dependent upon the combination of two things, viz. :—(*a*) The age-effect proper, and (*b*) the fact that for the higher ages the average of the durations of marriage are greater, and thus, throughout the range of observation, the conditions are not homogeneous.

35. **Initial and terminal non-linear character of the average issue according to duration of marriage.**—An inspection of Fig. 75, p. 268, and the results given in the preceding table, shew that there is a more or less systematic departure from linearity at the terminals of the graphs representing "issue according to duration of marriage." The table reveals the fact that the character of the differences, according to age, and for various durations of marriage, between the values according to formula (523), and the individual results are as follow :—

(i.) For the first year of duration of marriage, the computed total issue for ages under 35 is too great, and for ages over 35 is too small.

(ii.) For the second year of duration of marriage, the computed total issue is *invariably* too great, the maximum difference being at about age 24.

(iii.) For the third year of the duration of marriage, the computed issue is less than the actual for the younger ages, but soon becomes greater, the maximum difference occurring at about the age 43 or 44.

(iv.) The same remarks apply to the fourth year of the duration of marriage with the exception that the age is later than 45.

(v.) In the fifth year of the duration of marriage, the differences are small until the age of 40 is reached, when the computed result becomes markedly greater than the actual.

The relatively large differences for the various age-groups characterising the second year of the duration of marriage are due to the fact that the length of the period, which must necessarily intervene between a first and second birth, does not admit of so wide a "scatter" of the cases of maternity as to make the result uniform ; thus the average for the second year is in defect. This consequence is one which will (and does) tend to vanish for longer durations of marriage, owing to the fact that any want of coincidence of the intervals between birth and birth must more markedly characterise the points of time in proportion to their remoteness from the first year of duration of marriage. Owing to the fact that the period of

gestation alone is three-fourths of a year, and the period of lactation a considerable part of a year, and to the fact that so great a proportion of births appear in this year, it follows that the second year of duration must necessarily disclose a falling off in the apparent average. As time goes on, however, this apparent defect will tend to disappear, as will be clearly seen by a reference to Fig. 75, p. 268.

The character of the curves at their terminals for the longer durations may be fairly well ascertained by combining the terminal values. This has been effected as follows :—In the series shewn on Fig. 75 the two differences between the three last averages of the issue of curve for under 20, are taken, and similarly the four differences between the five last averages of the issue, etc., the number of values (averages of issue) being respectively 3, 5, 8, 9, 10, 12, and 12. The means of the differences, the numbers of which are respectively 7, 7, 6, 6, 5, 5, 5, 4, 2, 2, are taken, the results being as follows :— +0.230, +0.285 + 0.582, +0.106, + 0.153, —0.105, — 0.162, + 0.060, —0.246, +0.845, —0.489. The accumulated results compared with the successive multiples of 0.3643 furnish the co-ordinates of the average terminal shape. This gives :—

0	.364	.729	1.093	1.457	1.822	2.186	2.550	2.914	3.279	3.643	4.007
0	.230	.515	1.097	1.203	1.356	1.251	1.089	1.149	.903	1.748	1.259
Diff.	.134	.214	—.004	.254	.466	.935	1.461	1.765	2.376	1.895	2.748
Smth'd	.025	.100	.225	.400	.625	.900	1.225	1.600	2.025	2.500	3.025

The differences shew the amounts by which the successive points fall short of the line defined by the formula (523). As is shewn by the smoothed values, the defect from the linear condition, once it initiates, increases, on the average, very approximately as the square of the duration from the initiating point onward. This average defect η is expressed by the equation :—

$$(525) \ldots\ldots \eta = 0.025 \; I^2$$

I denoting the duration reckoned from the initiating point. This point may approximately be found as follows :—

Average age at birth*	18.9	22.9	27.5	32.4	37.3	41.9	46.3	
Initiation of droop†	6.0	10.0	15.0	18.0	24.0	29.0	33.?	
Difference	12.9	12.9	12.5	14.4	13.3	12.9	13.3	Aver. 13.1

* *i.e.*, Age of mother at birth of children. † Years of duration of marriage.

In these results the first line gives the average age of women at the time of maternity, and the second line gives the points where the droop from the linear relationship commences : the positions of these points being estimated from the graphs, Fig. 75, p. 268. The differences give a sensibly constant age, which is seen to average 13.16, hence the droop implies that the fecundity of those who are characterised by early marriage and late motherhood is less than the average for those who may be regarded as falling into the normal place.

Fig. 76. Fig. 79.

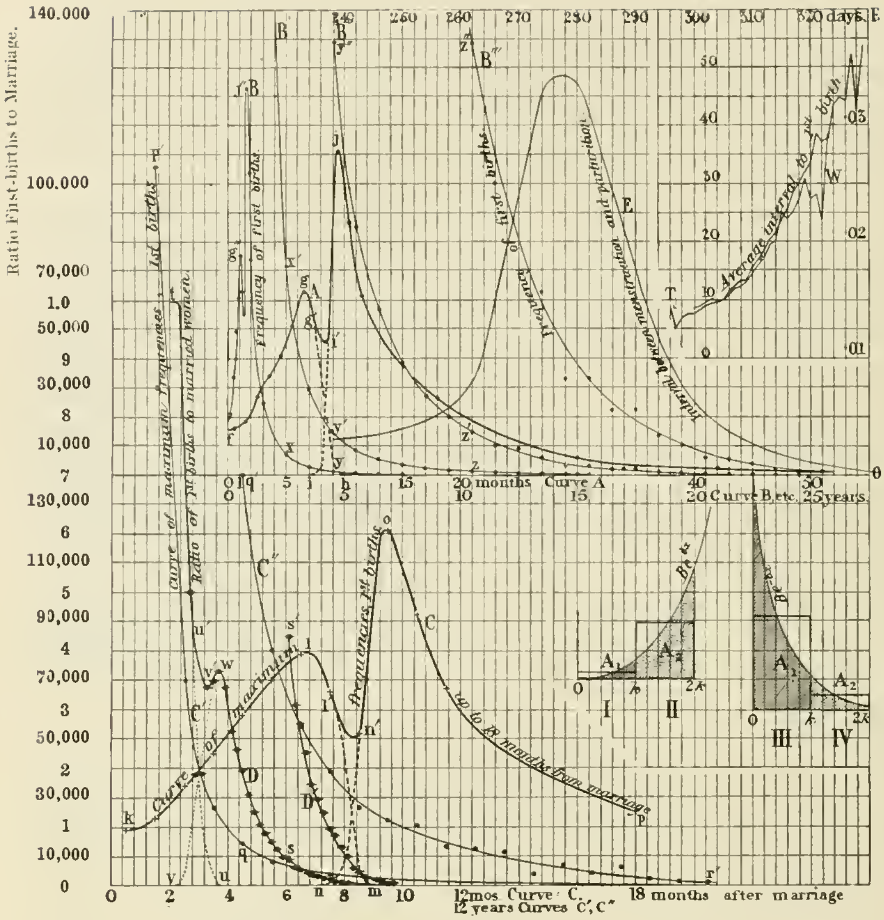


Fig. 77. Fig. 78.

Fig. 76.—Curve A denotes the frequency, according to duration of monthly-groups, of first-births, viz., the number of cases in a total of 1,000,000 first-births for all durations of marriage (see Table LXXVII., pp. 252-3). The curve f, g, g′, h, denotes the relative numbers attributable to prenuptial insemination, and the curve i, i′, and j, etc., the relative numbers attributable to post-nuptial insemination.

Curve B denotes the frequency, according to duration, of yearly groups, with a less extended lateral scale, the point g″ thereon corresponding to g on Curve A.

Curves B′, B″ and B‴ are plotted on a larger vertical scale, y′ and y″ being the same point as y, and z′ and z″ the same point as z.

Curve E is the curve of relative frequency of birth, according to the interval after the last menstruation, see Table LXXXIX., p. 278.

Fig. 77.—Curve C shews the relative maximum frequencies according to age (*i.e.*, for any age). The points l, l′ and m, and n, n′, o and p have the same significance as points g, g′ and h, and i, i′ and j in Fig. 76, curve A, and the point k corresponds to f.

Curves C′, C″, are an extension of curve C, the lateral scale being altered. The point p′ is the same as p, q′ as q, etc.

Curve D denotes the ratio, according to age, of first-births, to married women. It appears to be compounded of two curves, viz., u, u′ and v, v′, w, s. Curve D′ is plotted on a larger scale, the point s′ being identical with s.

Fig. 78 illustrates the formulæ for determining the exponential curves so as to make the shaded areas equal to the areas of the rectangles A_1 and A_2, in order to determine the positions of the centroid verticals, etc. See formulæ (496) to (510), pp. 264-5.

Fig. 79 is the graph of the approximate average intervals to between marriage and the "unprejudiced" first-births for New South Wales, 1893-1898, and for the Commonwealth, 1908-1914; the light zig-zag line marked W denoting the result for the former, and the heavy zig-zag line marked T denoting that for the latter. The figures denote months, and the lateral divisions denote two years' duration.

36. **The polygenesic, fecundity, and gamogenesic distributions.**—As we have seen, there are two ways in which records of *issue*, according to age and duration of marriage, come to hand, viz. :—

(i.) When, at the registration of births, the age, duration of marriage, and "*previous issue*" are also registered ; and

(ii.) When, at a Census, the age, duration of marriage, and total issue are ascertained.

There are certain differences between these. In (i.) the total age-range covered is that of the child-bearing period only ; in (ii.) the age-range is from the earliest age of maternity to the end of life. In (i.) the cases come under observation *during a period of time* ; in (ii.) they come under observation *at a given moment*. Hence, to deduce (ii.) from (i.) it is essential that the necessary records of births, migration, and deaths should extend over a long period of time, and even then, the deduction of (ii.) from (i.) is by no means simple. Both records are, however, of value statistically and both yield appropriate measures of fecundity, though on the other hand both require corrections if they are to represent what would have been furnished by a "constant population."

If, on a plane, the ages of mothers (x) be plotted as abscissae, and their duration of marriage (y) be plotted as ordinates, and if then verticals to this xy-surface be drawn denoting the number of cases of maternity, corresponding to each age and duration, the surface so defined may be called the *genesic distribution at maternity*, or simply (i.) *the polygenesic distribution*. Similarly if the verticals denote the number of children recorded at any moment as having been borne by women of any age and duration of marriage, the distribution may be called the *general genesic distribution*, or (ii.) the *fecundity distribution*.[1] The fecundity-distribution-contours, or lines denoting equal issue for various ages and durations of marriage, can be drawn by means of formula (523), together with the values of the constants given in Table XCI., the values of the durations (according to age) where the linear condition ends, see § 35, and formula (525). If 11 be assumed to be the earliest age of what may be called "extraordinary marriage," and 14 be assumed to be the earliest age of "ordinary marriage," and if also the generally approximate result, be adopted, viz., $0.6667 + 0.3643\ i$, the plan of the polygenesic surface will have for a limiting boundary the line $y = x - 11$; its surface will, for the major part, be (approximately) a plane, steepest at right angles to the axis of abscissae (age), and making an angle θ with the xy plane, the tangent of which angle is 0.3643. For any age x, the line on the surface denoting increasing durations of marriage,

[1] The assigning of the word "polygenesic" to the one, and "fecundity" to the other distribution, is, of course, somewhat arbitrary : the terms might, of course, have been interchanged.

rises uniformly till it attains the value $y = x - 14$. For greater durations than this the surface will droop. Between the axis and the contour-line representing say the third or fourth child, the surface is somewhat irregular.

If the distribution is based on the *ages at marriage* and the duration of marriage, it may appropriately be called the *gamogenesic distribution*. The abscissæ then are the ages of mothers when married (*i.e.*, "ages at marriage"), and the ordinates, as before, are the duration of marriage.

37. **Diminution of average issue by recent maternity.**—Returning to the results shewn in Tables XC. and XCI., for the second and subsequent years of duration of marriage, it may be noted that they are important in any attempt to ascertain what may be called the *unmodified fertility-ratio*. When the fertility-ratio is found by merely dividing the total number of cases of nuptial maternity at any age by the number of married women at the same age, the quotient is "modified" by the fact that they are *not all at equal risk*. If the fertility-ratio is to shew what is due to change of age alone, or rather, to change of age, unmodified by the effect of a recent birth, but unaffected as to all other factors, a certain proportion of the married women should be subtracted from the total. We shall first consider the question of estimating the diminution of average issue by recent cases of maternity.

Formula (523), shewing the general rate of increase in the average issue, (since it is derived only from all cases of maternity *coming under observation for each duration*), gives what may be called "the unmodified rate of increase" for what also may be called "the fertile section only" of the whole body of married women; see § 34, hereinbefore. Consequently the differences of average issue for successive durations of marriage, although an indication of, do *not* give a very exact measure of the proportions of women who are virtually removed from risk. These proportions are doubtless better defined by the differences between the observed average and the average issue computed upon the assumption of constant average rate of increase per year of duration. Hence the ratio of the diminution in the cases of maternity for any given age-group and for any given duration of marriage may at least approximately be found as follows :—

Let c'' be the average number of children (or average issue) on the supposition of a uniform increase, and c the actual number, each with suffixes to denote the duration of marriage and age. Then the diminution-ratio, that is the amount by which any previous births will have diminished the actual record of cases, will presumably be c/c''. But this diminution-ratio applies only to the cases in which maternity has occurred.

Consequently if the values of this fraction be formed, for successive years of duration, commencing not from marriage, but from the number for the first year of duration of marriage, it will furnish a rough estimate of the correction necessary, if it be desired to ascertain, from the number of cases actually occurring, the number of cases that would have occurred had the whole of the women in any age-group been at full average risk.

If to the values of c'', for duration 0 to 1, given in the top line of Table XC., successive multiples of 0.3643 be added, and the sums, so formed, be subtracted from the values on the second, third, etc., lines of that table, we shall obtain the figures shown on Table XCIII. on next page. These figures afford a fairly good indication of a systematic effect, according to duration, that is, of an effect which varies with age. This variation is not the same for each duration, and appears to change somewhat irregularly with age. The mean of the changes gives a fairly regular curve (see the upper part of Table XCIII.).[1] The individual graphs for the various durations, however, appeared to shew that the adoption of this general average for each series, was of doubtful validity, and for this reason a different linear change according to age was adopted for each duration.

In any attempt to estimate the diminution of the numbers at risk by means of the falling off in the average issue, according to duration, it is probably desirable to take the adjusted results in the upper part of Table XCIII. This will give — .186/.364, + .177/.729, etc., for age 18.92, —.217/.364, +.177/.364, etc., for age 22.87; and so on. The results are shewn in Table XCIV. If we call the tabular value c', the ratio ρ of the altered risk to the average risk is given by :—

$$(526)\ldots\ldots\rho = 1 + c'' / 0.3643 = 2.745\,(0.3643 + c'').$$

The value of $1-\rho$ will be required ; it is consequently :—

$$(527)\ldots\ldots 1 - \rho = -\,2.745\ c'''.$$

Since c'' is negative, if for any duration of marriage fewer women than the average have given birth to children (owing to a recent birth, etc), then this last expression is positive. Table XCIII. shows the deviations, according to age and durations of marriage up to four years ; from the general rate of increase.

[1] "The curve can be very closely represented by the curve $a + bx + cx^n$, where n is greater than 1. Smoothed, the values would be about + .000, —.031, —.072, —.124, —.183, —.265, —.422.

TABLE XCIII.—Shewing the Average Effect of a recent Maternity upon the Average Issue (Number of Children) Corresponding to Various Durations of Marriage, and of a Consequent Correction.

Mothers' Age-group. Years.	Excess (+) or Defect (—) in the Average Number of Children, on an Average (Linear) Increase according to Duration of Marriage. Duration of Marriage. Crude Results. *1–2.	2–3.	3–4.	4–5.	Mean.	Adjusted Results. †1–2.	2–3.	3–4.	4–5.	Average Age.
–19	—.120	+.190	+.046	—.007	+.027	—.186	+.177	+.078	+.049	18.92
20–24	.217	.143	.068	.210	—.054	.217	.106	.031	—.014	22.87
25–29	.295	.022	—.022	+.047	.062	.252	.023	—.024	.086	27.46
30–34	.307	—.059	.084	—.046	.124	.292	—.070	.086	.168	32.65
35–39	.326	.153	.147	.107	.183	.328	.154	.141	.241	37.29
40–44	.280	.304	.199	.279	.265	.363	.237	.197	.313	41.91
45–	—.355	—.326	—.449	—.558	.422	—.396	—.315	—.249	—.382	46.29
Aver. Difference for an age-difference of 10 yrs.						—.077	—.179	—.120	—.158	..
Mothers' Age-group.	**Average Increase in the Average Number of Children. Crude Results.‡**					**Adjusted Results.§**				**Age.**
19–	.244	.675	.220	.321		.245	.690	.240	.453	17.5
20–24	.147	.725	.289	.451		.147	.680	.275	.446	22.5
25–29	.069	.662	.340	.433		.088	.650	.310	.429	27.5
30–34	.057	.613	.339	.402		.053	.600	.345	.402	32.5
35–39	.030	.538	.370	.404		.032	.530	.380	.365	37.5
40–44	.084	.341	.469	.284		.019	.440	.415	.328	42.5
–45	.009	.394	.241	.255		.011	.330	.450	.271	47.5

Values of $1 - \rho = - 2.745\ c'''$

Mothers' Age.	$k=1$	$k=2.$	$k=3.$	$k=4.$
12	+.364	..	..	..
13	.385	—.777	..	..
14	.407	.728	—.376	..
15	.428	.679	.343	—.304
16	.449	.629	.310	.261
17	.470	.580	.277	.218
18	.491	.531	.244	.174
19	.512	.482	.211	.131
20	.534	.433	.178	.088
21	.555	.384	.146	.044
22	.576	.335	.113	—.001
23	.597	.286	.080	+.043
24	.618	.236	.047	.086
25	.639	—.187	—.014	.129
30	.745	+.059	+.151	.346
35	.850	.304	.316	.563
40	.956	.550	.480	.780
45	1.062	.800	.645	.997
50	+1.167	+1.041	+.810	+1.214

* These results are found by adding multiples of 0.3643 to the figures in the first row of Table XC., and then subtracting them from the figures for the corresponding duration in the successive columns.

† These results are the linear smoothings of the crude results. The linear adjustments are made by using the "average" ages, and can be regarded only as fairly satisfactory. The total number of cases of maternity analysed is, however, large; viz., 805,015.

‡ These rows are the differences of the columns in Table XC.

§ The adjustments follow no general law: the first is on a curve Ae^{bx}, the second is $A' - Bx^2$, the third, $A'' + B'x$, and the fourth $A''' - B'x - Cx^2$, the intervals $x_2 - x_1$, etc., between the age groups being taken as always of equal value, *i.e.*, the adjusted values are for 17.5, 22.5, etc.

The above table appears to shew that the period of time over which the influence of a case of maternity extends on the average, follows no simple law, and is by no means negligible for some years, especially as regards the later portion of the child-bearing period. The whole method is not quite satisfactory, but is the best available, until the record of the procreative history of a large number of married women is to hand, giving the intervals between marriage and the births of successive children preferably compiled for intervals of single months from at least one to sixty, and for somewhat larger intervals (quarters, half-years, or years), to the end of the child-bearing period. Such statistics would reveal accurately the characteristic of the frequency of maternity according to duration of marriage, and would allow of the ratio ρ referred to in formulæ (528, 529) hereinafter being more exactly ascertained.[1]

[1] As far as I am aware such a statistic has not been compiled, although it is of considerable importance.

38. **Crude fertility according to age corrected for preceding cases of maternity.**—The ratio (m/M), between the number of nuptial mothers (m) of a given age-group during a given period of time, to the total number (M) of married women of the same age-group, is not the true monogenous-fertility-ratio, inasmuch as the M married women are not homogeneous as regards the maternity-risk (ρ) to which they are subject. Obviously m/M is too low a value for women whose fertility remains in abeyance, and is too high a value for women who have just borne children. The survivors after the lapse of k years of the married women of age x last birthday are L_{x+k} / L_x.[1] Consequently if ρ_k is the average risk for the kth year after a birth (calling the year of birth 0), the corrected fertility ratio $(\mathfrak{p})$ is given by the equation :—

(528)......

$$\mathfrak{p}_x = \frac{m_x}{M_x - \left\{ m_{x-1} \cdot \frac{L_x}{L_{x-1}} (1-\rho_1)(1-r_{x-1}) + m_{x-2} \cdot \frac{L_x}{L_{x-2}} (1-\rho_2)(1-2r_{x-2}) + \text{etc.} \right\}}$$

$(1-kr_{x-k})$ denoting the rate at which the mothers of age $x - k$ have increased in k years. This may perhaps be ordinarily taken as the same at all ages, and as the rate of the population increases. The above formula may be put in the following form, viz. :—

(529)......

$$\mathfrak{p} = \frac{m_x}{M_x} \cdot \frac{1}{1 - \left\{ \frac{m_{x-1}}{M_x} \cdot \frac{L_x}{L_{x-1}} (1-\rho_1)(1-r_1) + \dots + \frac{m_{x-k}}{M_x} \cdot \frac{L_x}{L_{x-k}} (1-\rho_k)(1-kr_k) + \dots \right\}}$$

and the ratios of the m/M quantities in the denominator do not need to be very exactly computed. It will always be abundantly accurate for the purpose in view to assume that :—

$$(530)\ldots\ldots\ L_x/L_{x-k} = 1 - \tfrac{1}{2}k\,(q_x + q_{x-k})$$

a formula which is satisfactory through a fairly large range for k.[2] Since the quantity between the braces in (529) is positive and small, its effect is to increase the value of $\mathfrak{p}$. The correction is important in any attempt to ascertain the age of greatest fertility, consequently the values given in Table LXXIII., p. 242, are those with which we are mainly concerned : see columns ix. and xv. therein. The values of the factors (κ) of m/M in the denominator of (529) can be readily tabulated for say $r = 0.01$ and 0.03.

[1] L_x denoting the mean population living in the year of age x : as in the ordinary actuarial notation.

[2] For example from Australian Life Tables for 1901-1910, Report of Census, Vol. III., pp. 1217-8, we have for ages 40 and 30, from the L values 0.93986, and from the q values 0.93815, *i.e.*, for so large a value of k as 10, the error is less than 0.002.

The value of the L, ρ and r terms are as follows for Australia :—

TABLE XCIV.—Shewing the Factors Required to Correct the "Crude Fertility-ratio," for Preceding Cases of Maternity. Australia, 1908-1914.

Age of Mother.	Values of $(1-kr)\ L_x / L_{x-k}$ $r = .01$* ; $k = 1$ to 4.				Values of κ when $r = .01$. $r = .01$ and .03 ; $k = 1$ to 4.†			
	1.	2.	3.	4.	1.	2.	3.	4.
15	.9879	.9761	.9643	.9527	+.423 .414	—.662 .635	—.331 .311	—.290 .266
20	.9868	.9739	.9613	.9590	+.527 .516	—.422 .405	—.171 .160	—.084 .077
25	.9858	.9720	.9584	.9450	+.630 .617	—.182 .175	—.013 .012	+.122 .112
30	.9849	.9701	.9556	.9414	+.733 .718	+.057 .055	+.144 .135	+.326 .299
35	.9840	.9683	.9529	.9378	+.837 .820	+.294 .282	+.301 .282	+.528 .484
40	.9830	.9663	.9500	.9340	+.940 .921	+.531 .509	+.456 .428	+.728 .667
45	.9821	.9645	.9472	.9304	+1.043 1.022	+.767 .736	+.611 .573	+.927 .850

* To find the values for any other value, r' say, of r, multiply the tabular values by $(r'-r) / r$.

† To find the values for any other value of r, multiply by $(1-rk) / (1-.01k)$. Thus, for $r = .02$ the multipliers of the successive columns are 0.9899, 0.9796, 0.9691, 0.9583; and if $r = .03$ the successive multipliers are 0.9797, 0.9592, 0.9381, 0.9167.

The above values are very approximately given by :—

$$(531)\ldots\ldots(1-kr)\ L_x / L_{x-k} = 1 - 0.000188k\ (47.7 + x)\ ;^1$$

and those for the correcting factors ϵ by :—

$$(532)\ \ldots.\epsilon_1 = 0.02070\,(x + 5.43)\ ;\quad (532a)\ldots.\epsilon_2 = 0.04763\,(x-28.91)\ ;$$

$$(532b)\ldots.\epsilon_3 = 0.03140\ (x-25.54)\ ;\quad (532c)\ldots.\epsilon_4 = 0.04057\ (x-22.15).$$

Formula (525) may thus be written :—

$$(533)\ldots\ldots p'' = \frac{m_x}{M_x}\cdot\frac{1}{1-\frac{1}{M_x}(\kappa_2 m_{x-1}+\ldots\ldots+\kappa_k m_{x-k})},$$

k being the tabular value given in Table XCIV. (in which $r = .01$ and $r = .03$), and the probability of maternity ascertained by this last formula, will be free from the effect of recent cases of maternity : that is the crude probability must be multiplied by the fraction following m/M.

39. **Age of greatest fertility.**—When the probabilities according to age of maternity have been corrected so as to represent what would be given if all women were at equal risk, then the age of greatest probability may be regarded as the age of greatest fertility. Applying formula (533) to the data in Table LXXIII., p. 242, we have the following results about the maximum :—

[1] More exactly the values of the constant to be added to x are 47.60, 46.81, 47.63, and 48.63, and of the coefficients to be multiplied into k are 0.0001933, 0.0003866, 0.0005700, and 0.0007433.

TABLE XCV.—Shewing Corrections to the Fertility-ratio for Ages 13 to 23, when Allowance is made for Preceding Cases of Maternity.

Age of Mothers.	Values of κ when $k = 1$ to 4.				Factor K in $Kp = p'$		Fertility-ratio.	
	1.	2.	3.	4.	Crude.	Smoothed.	Crude.	Corrected.
13	+.374	—.727	—.371	—.342	1.039	1.001	.5?	.52?
14	.394	.681	.341	.304	1.011	1.013	.2055	.2076
15	.414	.635	.311	.266	1.013	1.024	.2269	.2299
16	.434	.589	.281	.288	1.012	1.036	.4063	.4112
17	.455	.543	.251	.190	1.048	1.048	.4316	.4521
18	.475	.497	.221	.152	1.066	1.059	.4776	.5093
19	.495	.451	.191	.114	1.077	1.071	.5022	.5109
20	.516	.405	.160	.077	1.092	1.083	.4540	.4956
21	.536	.359	.130	.039	1.074	1.094	.4375	.4700
22	.556	.313	.100	—.001	1.106	1.106	.4157	.4596
23	+.577	—.267	—.070	+.037	1.123	1.117	.3813	.4283

Although the values of κ are of the same order of magnitude, yet within the range shewn, the values of the successive κm-terms rapidly diminish, so that although there is no theoretical justification for stopping at $k = 4$, the inclusion of later terms would but slightly affect the result (at least in the second place decimals).

The factors K shew that about the age of maximum fertility the correcting factors to give the fertility, unprejudiced by previous cases of maternity, increase linearly with age, and are represented very approximately by the formula :—

$$(534) \ldots\ldots K = 1 + 0.01163\,(x - 12.91).$$

The values for these factors, so computed, are the smoothed values in the preceding table.

A smoothing, independent of that already given in Table LXXXIII., gave, as the maximum for the uncorrected fertility-ratio, 0.483 ; and a similar smoothing of the corrected values gave 0.517, the maxima and corresponding ages being :—

Uncorrected, age, 18.8,[1] 0.483 ; corrected, age, 19.0, 0.517.[2]

In the method outlined, of correcting the crude fertility-ratio (probability of maternity), equal "weight" is attributed to the values of k. An examination of Fig. 75 shews, however, that the "weight" to be attributed should probably decrease with increase in the value of k (that is with the number of years elapsed since a previous birth). Moreover, the change in the numbers of married women and cases of maternity is so rapid at the ages of maximum fertility that the age divisions should be less than one year, and the ages need to be very exactly given, which unfortunately they are not. For these reasons great exactitude in regard to the correction is at present impracticable.

40. **Fecundity-correction for infantile mortality.**—The frequencies of child-bearing as between two populations are, like their birth-rates, rigorously comparable as accurate measures of fecundity, only when their infantile mortality-rates are identical, and the crude frequencies require,

[1] The result in Table LXXXIII. was 18.23 years.
[2] The factor, according to (534) above, gives, on multiplying into, 0.483, 0.5168.

therefore, a correction, to reduce the risk of maternity to an equality ;[1] see Part XI., §§ 4-6, pp. 145-152. It has been shewn that the infantile mortality correction to birth-rate is, on the whole, about $\beta_0 = \beta(1+0.033\mu)$; see p. 145. If, therefore, there were two equal populations of say married females (M), of equal fecundity (f), but with different rates of infantile mortality, we should have for the cases of maternity (m) occurring therein, respectively :—

$$(535)\ldots\ldots m_1 = fM(1 + k_1\mu_1) \text{ and } m_2 = fM(1 + k_2\mu_2) ;$$

whence it follows that

$$(536)\ldots\ldots f = \frac{m_1}{M(1 + k_1\mu_1)} = \frac{m_2}{M(1 + k_2\mu_2)}$$

Thus the correction is always very small, and, in general, is practically negligible.

41. **Secular trend of reproductivity.**—The *crude reproductivity* may be measured by the ratio of the number of confinements to the number of persons at uniform risk ; thus the *crude nuptial reproductivity* is the ratio of nuptial confinements to the total number of married women, and similarly, the *crude ex-nuptial reproductivity* is the ratio of ex-nuptial confinements to the total "unmarried," which here will include the "divorced" and "widowed." The ratios are "crude," since no corrections have been applied for age-differences in the female population, and it is obvious from columns ix., x., xv., and xvi. of Table LXXIII., p. 242, that fertility greatly varies with age. For this reason, *whenever the age-distribution is not identical, the results are not strictly comparable :* they do not rigorously measure the degrees of reproductivity, or of malthusianism, operating. Consequently, for strict comparisons, a properly determined *index of initial reproductivity* would have to be computed, see §§ 3 to 6, pp. 235-239.

Nuptial and Ex-nuptial Maternity-Ratios, etc.

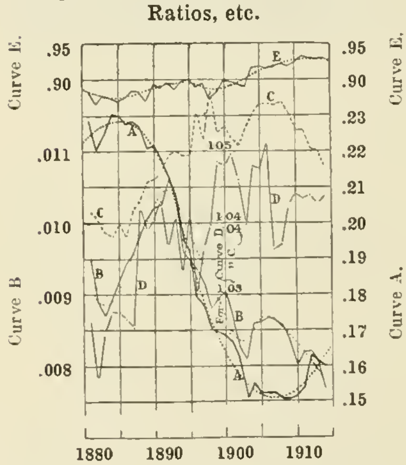


Fig. 80.

Curve A is the ratio of nuptial confinements to all married women.

Curve B is the ratio of ex-nuptial confinements to "unmarried" women of 12 years of age and upwards.

Curve C is the ratio of the ex-nuptial to the nuptial confinement rates, the range being between .038 and .059.

Curve D shews the variation in the average number at a birth.

Curve E shews the variations in the survival factor for the first year of life.

Neglecting this, however, for the present, and restricting the consideration to the crude initial nuptial

[1] It may be noted that after deducting the period of gestation and the puerperal period, there remains about one-sixth of a year during which mothers of the first-sixth of any year of record may give birth to a second child even in the same year, and the chance of this occurring is increased by the death of the child born.

and ex-nuptial reproductivities, the results are set out in Table XCVI. hereunder; see columns (ii.), (iii.), and (v.) thereof. The results are shewn also by curves A and B of Fig. 80, the former curve denoting the nuptial, and the latter the ex-nuptial frequency of maternity. The figure shews that while the nuptial and ex-nuptial rates by no means run identically, they yet exhibit, on the whole, similarity of trend, the ex-nuptial rate being roughly 0.05 of the nuptial. The exact fluctuations of the ratio of the ex-nuptial to the nuptial rate are indicated in column (v.) of Table XCVI., and are shewn as curve C in Fig. 80. The dotted lines on curves A and B shew the general trend of the phenomena.

TABLE XCVI.—Shewing the Secular Changes of Nuptial and Ex-nuptial Reproductivity. Australia, 1881 to 1914.

Year. (i.)	Ratio of Nuptial Confinements to Married Women.* (ii.)	Ratio of Ex-nuptial Confinements to Number of Unmarried Women.† (iii.)	Ratio of Births to Total Confinements. (iv.)	Ratio of Ex-nuptial to Nuptial Rates. (v.)	Infantile Mortality (Ratio of Deaths of Children during first 12 Months)‡ (vi.)	Survival Coefficients for end of First Year (vii.)
1881	.2285	.00950	1.00865	.0416	.1165	.8835
1882	.2206	.00891	1.00779	.0404	.1357	.8643
1883	.2245	.00870	1.00847	.0388	.1222	.8778
1884	.2305	.00893	1.00875	.0380	.1260	.8740
Mean 1–4 ..	**.2269**	**.00901**	**1.00842**	**.0397**	**.1251**	**.8749**
1885	.2301	00918	1.00873	.0399	1292	.8708
1886	.2274	.00946	1.00866	.0381	.1271	.8729
1887	.2285	.00957	1.00852	.0419	.1164	.8836
1888	.2271	.00983	1.01021	.0433	.1164	.8836
1889	.2206	.01008	1.00989	.0457	.1319	.8681
Mean 5–9 ..	**.2267**	**.00962**	**1.00920**	**.0418**	**.1242**	**.8758**
1890	.2216	.01021	1.01005	.0461	.1082	.8918
1891	.2181	.01026	1.01030	.0470	.1155	.8845
1892	.2133	.01060	1.00865	.0497	.1058	.8942
1893	.2072	.01034	1.01008	.0499	.1149	.8851
1894	.1947	.00961	1.00931	.0494	.1031	.8969
Mean 0–4 ..	**.2110**	**.01020**	**1.00968**	**.0484**	**.1115**	**.8885**
1895	.1916	.00947	1.01008	.0494	.1012	.8988
1896	.1788	.00935	1.00900	.0558	.1126	.8874
1897	.1770	.00914	1.01066	.0517	.1048	.8952
1898	.1700	.00879	1.00997	.0586	.1272	.8728
1899	.1697	.00894	1.01086	.0527	.1167	.8833
Mean 5–9 ..	**.1774**	**.00814**	**1.01011**	**.0538**	**.1125**	**.8875**
1900	.1691	.00905	1.01078	.0535	.1002	.8998
1901	.1668	.00865	1.01095	.0519	.1037	.8963
1902	.1625	.00826	1.01060	.0508	.1071	.8929
1903	.1513	.00807	1.00997	.0533	.1105	.8895
1904	.1554	.00859	1.01079	.0553	.0825	.9175
Mean 0–4 ..	**.1610**	**.00852**	**1.01062**	**.0530**	**.1008**	**.8992**
1905	.1524	.00861	1.01076	.0565	.0819	.9181
1906	.1527	.00868	1.01112	.0568	.0836	.9164
1907	.1527	.00864	1.00962	.0566	.0814	.9186
1908	.1506	.00857	1.00969	.0569	.0780	.9220
1909	.1506	.00837	1.01024	.0550	.0718	.9282
Mean 5–9 ..	**.1518**	**.00857**	**1.01029**	**.0565**	**.0793**	**.9207**
1910	.1511	.00801	1.01040	.0530	.0751	.9249
1911	.1541	.00818	1.01033	.0531	.0680	.9320
1912	.1632	.00821	1.01037	.0503	.0708	.9292
1913	.1609	.00805	1.01025	.0500	.0720	.9280
1914	.1598	.00766	1.01038	.0479	.0713	.9287
Mean 0–4 ..	**.1578**	**.00802**	**1.01035**	**.0509**	**.0714**	**.9286**

* That is, to all married women, irrespective of age.

† That is, to "never-married," "widowed," and "divorced," of 12 years of age and upwards, taken together.

‡ The infantile mortality as given is *not* the ratio of deaths registered as under one year of age, in any year, to the births registered in the same year, but are those given in a paper "On the improvement in infantile mortality, etc.," read before the Australasian Medical Congress in September, 1911 (see p. 672 Journ.), and are related to the number of births of the "equivalent year."

42. **Crude and corrected reproductivity.**—It has been shewn in Part XI., § 6, see Table XXXV., that the crude birth-rate gives only the *initial* reproductivity, and that, owing to the measure of infantile

mortality, the residual, after the first 12 months have elapsed, is more significant than the birth-rate as regards the increase of the population. The necessary correction is secured by multiplying by a "*survival factor.*" The principle may be extended for various purposes. Thus survival factors (σ) may be calculated for the commencing school-age, the ages of puberty or nubility, the commencing age of military service, the age of highest average economic efficiency, and so on. In actuarial notation these factors are denoted by l_x/l_0, and for brevity's sake may be denoted by σ_x. To compare two populations for survivals, S, up to any age x, we have, therefore, B denoting the births :—

$$(537) \ldots\ldots S_x = Bl_x/l_0 = B\sigma_x = B - D_x$$

in which D_x denotes the aggregate of the deaths (of the native-born) up to age x. When $x = 1$, the values of σ are unity, less the rate of infantile mortality taken for the "equivalent year." For rates, these quantities must be divided by the mean population of the period covered by the births. The more rigorous treatment of this question has already been dealt with in Part XI., §§ 7 to 9, pp. 152-180 ; see also Tables XXXVI. and XXXVII. The infantile mortality varies, however, considerably from year to year, see column (vi.) in Table XCVI., which gives the rates calculated approximately for the "equivalent year."[1] If γ denote the infantile mortality (see p. 151, hereinbefore), σ being the survival factor, then we have :—

$$(538) \ldots\ldots \sigma = 1 - \gamma\ ; \text{ or } \gamma = 1 - \sigma\ ;$$

as on (352), p. 151. This, of course, differs according to sex, with time, as is shewn in Table XCVI., and according to locality. The highest value of the survival-factor for Australia was 0.9320 in 1911. For the period 1901-10 for the Commonwealth of Australia it was 0.90490 for males, and 0.92047 for females,[2] corresponding to infantile mortalities of 0.09510 and 0.07953. We thus arrive at the conception of a *survival-value* for a birth-rate, that is, the birth-rate reduced to its value at age x, and this survival-value may be averaged for the whole of life, *i.e.*, integrated for all ages. Such an integral will constitute the best general measure of the reproductivity. It is equal to the average period lived multiplied by the birth-rate. Or if ω denote the greatest possible age, then :—

$$(539) \ldots . \Sigma_0 = \beta \int_0^\omega \sigma_x dx$$

and Σ_0 is the reproductivity of the population taken as a whole. If σ_0 be unity, and the unit of x be one year, then the value of (538) will be the

[1] *Vide* a paper (by the author): "The improvement in infantile mortality ; its annual fluctuations and frequency according to age, in Australia." Journ. Australasian Medical Congress, Sydney, Sept. 1911, pp.670-679.

[2] See Life Tables, Census Report, Vol. III., pp. 1215 and 1217.

birth-rate multiplied into the number of years expressing the length of life lived on the average ; consequently the product of the birth-rate into the "expectation of life at age 0," may be taken as the most serviceable expression of the reproductivity.[1]

The value given by (538) may be regarded as the *crude reproductivity.* The birth-rate β is ordinarily computed as for the total population, but may also be based upon the total female population, upon the female population of child-bearing ages, or upon the married of child-bearing ages plus a reduced number of the unmarried, equating them to the nuptial condition. Let the ratio of the fertility of women at full risk (or otherwise if desired), at any age x, to the fertility at the age at which it is a maximum be denoted by f_x : then the actual number of married women of all ages may be reduced to an *equivalent number* of women at the age of maximum fertility by multiplying by this quantity. With these can be included also the unmarried, with whom in Australia the fertility is about one-twentieth of that of the married. The corrected reproductivity may be given in the form of a birth-rate, viz., β_e :—

$$(540)\ldots\ldots\beta_e = B \,/\, \Sigma\,(f_x M_x + f'_x U_x)$$

in which Σ denotes "sum," f and f' are the ratios for the fertilities of the married and unmarried respectively, referred to the greatest fertility of the married, and M and U are respectively the numbers of the married and the unmarried, who together give birth to B children. This measures the ratio of the actual births to a *fictitious* number of mothers of highest fertility, and hence birth-rates so computed shew the variations of the extent to which potential fertility is actualised. These, of course, may be further reduced to their survival values.

The mode of comparing reproductive efficiency by means of an index, viz., the genetic index or first natality index, has already been indicated ; see § 5, p. 237, hereinbefore.

43. **Progressive changes in the survival coefficients.**—The survival-factors are by no means constant, as is shewn in column vii. of Table XCVI. As tabulated, they are merely unity, less the ratio of the deaths under 12 months to the births in the same year. This, as shewn before,

---

[1] Actuarially, the quantity :—

$$\overset{\circ}{e}_x = T_x \,/\, l_x = \int_x^{\omega} l_x \, dx \div l_x$$

when $x = 0$, may, when multiplied by the birth-rate, be adopted as the measure of the reproductivity of a population. Since this is obtained from the mortalities at successive ages, it is not quite homogeneous, as it is affected by the vitality of migrants, and, moreover, the mortality of the older part of the population is affected by their earlier history, and may not therefore represent future experience. If $l_0 = 1$, then $\overset{\circ}{e}_0 = T_0 = \Sigma_0 \,/\, \beta$.

is not quite correct, see pp. 155-160, but the correction is of no moment for the present purpose. It is worthy of note that the infantile mortality is roughly about 0.5522 of the rate of confinements of married women, as is shewn by comparing the means. The means (see Table XCVI.) 0.2269, 0.2267, etc., multiplied by the above fraction gives the following results :—

Period	1881-4	1885-9	1890-4	1895-9	1900-4	1905-9	1910-4
Infantile mortality ..	.1251	.1242	.1115	.1125	.1008	.0793	.0714
As computed from the nuptial confinement rate	.1253	.1252	.1165	.0980	.0889	.0838	.0871
Survival factor divided by ratio of nuptial confinements	.5513	.5479	.5284	.6342	.6261	.5224	.4525

The ratio is therefore not uniformly constant.

The infantile mortality is decreasing, but nevertheless shews a fairly definite fluctuation, see curve E, Fig. 80, which shews it on a large vertical scale ; its limiting value is, of course, unity.

XIV.—COMPLEX ELEMENTS OF FERTILITY AND FECUNDITY.

1. **General.**—In dealing with the more complex elements of fertility and fecundity, it will generally be necessary to distinguish between the nuptial and ex-nuptial cases, and since their frequency is very different, some simple method of correlating and comparing the two will have to be devised.[1]

Often it is necessary to distribute unspecified cases, since, in double-entry tabulations, the cases are often *partially* specified, and the neglect of partially-specified and wholly-unspecified cases will often lead to material error.

There is another general matter of importance, viz., the corrections required in statistics of duration, if they are required to represent the results which, other things being equal, would have been furnished by a constant population. This will receive attention in § 3, pp. 298-9.

2. **Correspondence and correlation.**—It is often possible to see the essential identity of two curves by mere change of scale, or by systematic deformations (anamorphosis) of one in order to bring it into agreement with another. This fact is of value in the graphs of various vital phenomena.

For example, any attempt to make the widest possible comparisons of population phenomena requires the construction of world-norms for the human race. But such an attempt involves the consideration of physiological and general correspondence of human developments. In connection with marriage, fertility, fecundity, etc., and their significance, for instance, this demands the consideration of the following, viz.:—

(*a*) The average ages of puberty, nubility, etc.
(*b*) The frequency-distribution about those ages;
(*c*) The fertility and fecundity at different ages;
(*d*) The characteristics of the decay of fecundity at the end of the fertile period.

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[1] The determination of a typo-formulæ to be adopted for any two curves, the ascertaining of their constants, and of the "skewness" of each curve will serve to exhibit their degree of correlation. This can also be expressed by a correlation coefficient; see "Statistical Methods," by C. B. Davenport, 1904, and the monographs of Prof. Karl Pearson, W. F. Sheppard, G. U. Yule, De Vries, W. Palin Elderton, Gini, Savorgnan, and others.

Suppose, for example, curve A, Fig. 81, represents the average fertility according to age of women of one part of the world and B that of another part. Let x, x', x'', etc., denote the abscissa of the initial point, that of the mode, and that of the terminal point of the curve A, or of curve B, the particular curve being indicated by the suffix a or b. Then the simplest correspondences are those where $x_a/x_b = x'_a/x'_b = x''_a/x''_b$, etc., or where $x_b - x_a = x'_b - x'_a =$ etc., *i.e.*, where the abscissæ of the corresponding critical points of the curves are in a constant ratio, and the ordinates are also in a constant ratio, or where the abscissæ of the critical points differ by a constant. Correspondence of this character may be called *planar*, because the curve B can be derived from the curve A by parallel linear projection on to a plane inclined to that on which A lies. If the two curves in question be represented by $y_a = F_a(x)$; $y_b = F_b(x)$ then planar correspondence may be defined as follows :—

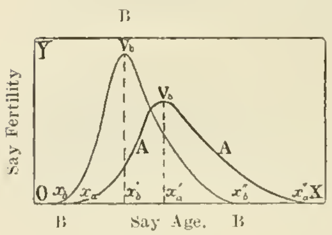

Fig. 81.

The points on curve B are in *planar correspondence* with those on A when—

$$(541)\ldots\ldots y_b = k\,F_a(mx_a + q)$$

k, m, and q being constants : when k or m or both are functions of x_a, then the correspondence is *non-planar*. If these functions of x_a are not simple, the correspondence becomes less significant.

This method of envisaging the problem has advantages over the system of determining a mere numerical "coefficient of correlation,"[1] because it is often possible to construct one curve from the data of the other. Moreover, it is not without value to examine how far the graphs of phenomena, which might have been imagined *a priori* to be identical, or convertible by oblique projection with change of scale, differ. Later nuptial and ex-nuptial fertility, according to age, will be compared.

3. **Corrections necessary in statistics involving the element of duration.**—The type of corrections necessary to be applied to the data of statistics involving the element of duration, depends upon the purpose in view. Two types are of special importance, that which aims at presenting the results, in the form in which they would have been given by (a) a constant population, and (b) by a population increasing according to some definite law, which for general comparative purposes is preferably

[1] See Galton's graphic method, F. Galton, 1888, Proc. Roy. Soc. Lond., XLV., 136–145. Davenport, Statistical methods, p. 44, 2nd Edit., Lond., 1904. See also Pearson's, Yule's, and other papers on the subject.

the *norm of increase, i.e.*, the characteristic of the increase of the whole of the populations to be compared. The latter involves the smaller corrections, and has the advantage that for many purposes the corrections will be negligible. Let it be supposed that the population is an *increasing* one: the data will then be characterised as follows:—

(i.) The data for longer durations, drawn therefore from a smaller population, will be smaller (all other things being equal) than would characterise a constant population of the size from which the more recent data are drawn. Hence the necessary correction is a factor $1 + e$, where e is positive.

(ii.) If the numbers of individuals have been taken into account for earlier dates, they can be deduced from the survivors, provided (a) that a correct mortality table is available, and (b) that migration has introduced no (material) modification.

(iii.) If the data are related to events occurring with a varying rate (as in cases of birth, marriage, death, etc.), the rate at which they occur must be determined according to the duration in question.

The type-formula for correction is as follows:—Let N denote the number given at any point of time, that is, let N denote the *survivors* after the duration i, from N' persons; then if, in origination, N may be presumed to vary with the population, we shall have, on making allowance for the fact that these are only survivors, and that what is required is a result which shall either coincide (i.) with the final magnitude of the population, viz., at the date from which i is reckoned, or (ii.) with a definite rate of population growth (the rate of normal increase):—

(542). . $N' = Ne^{\rho i} L_{x-1}/L_x = Ne^{\rho i}[1 + \frac{1}{2}(q_{x-1} + q_x)]$,* approximately.

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* See formula (530), p. 289. The notation is the ordinary actuarial notation. It is fairly obvious that L_{x-i}/L_x must equal $1 + \frac{1}{2}(q_{x-i} + q_x)\, i$ approximately. It will be found that, through a large range, this latter and arithmetically more convenient form is sufficiently accurate for correction purposes to the data of statistics of duration. For example, if 12 be taken as the lowest age (it is the lowest age of least mortality for Australian females), and successive intervals of 10 years from this be also taken, the following results are obtained, viz.:—

x-i and x ..	12–22	12–32	12–42	12–52
Exact formula ..	1.03114	1.0933	1.1861	1.3133
Approx. formula ..	1.03110	1.0893	1.1500	1.2888

Even the final difference is ordinarily of no moment, since, as a rule, the numbers to which it would have to be applied are very small.

In this ρ will denote in case (i.) the absolute rate of increase, and in case (ii.) the excess over the normal rate of increase. Certain events, however, for example births, marriages, and deaths, migration, etc., occur with a rapidity which fluctuates on either the positive or negative side of the general rate of increase of the population, in which case it may be necessary to introduce, into equation (542), a factor depending on the fact in question.

4. **Distribution of partially and wholly unspecified quantities in tables of double-entry.**—If a series of quantities, A, B, C, etc., and A′, B′, C′, etc., fully specified so as to permit of proper double-entry, and others, a, a', etc., and α, α', etc., specified so as to permit only of single entry, and again a third set ω not specified, so as to permit of entry under either of two series of headings, be tabulated or arranged as hereunder, and totalled, the result will be as shewn symbolically in the following table :—

TABLE XCVII.—Scheme of a Double-entry Tabulation of Defectively Specified Data.

Arguments	y	y'	y''	y'''	etc.	etc.	Specified as regards x only.	Totals.
x	A	B	C	D	etc.	etc.	a (b)	$S + a\ (S + b)$
x'	A′	B′	C′	D′	etc.	etc.	a' (b')	$S' + a'\ (S' + b')$
x''	A″	B″	C″	D″	etc.	etc.	a'' (b'')	$S'' + a''\ (S'' + b'')$
etc.	etc.	etc.	etc.	etc.	etc.	etc.	etc.	etc.
Specified as regards y only	$\alpha\ (\beta)$	$\alpha'\ (\beta')$	$\alpha''\ (\beta'')$	$\alpha'''\ (\beta''')$	etc.	etc.	ω (0)	$[\alpha+\alpha'+\ldots]+\omega$ $(\beta+\beta'+\ldots)+0$
Totals	$T + \alpha$	$T'+\alpha'$	$T''+\alpha''$	$T'''+\alpha'''$	etc.	etc.	$[a+a'+\ldots]+\omega$ $(b+b'+\ldots)+0$	$\Sigma S+\Sigma a+\Sigma\alpha+\omega$ $\Sigma T+\Sigma a+\Sigma\alpha+\omega$

In this type-table, the horizontal and vertical totals of the fully-specified quantities are respectively S, S′, etc., and T, T′, etc., but the aggregates of the rows are $S + a$, etc., and of the columns are $T + \alpha$, etc. (*i.e.*, for the fully specified quantities together with those specified as regards one particular only). The totals $T + \alpha$ are specified as regards the "arguments" in the horizontal headings, and the totals $S + a$ are specified as regards the "arguments" in the vertical headings. Thus the grand total is $\Sigma S\ (= \Sigma T) + \Sigma a + \Sigma\alpha + \omega$, and this is the sum of either of the series of totals, viz., that of the final column or that of the final row.

In order to distribute the quantity wholly unspecified, it is necessary to add a portion of ω to the (vertical) columns, and a portion thereof to the (horizontal) rows, so that the corrected values of A, B,...... A′, B′...., etc., shall equal the grand total, and so that the adjustment

shall be the most probable. Such adjustment can be effected as follows: It is assumed that the division of the quantity ω into two parts, viz., ω' and ω, proportional to the aggregates of the a and α quantities respectively, is the most probable apportionment of the doubly-unspecified quantity among the two, and further, that if these divisions, ω' and ω, be again subdivided proportionally to the individual values of a, a', etc., and α, α', etc., the result will be the most probable subdivision. Let—

$$(543)\ldots\ldots\omega = \omega' + \omega\ ; \text{ and } \Omega = \Sigma a + \Sigma \alpha;\ \text{then}$$

$$(544)\ldots\ldots\omega' = \Sigma a\,.\,\omega\ /\ \Omega\ ; \text{ and } \omega = \Sigma \alpha\,.\,\omega\ /\ \Omega\ ;$$

consequently the amounts of the corrections to the a and α quantities are ascertained by multiplying each of them by the ratio ω/Ω, or what is the same thing, the required result is attained by multiplying by this factor increased by unity. Calling the adjusted numbers b and β respectively, their values are:—

$$(545)\ldots\ldots b = a\,(1 + \omega\ /\ \Omega)\ ;\ \beta = \alpha\,(1 + \omega\ /\ \Omega).$$

Similarly, if these b and β quantities are distributed proportionally to the A, B, C, etc., quantities, and the A, A', A , etc., quantities respectively, the required corrections are:—

$$(546)\ldots \mathrm{A} + \mathrm{a} = \mathrm{A}\,(1 + \frac{b}{\mathrm{S}} + \frac{\beta}{\mathrm{T}})\ ;\ \mathrm{B} + \mathrm{b} = \mathrm{B}\,(1 + \frac{b}{\mathrm{S}} + \frac{\beta'}{\mathrm{T}'});\text{etc.}$$

$$(547)\ldots \mathrm{A}' + \mathrm{a}' = \mathrm{A}'\,(1 + \frac{b'}{\mathrm{S}'} + \frac{\beta}{\mathrm{T}})\ ;\ \mathrm{B}' + \mathrm{b}' = \mathrm{B}'\,(1 + \frac{b'}{\mathrm{S}'} + \frac{\beta}{\mathrm{T}});\ \text{etc.}$$

and so on. The additive quantities, $\mathrm{A}b\ /\ \mathrm{S}$, $\mathrm{A}\beta\ /\ \mathrm{T}$, etc., are most readily computed separately, and are then added to the fully-specified quantities. By the process indicated, both series of singly-specified quantities, and the unspecified quantities are suitably distributed, the adjusted table consisting of the values A + a, B + b, etc.; and A' + a', B' + b', etc.

The process indicated is also valid when the distribution should be made on other bases.

Let $\mathrm{a} = \mathrm{a}_1 + \mathrm{a}_2$; $\mathrm{b} = \mathrm{b}_1 + \mathrm{b}_2$; etc., $\mathrm{a}' = \mathrm{a}'_1 + \mathrm{a}'_2$; etc., etc., the subdivisions being the values of $\mathrm{A}b\ /\ \mathrm{S}$, $\mathrm{A}\beta\ /\ \mathrm{T}$, etc. Then, if the fundamental supposition that the corrections are proportional to A, B, etc., A', B', etc., be not satisfactory, any function of these quantities

may be substituted, in which case S and T will be $\Sigma\phi(A)$, and $\Sigma\psi(A)$, the former denoting the sum of the values of ϕA, ϕB, etc., and the latter the sum of ψA, $\psi A'$, etc. The process is identical in all respects with the preceding one, when the substitutions of ϕA for A, etc., have been made.

In general, this method of distribution not only gives results of a very high degree of probability,[1] but has also the advantage of being arithmetically very convenient.

5. **Unspecified cases follow a regular law.**—In general, the number of unspecified cases in any compilation exhibit great regularity. It will be sufficient to take two examples, which may be obtained from Tables CXIII. and CXIV. hereinafter.

According to the former Table, out of 733,773 wives, 21,151 made no statement as to the duration of marriage, but stated the number of children borne by them; 12,073 stated the duration of marriage, but omitted to state how many children were borne by them, and 3747 gave no information as regards either particular. See Census Report, Vol. III., pp. 1140-1. In the latter table, out of the same number, 5432 stated the number of children borne by them, but did not state their ages; 15,477 stated their ages, but did not state the number of children borne by them; and 343 gave no information as regards either particular: see Census Report, Vol. III., pp. 1136-7.

The regularity of distributions of the partially-specified cases is shewn by forming the ratios of the unspecified to the completely specified in the same category. The results are as follow:—

TABLE XCVIII.—Exhibiting the Regularity of the Ratios of those who Fail to Specify Particulars completely to those who do not so Fail.

Ratio to total who fully Specify, of those who omit to State	Particulars Specified—Issue to the Number of—										
	0	1	2	3	4	5	6	7	8	9	10
Duration of Marriage	.0253	.0220	.0232	.0273	.0303	.0347	.0369	.0386	.0406	.0421	.0433
Age ..	.0066	.0068	.0069	.0077	.0081	.0084	.0086	.0079	.0087	.0085	.0089
	Particulars Specified—Issue to the Number of—										
	11	12	13	14	15	16	17	18	19	20	21, etc.
Duration of Marriage	.0452	.0539	.0554	.0464	.0465	.0370	.0652	.0490	.0488	.0625	.1081
Age ..	.0094	.0081	.0094	.0066	.0088	.0105	.0074	.0236	.0000	.0000	.0208

[1] The ground of assurance as to this is indicated in the next section.

Table XCVIII.—Exhibiting the Regularity of the Ratios of those who Fail to Specify Particulars completely to those who do not so Fail—*continued.*

Ratio to total of those who fully Specify, of those who Omit to State	Duration of Marriage.									
	0–4	5–9	10–14	15–19	20–24	25–29	30–34	35–39	40–44	45–
No. of Children borne	.0216	.0204	.0174	.0104	.0142	.0137	.0130	.0145	.0124	.0151

	Ages of Wives.											
	14	15	16	17	18	19	20	21–24	25–29	30–34	35–39	40–44
No. of Children borne	.0000	.0000	.0148	.0105	.0107	.0161	.0231	.0215	.0209	.0198	.0201	.0212

	Ages of Wives.											
	45–49	50–54	55–59	60–64	65–69	70–74	75–79	80–84	85–89	90–94	95–99	100–04
No. of Children borne	.0207	.0236	.0238	.0263	.0290	.0349	.0378	.0642	.0824	.0625	.0000	2.0000

From the above results it is evident that the number who fail to specify "duration of marriage" is a fairly definite function of the "number of children borne," and also that the number who fail to specify the "number of children borne" is also a fairly definite function of the "duration of marriage," (see the upper part of the Table). Also, the lower part of the table shews that those who omit to state the number of children borne is a fairly definite function both of the "duration of marriage," and—as might consequently be expected—of the "age of the wives." These facts justify, *pro tanto*, the distribution of the unspecified cases, and there is little reason to doubt the result, after distribution, has a much higher degree of probability than that which rejects all partially-specified cases.

Other tabulations disclose, in an equally striking way, the regularity of the numbers of the unspecified, and confirm the desirability of adjusting tabulations generally, in the manner indicated, before using the results. The use of the fully specified tabulation as proportionally correct is obviously *not* satisfactory.

6. **Number of children at a confinement—a function of age.**—By dividing for each age the number of cases of confinement into the number of children born, during a sufficiently long period, the average number of children at a confinement is found to vary with age: that is, B denoting children born, and M the number of their mothers :—

$$(548)\ldots\ldots 1 + \epsilon_x = B_x \,/\, M_x = f(x)$$

ϵ denoting the excess over unity. Seven years' experience give the following results, the figures, however, being confined to cases of twins, that is, the third child in cases of triplets is not taken into account :—

TABLE XCIX.—Shewing Excess due to the Occurrence of Multiple Births according to Age in the Average Number of Children Born per Confinement, Australia, 1907–14.

Age.	Nuptial Confinements. Confinements.	Excess* over 1 Child.	Ratio. Crude.	Ratio. Smoothed.	Ex-nuptial Confinements. Confinements	Excess* over 1 Child.	Ratio. Crude.	Ratio. Smoothed.
12	0	..	..	.0000	5	..	..	.0000
13	4	..	..	.0006	21	..	..	.0007
14	30	..	..	.0013	126	..	..	.0014
15	170	..	..	.0019	537	1	.0019	.0021
16	1,138	2	.0018	.0026	1,500	2	.0013	.0028
17	3,962	12	.0030	.0032	2,980	11	.0037	.0035
18	9,761	35	.0036	.0038	4,504	15	.0033	.0042
19	18,071	94	.0052	.0045	5,317†	22	.0041	.0049
20	25,159	148	.0059	.0051	5,272	30	.0057	.0056
21	35,326	203	.0057	.0058	5,008	32	.0064	.0063
22	43,353	254	.0059	.0064	4.231	36‡	.0085	.0070
23	50,322	333	.0066	.0070	3,848	30	.0078	.0077
24	53,175	394	.0074	.0077	3,182	26	.0081	.0084
25	54,259	453	.0083	.0083	2,548	19	.0075	.0091
26	55,006†	447	.0081	.0090	2,161	20	.0093	.0098
27	53,735	494	.0092	.0096	1,785	27	.0151	.0105
28	53,244	509	.0096	.0102	1,699	20	.0118	.0112
29	49,200	539	.0110	0.109	1,410	7	.0050	.0119
30	47,980	555	.0116	.0115	1,356	17	.0125	.0126
31	40,199	484	.0120	.0122	851	10	.0118	.0134
32	41,528	565‡	.0136	.0128	956	13	.0136	.0146
33	37,426	508	.0136	.0134	812	15	.0185	.0162
34	34,362	486	.0141	.0141	779	13	.0167	.0186
35	31,349	445	.0142	.0147	688	17	.0247	.0200
36	29,399	496	.0169	.0154	636	12	.0189	.0190
37	26,213	419	.0160	.0160	544	7	.0129	.0160
38	24,664	380	.0154	.0163	555	9	.0162	.0135
39	20,790	326	.0157	.0158	436	5	.0115	.0115
40	17,023	232	.0136	.0145	383	6	.0016	.0102
41	12,252	173	.0141	.0129	201	3	.0149	.0083
42	11,012	126	.0114	.0114	205	0	.0000	.0068
43	7,457	85	.0114	.0101	155	1	.0065	.0056
44	4,746	37	.0078	.0088	85	1	.0118	.0045
45	2,755	21	.0076	.0075	58	0	.0000	.0036
46	1,389	10	.0072	.0063	36	1	.028	.0028
47	684	4	.0058	.0052	17	..	..	.0022
48	310	1	.0032	.0042	12	..	..	.0016
49	106	0	.0000	.0032	7	..	..	.0011
50	34	0	.0000	.0023	5	..	..	.0007
51	12	0	.0000	.0016	1†	..	..	.0004
52	6	1	.1666	.0009	0	..	..	.0002
53	4	..	..	.0005	1	..	..	.0001
54	3	..	..	.0002	..	..	..	..
Total	897,618	9,271	.001032	..	54,913	428	.000778	..

* Triplets are included in the result. † Maximum for confinements. ‡ Maximum excess for multiple births.

The results shewn indicate that the increase with age, x, is as follows:

$$(549)\ldots\ldots\beta_n = 1 + 0.00064\,(x - 12)\,;\ \text{up to age 37.}$$

$$(550)\ldots\ldots\beta'_e = 1 + 0.00070\,(x - 12)\,;\ \text{up to age 30 (?)}$$

The excess of over is the quantity ϵ in (548). Probably the age 37 could be adopted in both cases. For later ages, Table XCIX. must be consulted, and the values are uncertain. As the numbers are small this fact is, however, of little moment. Later, the case will be more fully analysed, for example, in regard to the duration of marriage, etc. Curves A and B, Fig. 82, shew the nuptial and ex-nuptial results respectively. The nuptial maximum is 0.0163 for age 38.4, and the ex-nuptial maximum is 0.0201 for age 35.5. The numbers after age 44 in the nuptial, and after age 36 in ex-nuptial cases are so small as to make the results for later ages doubtful, and the somewhat wide dispersion of the ex-nuptial results then probably is large for the later ages.

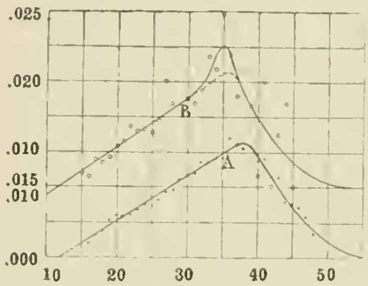

Fig. 82.

Curve A shews by a continuous line the smoothed curve of the excess over 1 at a birth, according to the age of the married mother; and Curve B similarly shews the results for unmarried mothers. The dots and circles show the crude results.

7. **Relative frequency of multiple births.**—For the period 1881 to 1915, the relative frequencies of twins, triplets, and quadruplets were as follow:—

TABLE C.—Multiple Births, Australia, During 35 Years, 1881–1915, and for Other Places.

	Population Aggregate.*	Female Population Aggregate.*	Confinements.	Cases of Twins.	Cases of Triplets.	Cases of Quadruplets.	Cases of Quintuplets.
Numbers	113,900,167	53,955,512	3,329,594	32,636	276	5	0
Ratios ..	34,208,424	16,204,832	1,000,000	9,802	82.9	1.50	0
,,	..	..	102.02	1	.00846	.00015	0
,,	..	..	12,064	118.25	1	.0188	0
,,	..	..	665,919	6,527	53.2	1	0
Year.	**Authority.**	**Total Births.**	**Confinements**	**Cases of Twins.**	**Cases of Triplets.**	**Cases of Quadruplets.**	**Cases of Quintuplets.**
	Neefe†	50,000,000	1,000,000	12,080	156	1.8	..
1871-80	Prinzing‡	63,000,000	1,000,000	11,677	143	1.3	..
1872-80	Knibbs	German Empire	1,000,000§	12,856	124	1.33	0.25¶

* Sum of the mean annual populations of the Australian States for which the necessary birth statistics were taken out.

† Zur Statistik der Mehrgeburten. Jahr. f. Nat. u. Stat., 1877, Bd. XXVIII., p. 174.

‡ Medizinischen Statistik. H. Prinzing, p. 65.

§ Confinements 12,013,134; Twins 154,444; Triplets, 1489; Quadruplets, 16; in the German Empire.

¶ Based on 15,965,391 children born, excluding still-births about 15,758,822.

Quintuplets have been reported by Volkmann,[1] Düsseldorf; by A. Bernheim,[2] Philadelphia; by Horlacher,[3] Württemberg; by Nyhoff,[4] Groningen; in 30 cases collected by the last-named, the majority were born at between 4 and 5 months.

Sextuplets are reported by Vassali,[5] and Vortisch, Alburi,[6] and *sextuplets* at Hameln in Westphalia in 1600[7]; no cases, however, so far as I am aware, have been reported in Australia.

The observed frequency of multiple births is as follows:—

TABLE CI.—Relative Frequency of Twins in Various Countries.*

Country.	Period.	Frequency.	Country.	Period.	Frequency.
Australia	..	..	Switzerland ..	1881–1900	.0126
Spain	1863–70	.0087	Germany ..	1901–1902	.0127
Roumania ..	1871–80	.0088	Baden ..	1891–1900	.0128
France	1899–1902	.0109	Prussia ..	,,	.0129
Belgium	1890, 5, 1900	.0111	Netherlands ..	,,	.0129
			Hungary ..	,,	.0131
Italy	1891–1900	.0117	Württemberg	,,	.0132
Russia	1887–91	.0121	Norway ..	1876–1880	.0133
Bavaria	1891–1900	.0123	Sweden ..	1871–80	.0146
Saxony	,,	.0123	Finland ..	1891–1900	.0147
Austria	1896–1900	.0126			

* The results other than for Australia are given in H. Prinzing's "Handbuch der medizinischen Statistik, p. 64.

The frequencies, however, have wide ranges of values. Thus, in Italy, they ranged in the period 1892-1899 through .0080 for Basilicata, to .0148 for Venice. For rough approximations the order of frequency with which twins, triplets, etc., occur, is as follows:—

Confinement.	Twins.	Triplets.	Quadruplets.	Quintuplets
1	.01	.0001	.000001	.0000002

8. **Uniovular and diovular multiple births.**—Observations as to the frequency of what may be called uniovular and diovular production of twins shew (i.) that the sexes are the same where the twins are produced by the division of a single ovum; (ii.) that this occurs in about one-fifth or one-fourth of the cases, these being recognised by the fact that they have common chorion; and (iii.) that where the twins are produced from two ova, the sexes may be identical or otherwise, these being recognised by the fact that the chorion is divided.

[1] Zentral bl. f. Gyn., 1879, p. 17.
[2] Deutsche med. Wochenschrift, 1899, p. 274.
[3] Horlacher, Württ., Korr. Bl. 1840.
[4] Zeitschr. f. Geb. u. Gyn., 1903, Bd. lii., p. 173.
[5] Anatom. Anzeiger, Bd. x., No. 10.
[6] Münch. med. Wochenschr., 1903, No. 38, pp. 1639-40 a photograph is given. Date of birth, 9th January, 1600.
[7] Deutsche med. Wochenschr., No. 19, 1899, p. 312.

Statistics for an examination of this question are not available in Australia, but are available for the German Empire. The data for 1906 to 1911 inclusive are as follows :—

TABLE CII.—Frequency of Multiple Births (German Empire, 1906 11).

Confinements.	2 Males.	Pairs.	2 Females.	3 males.	2m.,1f.	1m.,2f.	3 Females.	Males Born.	Females Born.	Total Quintuplets.*
12,013,134	49,425	58,382	46,637	343	390	395	361	28	36	
Children Born.	Total Cases of Twins.			Total Cases of Triplets.				Total Cases of Quadruplets.		
12,170,604	154,444			1,489				16.		3.
10,000,000†	128,563			1,239.5				13.3		2.54

* This is based upon 15,965,391 children born ; or about 15,758,800 confinements in 1872 to 1880, during which time 4 quintuplets were born. † This would give the proportion 3.05. As is evident for the number of children, the twins must be multiplied by 2, the triplets by 3, etc.

The proportion (ξ) of uniovular cases can be deduced at once from the preceding figures. Let μ denote the masculinity, defined as the ratio of the difference of the pairs of males and pairs of females to their sum ; see (335), p. 132. Obviously, the uniovular cases are in the ratio $(1+\mu)$ pairs of males to $(1-\mu)$ pairs of females. The diovular cases are in the same ratio as regards the same pairs, and the mixed pairs are equal to both combined, that is they are :—

TABLE CIII.—Theoretical Distribution of Diovular and Uniovular Cases Among Cases of Twins.

Total T	2 males	:	Male and female	+	Female and male	:	2 females
$T(1-\xi)$	$1+\mu$	:	1	+	1	:	$1-\mu$
$T\xi$ ..	$1+\mu$	:	0	+	0	:	$1-\mu$

Of the total there are ξ uniovular and $(1-\xi)$ diovular cases : consequently—

$$(551)\ldots\ldots\xi = \frac{M+F-P}{M+F+P}\,;\ \text{and}\ \mu = \frac{1}{2}\cdot\frac{M-F}{M+F}$$

M denoting the number of pairs of males, F the pairs of females, and P the cases of one of each sex. The above results thus give $\xi = 0.24397$ and $\mu = 0.029023$.

Direct observations according to Weinberg[1] and Ahlfeld[2] gave respectively for the relative frequency of uniovulate cases 0.21 and .0172, but it would appear from the preceding result that a sufficiently extended number of cases could be expected to give a higher ratio.

[1] Beiträge zur Physiologie und Pathologie der Mehrlingsgeburten beim Menschen. Archiv f. ges. Physiol., 1901, Bd. lxxxviii, p. 346 ; Neue Beiträge zur Lehre von den Zwillingen. Zeit. f. Geb. u. Gyn., 1903, Bd. xlviii., H. 1.

[2] Zeit. f. Geb. u. Gyn., 1902, Bd. xlvii., p. 230.

A similar investigation may be applied to the more limited results for triplets. Neglecting the masculinity tendency, it is obvious that for the triovular and diovular cases the proportions of cases in each category will be respectively :—

TABLE CIV.—Theoretical Distribution of Diovular and Triovular Cases Among Triplets.

Total T	3 males : (M)	2 males and 1 female : (P)	1 male and 2 females : (Q)	3 females (F)
$T(1-\xi')$	.125	.375	.375	.125
$T\xi'$ ..	.25	.25	.25	.25*

* It is assumed that when the births m.f.m and f.m.f occur, the chance of the two males or two females being uniovular is zero. If this condition were not physiologically impossible, it is easy to see (by exhaustive enumeration) that the probabilities of the four cases would be 0.2 : 0.3 : 0.3 : 0.2.

An examination of the individual figures for each year shews that the differences are too great to give any ground for deducing masculinity to be other than zero. Hence we may take means adopting :—

352 : 392.5 : 392.5 : 352 instead of 343 : 390 : 395 : 361.

and this gives for the series of triovular and diovular births respectively :

20.25 : 60.75 : 60 : 75 : 20.25 and 331.75 : 331.75 : 331.75 : 331.75,

or 162 triovular and 1327 diovular births in all ; or ratio of diovular cases of no less than 0.8912 of the total, the triovular being 0.1088. Thus it follows that *triovulation is a much rarer occurrence than the production of uniovular twins,* that is, the ratio of triovulation in triplets to diovulation is 8.20. From the above we obtain by symmetrically including all the data :—

$$(552)\ldots\ldots\xi' = \{3(M+F)-(P+Q)\}\,/(M+P+Q+F).$$

Thus, according to the recent experience of the German Empire, we have for 10,000,000 cases of confinement, 31,365.5 cases of uniovulation production of twins among the twins, and 1104.6 cases occurring among the triplets. We may assume at least the same ratio for the cases of quadruplets and quintuplets, which will give, say, 14.1 for both combined.[1] Hence the ratio ζ of occurrence for all cases of uniovular production of twins (*i.e.*, appearing as twins or as portion of triplets, &c.) :—

$$(553)\ldots\ldots\zeta = 0.0032484.$$

or, say, 13 cases in 4000, or 1 case in 308.

[1] In quadruplets there are 16 possible orders in which births may occur, and in these 24 possible cases of uniovulation. Since, however, the number of males and females are unequal—28 and 36—the possible cases have not occurred, and hence we may regard the 16 quadruplets and 3 quintuplets as roughly expressing the probable number of cases. Schrœder (Lehrbuch der Geburtshülfe, 10° aufl.) gives for twins 1 : 89, triplets 1 : 7910, quadruplets 1 : 371126.

9. **Small frequency of triovulation.**—The preceding analysis appears to shew that the triovular cases are only 162 in 12,013,134 confinements. The probability of triovulation, ζ', therefore, would appear to be :—

$$(554)\ldots\ldots\zeta' = 0.00001348.$$

or, say, 1 case in 74,000 confinements, though triplets occur at the rate of 1 case in 8068 confinements in the German Empire. This subject might well form the result of more definitive study when the data are adequate.

10. **Nuptial and ex-nuptial probability of twins according to age.**—The probability, in any nuptial or in any ex-nuptial confinement, of the occurrence of twins has been analysed from an aggregate of the Australian data from 1908 to 1914, both inclusive. It must, of course, be in substantial agreement with the result found for ϵ in § 8. Table CV., columns (ii.) and (vi.), give the number respectively of nuptial and ex-nuptial confinements (totals 897,618 and 54,913) occurring in Australia in 8 years, and the numbers of twins corresponding to each, viz., 9187 and 422. These are shewn by curves A and B, Fig. 83, the dots denoting the individual results for nuptial cases, and the firm lines the smoothed results; the values for the latter being given in column (v.) of the table. The ex-nuptial cases are denoted by circles, and where the numbers were small, the quinquennial aggregates only were graphed. The rate of increase per year of age up to age 37 is for nuptial and ex-nuptial cases respectively:

Frequency of Twins according to Age.

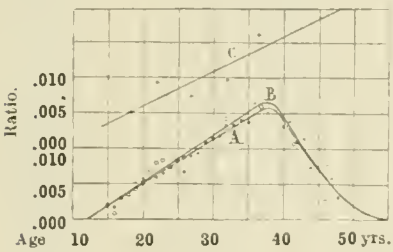

Fig. 83.

Curve A represents the ratio of the number of cases of at least two births to the number of nuptial confinements.

Curve B represents the same ratio for ex-nuptial confinements.

Curve C represents the number of cases of three or more at a birth to the number of cases of two or more.

$$(555)\ldots\ldots\epsilon_2 = 0.000632\ (x - 12)\ \text{and}\ \epsilon'_2 = 0.000668\ (x - 12)$$

x being the age of the mother. Beyond the age in question the results can be taken from the table. The ratios for all ages are—nuptial, 0.010234, and ex-nuptial, 0.00768. The general result is (i.) that *with increase of age* (and possibly duration of marriage) *the frequency of twins increases linearly*, till the end of the ordinary child-bearing period is approached, and (ii.) *this increase is slightly greater for ex-nuptial cases*, viz., about 5.7 per cent. greater. The ex-nuptial relative frequency of

twins for all ages combined is exactly 0.75 the nuptial relative frequency. Since in the ex-nuptial cases the confinements are probably on the whole not repeated, the result would appear to be due to age. This matter will be further considered later.

TABLE CV.—**Shewing Probability according to Age of the Occurrence of Nuptial and Ex-nuptial Twins, and of Triplets, based on 8 Years' Australian experience, 1907-1914.**

Age.	Nuptial Confinements.	Cases of Nuptial Twins.	Frequency of Nuptial Twins. Crude	Frequency of Nuptial Twins. Smo'thed.	Ex-Nuptial Confinements.	Cases of Ex-Nuptial Twins.	Frequency of Ex-nuptial Twins. Crude.	Frequency of Ex-nuptial Twins. Smo'thed.	All Twins.	All Triplets	Ratio of Triplets to Twins.
(i.)	(ii.)	(iii.)	(iv.)	(v.)	(vi.)	(viii.)	(vii.)	(ix.)	(x.)	(xi.)	(xii.)
12	0	0	0	.0000	5	..	..	.0000	..	..	.0030
13	4	0	0	.0006	21	..	..	.0007	..	..	.0035
14	30	0	..	.0013	126	0	.0000	.0013	..	..	.0039
..	**34**	..	..	..	**152**	..	..	..	..	..	..
15	170	0	..	.0019	537	1	.0019	.0020	..	..	.0044
16	1,138	2	.0018	.0025	1,500	1	.0007	.0027	..	..	.0049
17	3,962	12	.0030	.0032	2,980	9	.0030	.0033	..	..	.0054
18	9,761	36	.0037	.0038	4,504	16	.0036	.0040	..	..	.0058
19	18,071	94	.0052	.0044	5,317	23	.0043	.0047	..	..	.0063
..	**33,102**	**144**	..	..	**14,838**	**50**	**.0337**	..	**194**	**1**	**00.52**
20	25,159	147	.0058	.0051	5,272	27	.0051	.0053	..	1	.0068
21	35,326	202	.0057	.0057	5,008	33	.0066	.0060	..	3	.0072
22	43,353	254	.0059	.0063	4,231	34	.0080	.0067	..	1	.0077
23	50,322	329	.0065	.0069	3,848	32	.0083	.0073	..	6	.0082
24	53,175	392	.0074	.0076	3,182	24	.0075	.0080	..	3	.0086
..	**207,335**	**1,324**	..	..	**21,541**	**150**	**.0696**	..	**1,474**	**14**	**.0095**
25	54,259	452	.0083	.0082	2,548	21	.0082	.0087	..	1	.0091
26	55,006	434	.0079	.0088	2,161	19	.0087	.0094	..	3	.0096
27	53,735	487	.0091	.0095	1,785	25	..	.0100	..	8	.0101
28	53,244	506	.0095	.0101	1,699	23	..	.0107	..	5	.0105
29	49,200	538	.0109	.0107	1,410	8	..	.0114	..	2	.0110
..	**265,444**	**2,417**	..	..	**9,603**	**96**	**.0100**	..	**2,513**	**19**	**.0076**
30	47,980	548	.0114	.0114	1,356	14	..	.0120	..	7	.0115
31	40,199	485	.0121	.0120	851	13	..	.0127	..	4	.0119
32	41,528	559	.0135	.0126	056	11	..	.0134	..	8	.0124
33	37,426	505	.0135	.0133	812	15	..	.0140	..	3	.0129
34	34,362	488	.0142	.0139	779	15	..	.0147	..	4	.0133
..	**201,495**	**2,585**	..	..	**4,754**	**68**	**.0143**	..	**2,653**	**26**	**.0098**
35	31,349	436	.0140	.0145	688	17	..	.0154	..	10	.0138
36	29,399	488	.0166	.0152	636	9	..	.0160	..	12	.0143
37	26,213	414	.0158	.0158	544	7	..	.0167	..	5	.0148
38	24,664	377	.0153	.0161	555	8	..	.0168	..	4	.0152
39	20,790	324	.0156	.0156	436	5	..	.0163	..	3	.0157
..	**132,415**	**2,039**	..	..	**2,859**	**46**	**.0161**	..	**2,085**	**34**	**.0163**
40	17,023	226	.0133	.0143	383	6	..	.0150	..	6	.0162
41	12,252	171	.0140	.0127	201	3	..	.0134	..	2	.0166
42	11,012	123	.0112	.0112	205	0	..	.0118	..	2	.0171
43	7,457	85	.0114	.0099	155	1	..	.0104	..	0	.0176
44	4,746	36	.0076	.0086	85	1	..	.0090	..	1	?
..	**52,490**	**641**	..	..	**1,029**	**11**	**.0107**	..	**652**	**11**	**.0169**
45	2,755	21	.0076	.0074	58	0	..	.0077	..	..	?
46	1,389	10	.0072	.0062	36	1	..	.0064	..	..	?
47	684	4	.0060	.0051	17	0	..	.0052	..	..	?
48	310	1	.0032	.0041	12	0	..	.0042	..	..	?
49	106	0	..	.0031	7	0	..	.0032	..	..	?
..	**5,244**	**36**	..	..	**130**	**1**	**.0077**	..	**37**	..	**.0000**
50	34	..	..	.0022	5	..	..	.0023	..	..	?
51	12	..	..	.0016	1	..	..	.0016	..	..	?
52	6	1	.17	.0009	0	..	..	.0009	..	..	?
53	4	..	..	.0005	1	..	..	.0005	..	..	?
54	3	..	..	.0002	..	..	..	.0002	..	..	?
.. Not Stated	**59**	**1**	..	..	**7**	..	..	..	**1**	..	**.0000**
Totals	897,618	9,187	.01023	..	54,913	422	.00768	..	9,609	105	.01093

11. **Probability of triplets according to age.**—The results of the 8 years, 1907-14, gave the following results for nuptial and ex-nuptial twins and triplets, viz. :—

	Twins.			Triplets.		
	Nuptial.	Total.	Ex-nuptial.	Nuptial.	Total.	Ex-nuptial.
Numbers	9,187	(9,609)	422	98	(105)	7
Ratio ..	1.0000	..	.0459	1.000	..	0.071

The numbers are too small, however, to establish that the frequency of the occurrence of triplets ex-nuptially is between 50 and 60 per cent. greater than nuptially.

If the frequency be related to the number of twins, it is roughly given by the smoothed results in column (xii.) of Table CV. We shall call the probability P_3/P_2 say, τ_3. Thus we shall have :—

$$(556) \ldots \tau_3 = 0.0030 + 0.00047\,(x - 12); \text{ or } = 0.00047\,(x - 5.6)$$

the second form, however, being without meaning till the age of child-bearing. The firm line, curve C, on Fig. 83, denotes the increase; the crosses represent the group results used in deducing this.

12. **Probability of twins according to duration of marriage.**—Given a birth, the probability of a second child being born is found by dividing the number of twins, including triplets, by the number of confinements tabulated according to duration of marriage. Thus, column (v.) in Table CVI. is found by dividing the figures in column (iii.) by those in column (ii.). The crude results are shewn by the dots in Fig. 84, and the smoothed results by the firm line, curve A. For the form of the initial part of the curve see § 14, and also Fig. 85 hereinafter.

13 **Probability of triplets according to duration of marriage.**—The probability of a third child being born may, as before, be referred to the number of cases where a second child has been born. This probability is found by dividing the number of triplets by the number of twins, including the triplets, etc. But the numbers to be dealt with are so small and irregular that the expedient was adopted of forming groups of eleven. As no correction was applied for the systematic error of the grouping, the curve represents the ratio of 11-year groups of duration of marriage, the argument being the central years of the group. The results are shewn on Fig. 84, curve B, and the data are shewn in Table CVI., and seem to indicate the change with duration of marriage is sensibly a linear one through for the major part (presumably) of the child-bearing period.

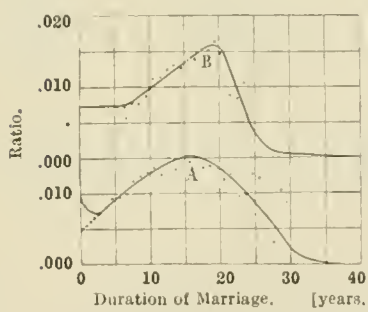

Fig. 84.

Curve A denotes the frequency of the birth of two or more children to the number of confinements.

Curve B denotes the ratio of 11-year means of the number of triplets to the number of cases of two or more children.

TABLE CVI.—Probability of Twins* and Triplets† according to Duration of Marriage. Australia, 1908-1914.

Duration of Marriage.	Confinements	Twins including Triplets.	Triplets.	Ratio of Twins to Confinements.		Ratio of Triplets to Twins (Groups of 11).	
				Crude.	Smoothed	Crude.	Smoothed
(i.)	(ii.)	(iii.)	(iv.)	(v.)	(vi.)	(vii.)	(viii.)
0–1	134,171	1,129	9	.0084	.0084	..	.0073
1–2	61,213	460	3	.0075	.0075	..	.0073
2–3	64,229	465	4	.0072	.0072	..	.0073
3–4	70,317	564	3	.0080	.0080	..	.0073
4–5	59,407	551	2	.0093	.0090	..	.0073
5–6	53,275	504	4	.0095	.0098	.0074	.0073
6–7	47,250	468	1	.0099	.0106	.0072	.0075
7–8	41,713	492	3	.0118	.0113	.0078	.0080
8–9	37,115	466	7	.0125	.0120	.0077	.0087
9–10	32,170	417	3	.0130	.0126	.0088	.0095
10–11	29,607	404	5	.0136	.0132	.0112	.0102
11–12	25,887	328	2	.0127	.0138	.0115	.0109
12–13	23,372	352	5	.0151	.0143	.0125	.0117
13–14	20,339	273	2	.0134	.0148	.0130	.0124
14–15	17,572	281	6	.0160	.0152	.0128	.0131
15–16	15,217	228	9	.0150	.0154	.0138	.0138
16–17	13,271	196	2	.0148	.0152	.0139	.0146
17–18	11,617	159	1	.0137	.0149	.0153	.0155
18–19	10,073	139	0	.0138	.0145	.0152	.0158
19–20	8,520	117	2	.0137	.0139	.0164	.0158
20–21	7,424	89	2	.0120	.0132	.0149	.0149
21–22	5,988	76	0	.0127	.0124	.0087	.0121
22–23	4,726	46	1	.0097	.0114	.0083	.0095
23–24	3,561	35	0	.0098	.0103	.0105	.0068
24–25	2,664	34	0	.0128	.0092	..	.0043
25–26	1,809	22	0	.0122	.0080	..	.0028
26–27	1,146	8	0	.0070	.0067	..	.0016
27–28	643	2	0	.0031	.0054	..	.0010
28–29	383	4	1	.0104	.0041	..	.0006
29–30	192	..	0	..	.0028	..	.0003
30–31	77	..	..	..	.0016	..	.0002
31–32	45	..	..	..	.0010	..	.0002
32–33	16	..	..	..	.0006	..	.0001
33–34	5	..	..	..	.0004	..	.0001
34–35	0	..	..	..	.0003	..	.0001
35–36	1	..	..	..	.0002	..	.0000
Totals	805,015	8,308	77	.010320	..	.00927	..

* That is, of two or more occurring at a birth. † That is, of third child in any case where two are born.

14. **Remarkable initial fluctuation in the frequency of twins, according to interval after marriage.**—There is no known ground for supposing that the ratio of the number of twins to the number of confinements in which they occur, can in any way depend on the interval after marriage, at least, if that interval be small. The results in Tables CVII. and CVIII.

hereunder for the years 1908 to 1915 and 1908 to 1914 respectively, shew, however, that apparently the dependence exists. The average for the first three months after marriage equals that of the third three months, and both are very much above the average. The second and fourth periods of three months are about equal. These results are shewn by curve C on Fig. 85.

TABLE CVII.—Shewing Variation in the Frequency of Twins during the First 24 Months after Marriage. Australia 1908-1915.

Year.	Twins Born during Interval after Marriage of Months					Confinements during Intervals after Marriage of Months					Ratio of Twins during Intervals after Marriage of Months				
	0–3	3–6	6–9	9–12	12–24	0–3	3–6	6–9	9–12	12–24	0–3	3–6	6–9	9–12	12–24
1908	16	24	34	56	60	1,533	3,152	4,006	7,007	6,298	.0104	.0076	.0085	.0080	.0095
1909	21	26	44	62	48	1,799	3,556	4,139	7,307	6,973	.0116	.0073	.0106	.0085	.0068
1910	19	29	58	59	48	1,888	3,659	4,474	7,500	6,919	.0101	.0079	.0129	.0079	.0069
1911	15	31	49	64	56	1,987	4,075	5,220	7,877	7,400	.0076	.0076	.0094	.0081	.0075
1912	27	32	60	61	60	2,119	4,458	5,827	8,899	8,518	.0127	.0072	.0103	.0069	.0071
1913	17	34	61	66	65	2,107	4,502	5,916	9,301	9,142	.0081	.0076	.0103	0071	.0071
1914	14	32	58	60	63	2,080	4,268	5,897	9,185	9,247	.0067	.0075	.0098	.0065	.0069
1915	28	46	51	76	82	2,023	4,149	5,828	8,795	8,953	.0099	.0111	.0088	.0086	.0091
Totals	157	254	415	504	482	15,536	31,819	41,307	65,871	63.450	.01010	.00798	.01005	.00765	.00760

Thus the proportion of twins for all pre-nuptial conceptions is high. It is to be noted, however, that the proportion of ex-nuptial twins over all is low (see Table CV.), and it is not unlikely that the initial high rate, and, in general, the higher rate for the cases due to pre-nuptial insemination is due to the transfer, owing to the *peithogamic* influence, of what might have been ex-nuptial to the nuptial cases. To obtain the fluctuation more exactly, the results were taken out monthly, from 1908 to 1914, according to interval after marriage.

TABLE CVIII.—Shewing Variations in the Frequency of Twins for each Interval of One Month after Marriage (First Births only), and of Triplets. Australia 1908-14.

Interval* ..	0–1	1–2	2–3	3–4	4–5	5–6	6–7	7–8	8–9	9–10	10–11	11–12	0–12
Twins ..	39	40	50	55	70	83	85	109	170	195	146	87	1,129
Confinements	3,529	4,059	5,925	7,455	9,055	11,160	13,870	11,545	10,064	24,434	19,047	13,595	183.738
Ratio ..	.0110	.0098	.0084	.0073	.0077	.0074	.0061	.0094	.0169	.0080	.0076	.0064	.00844

Interval† ..	1–2	2–3	3–4	4–7	7–11	11–26	1–26	0–26	Interval†	0–1	1–26
Twins ..	400	141	58	59	17	7	682	1,811	Triplets	8	6
Confinements‡	54,497	15,801	6,458	6,413	2,209	905	86,283	220,021	Twins	1,129	682
Ratio	.0073	.0089	.0091	.0092	.0077	.0078	.00790	.00823	Ratio	.0071	.0088

* Months. † Years. First births.

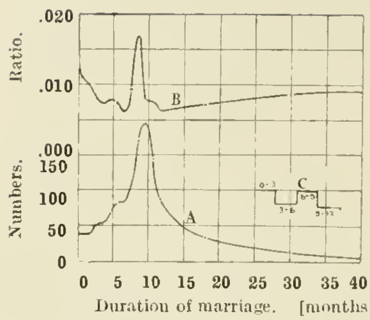

Fig. 85.

Curve A denotes the actual number of twins in Australia during 7 years' experience.

Curve B denotes the ratio of cases of births of 2 or more children to cases of confinement.

Curve C denotes, similarly to curve B, the group ratios for three months, however, instead of one.

The ratio for 1-4 is .0078, and for 5-26 is .0087. The numbers for the lesser subdivisions are doubtless too small to rely on the results. The results shewn are for *first births* only; but for the smaller durations the distinction is without meaning. Fig. 85 shews the results, curve A denoting the actual number of twin births, and curve B the frequency with which twins occur.

15. **Frequency of twins according to order of confinement.**—From the frequency of the occurrence of twins according to previous issue, an estimation according to order of confinement can be made by taking account of the probability of twins or triplets, &c. From the frequency according to previous issue, it may be deduced that the probability of twins is approximately as follows:—

Previous Confinements	0	1	2	3	4	5	6	7	8	9	10
Probability (about)	.0082	.0096	.0107	.0117	.0124	.0130	.0134	.0136	.0138	.0139	.0140

We have also from the general result that the frequency of single births, twins, and triplets in Australia was, for 1908-14,

$$\left.\begin{array}{ccccc} 799831 & : & 8247 & : & 77 \\ 1 & : & 0.010311 & : & 0.000096 \end{array}\right\} \quad \begin{array}{l} \text{or roughly, say,} \\ 10{,}000 : 100 : 1 \end{array}$$

The probability of twins occurring twice, ${}_2p_2$, is, therefore, approximately identical with that of the occurrence of triplets, p_3, that is:—

$$(557)\ldots\ldots{}_2p_2 = p_2^2 = p_3, \text{ approximately.}$$

The number entered under 0 will be correct. That is, the cases "according to previous issue," and "according to previous confinements" are identical. But in every case where there were twins or triplets, etc., at the first birth, the cases would be tabulated under "previous issue," 2 or 3, etc., respectively, instead of under 2; and similarly *mutatis mutandis* for all later columns in the "according-to-previous-issue"

tabulation. We therefore must add the appropriate numbers, and deduct equal numbers from later columns. The precision of the result will, of course, never be of a high order.

The data are given in the upper part of Table CIX., and the approximate restatement according to the order of confinement forms the lower part of the table.

TABLE CIX.—Frequency of Multiple Births according to Previous Issue. Australia 1908-14.

Previous Issue (upper table), or Order of Confinement (lower table.)

Numbers.		0	1	2	3	4	5	6	7	8	9
Cases of at least 2 children ..		1,811	1,357	1,325	1,094	834	591	477	306	227	127
Cases of at least 3 children ..		12	10	7	7	8	5	9	2	4	1
Mothers of at least 1 child		220,807	167,091	125,779	92,116	65,343	46,156	31,733	21,918	14,727	9,671
Ratio of twins to mothers		.00820	.008121	.01053	.01188	.01276	.01280	.01503	.01396	.01541	.01313
According to order of Confinement	Twins	1,811	1,386	1,337	1,096	831	590	467	302	218	122
	Mothers	220,807	169,851	126,377	92,083	65,099	45,683	31,253	21,467	14,287	9,254
	Corresponding Ratio	.00820	.00816	.01058	.01190	.01277	.01292	.01494	.01407	.01526	.01318
Ratio Triplets Smoothed ..		.000055	.000062	.000074	.000088	.000106	.000130	.000158	.000193	.000235	.000286

Numbers.		10	11	12	13	14	15	16	17	18	19	20	21	22
Cases of at least 2 children ..		79	39	21	9	6	2	1	0	0	0	1	0	1
Cases of at least 3 children ..		0	0	0	0	0	0	1	0	0	0	0	0	0
Mothers of at least 1 child		5,694	3,181	1,665	814	388	144	59	25	6	3	1	0	1
Ratio of twins to mothers		.01387	.01226	.01261	0.01388									
According to order of Confinement	Twins	74	37	19	8	5	2	1	0	0	0	1	0	1
	Mothers	5,378	2,964	1,530	740	340	127	52	21	5	2	0	1	1
	Corresponding Ratio	.01376	.01248	.01242	.01081	.01471	.01575	.01923						

Since the correction system affects the number of twins and the mothers in the same way, it obviously cannot produce any appreciable difference in the ratios, though it may alter the numbers. This is seen in the results given in the table above. If the number of triplets be smoothed, the result shown in the final line is obtained. But the numbers are too small to lead to any reliance upon their value, though they confirm in a general way the dictum that multiple fecundity increases with the issue, thus also with age and duration of marriage.

TABLE CX.—Shewing Secular Variation in the Frequency of Twins and Triplets. Australia, 1881-1915.

Year.	No. of Confinements (Total).*	Cases of 2 or more Children.	Cases of 3 or more Children.	Cases of 4 or more Children.	Ratio of Twins† to Confinements. Crude.	Ratio of Twins† to Confinements. Smoothed.	Ratio of Triplets to Twins, etc.‡ Crude.	Ratio of Triplets to Twins, etc.‡ Smoothed.
(i.)	(ii.)	(iii.)	(iv.)	(v.)	(vi.)	(vii.)	(viii.)	(ix.)
1881	63,818	545	7	0	.00854	.0080	..	..
2	64,069	496	3	0	.00774	.0082	..	..
3	68.135	575	2	0	.00843	.0084	..	..
4	72,832	629	8	0	.00863	.0086	..	..
5	76.026	661	3	0	.00869	.0087	.0063	.0063
6	79,009	682	2	0	.00863	.0088	.0056	.0056
7	83,085	704	4	0	.00847	.0090	.0054	.0056
8	86,393	875	6	1	.01012	.0096	.0066	.0062
9	87,195	859	3	0	.00985	.0099	.0068	.0070
1890	91,030	910	5	0	.00999	.0102	.0072	.0076
1	91,734	941	4	0	.01025	.0103	.0083	.0081
2	91,980	784	12	0	.01023	.0102	.0082	.0082
3	90,379	899	11	1	.00994	.0100	.0080	.0081
4	86,384	797	7	0	.00922	.0096	.0081	.0081
5	91,225	907	12	1	.00994	.0094	.0085	.0085
6	86,526	775	4	0	.00895	.0094	.0089	.0088
7	90,614	960	5	1	.01059	.0099	.0085	.0089
8	88,993	883	4	0	.00992	.0104	.0086	.0088
9	90,244	971	9	0	.01075	.0107	.0088	.0087
1900	92,057	985	7	0	.01069	.0108	.0084	.0086
1	92,826	1,005	11	0	.01082	.0107	.0089	.0088
2	92,852	972	12	0	.01046	.0104	.0088	.0092
3	89,060	877	10	1	.00984	.0102	.0098	.0095
4	93,973	1,005	9	0	.01069	.0104	.0093	.0097
5	95,060	1,012	11	0	.01064	.0107	.0099	.0099
6	97,867	1,083	5	0	.01106	.0107	.0100	.0100
7	100,161	951	13	0	.00949	.0102	.0099	.0099
8	110,491	1,065	6	0	.00963	.0098	.0100	.0097
9	112,921	1,142	14	0	.01011	.0100	.0096	.0096
1910	115,609	1,189	13	0	.01028	.0102	.0092	.0093
1	120,957	1,236	14	0	.01021	.0102	.0093	.0089
2	131,726	1,350	16	0	.01024	.0101	..	..
3	134,343	1,369	8	0	.01019	.0101	..	..
4	136,576	1,406	11	0	.01029	.0102	..	..
5	133,444	1,417	10	0	.01061	.0104	..	..
Tot'ls	3,221,594	32,917	281	5§	.010217	..	.00853	..

* That is, nuptial and ex-nuptial. † Including triplets and quadruplets. ‡ That is, the ratio of 9-year groups of triplets including quadruplets to 9-year groups of twins, including triplets. § Ratio of quadruplets to triplets = 0.018.

16. **Secular fluctuations in multiple-births.**—The ratio of multiple births to confinements would appear *a priori* to be independent of time, but it will be seen from Fig. 86 that there are indications of a definite secular fluctuation, see also Table CX. above. The number of confinements which constitutes the basis of the experience is more than doubled in the 35 years under review (see column ii.), and the number of twins (which includes triplets and quadruplets) is large. The aggregate experience includes 3,221,594 confinements, in which there were a total of 32,917 births of two or more children, a total of 281 births of 3 or more children, and 5 quadruplets. These give the ratios shewn in the table. In Fig. 85, curve A is the smoothed secular fluctuation-curve of the twins; curve B that of the triplets (which

Secular Fluctuation in Relative Frequency of Births and Twins and Triplets.

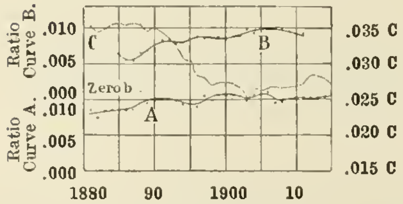

Fig. 86.

Curve A denotes the smoothed secular fluctuation of the ratio of births of two or more to the number of confinements.

Curve B denotes the ratio of 11-year groups of births of three or more to the number of births of two or more.

Curve C denotes the crude birth rate and number of births per unit of the general population.

were grouped in nines); while curve C shews the fluctuations of the crude birth-rates for the same years. The individual values are shewn by dots. It will be observed that on the whole the frequency of twins and triplets rises as the frequency of births diminishes.

17. **Comparison of nuptial and ex-nuptial fertility.**—In columns (x.) and (xvi.) of Table LXXIII., p. 242 hereinbefore, the crude and smoothed ratios for ex-nuptial fertility, attributed wholly to the "never married," were given. The crude results are repeated in column (ii.) of Table XCI. hereunder. If attributed to the "unmarried," which includes the widowed and divorced, the results in column (iii.) are obtained, and the corresponding smoothed results are shewn in column (iv.). Reference to the table shews that the maximum fertility is nuptially attained at about the year of age 18.3 to 19.3, and is about 0.484. The maximum fertility is ex-nuptially attained, however, only at about age 21.5 to 22.5, and is about 0.0182; that is to say, the maximum is about 3.2 years later, and the proportion at the maximum is only 0.0376, or say 3/80ths. For all ages from 12 to 57 we have for nuptial-fertility ratio 0.1704, and for the ex-nuptial ratio 0.00993. Hence the proportion of the averages is 0.05828. It is obvious that the initial parts of the curves representing the nuptial and ex-nuptial fertility-ratios are not likely to be identical, because the nuptial denominator for early ages will be small, and the ex-nuptial denominator will be large. Curves A and C, Fig. 87, denote respectively the nuptial and ex-nuptial curves. By the process indicated in § 2, p. 298, the results in columns (vi.) and (vii.) of Table CXI. are obtained; these are shewn in Fig. 87 by curve C; hence the curves are not in planar correspondence. If, however, the curve A be corrected for the effect of previous births, the two curves come into closer correspondence[1]; that is, ex-nuptial fertility has, in general, nearly the same characteristics as nuptial fertility, excepting that the greater measure of restraint operates to make the maximum occur later, and to enormously reduce the ratio.

Nuptial and Ex-nuptial Fertility-ratios.

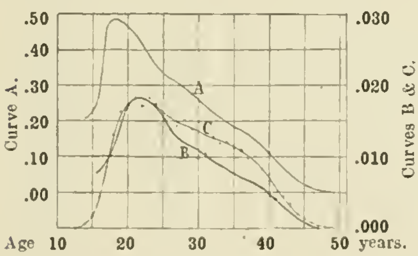

Fig. 87.

Curve A denotes the nuptial fertility ratio.

Curve B is the oblique projection of curve A.

Curve C is the ex-nuptial fertility ratio.

[1] It is obvious that the ex-nuptial curve does not need the same correction, since oft-repeated ex-nuptial maternity is not likely to occur.

TABLE CXI.—Comparison of Nuptial and Ex-nuptial Fertility-ratios according to Age. Australia 1907 to 1914.

Age of Mother.	Ratio of Ex-nuptial Births to— the "Never Married."	Ratio of Ex-nuptial Births to— the "Unmarried."		Ratio of Nuptial Births to the Married.	Ex-nuptial Rate Computed by Oblique Projection.*	
	Crude.	Crude.	Smoothed.	Smoothed.	Rate.	Age.
(i.)	(ii.)	(iii.)	(iv.)	(v.)	(vi.)	(vii.)
12	.0000	.0000	.0000	..	..	..
13	.0001	.0001	.0001	..	..	..
14	.0004	.0004	.0004	.207	.0077	15.4
15	.0016	.0016	.0016	.227	.0085	16.5
16	.0043	.0043	.0043	.301	.0113	18.0
17	.0085	.0085	.0085	.458	.0171	20.1
18	.0131	.0131	.0131	.483	.0181	21.3
19	.0162	.0162	.0158	.479	.0179	22.3
20	.0172	.0172	.0174	.464	.0174	23.2
21	.0181	.0181	.0181	.443	.0166	24.0
22	.0173	.0171	.0181	.416	.0156	24.9
23	.0183	.0182	.0177	.381	.0142	25.7
24	.0176	.0174	.0172	.352	.0132	26.4
25	.0163	.0161	.0163	.333	.0124	27.3
26	.0157	.0154	.0154	.319	.0119	28.1
27	.0147	.0143	.0149	.307	.0115	29.0
28	.0157	.0152	.0145	.293	.0110	30.0
29	.0145	.0139	.0141	.274	.0102	30.9
30	.0157	.0150	.0136	.256	.0096	31.7
31	.0111	.0104	.0131	.241	.0090	32.6
32	.0138	.0128	.0127	.225	.0084	33.5
33	.0131	.0119	.0123	.210	.0079	34.4
34	.0135	.0121	.0119	.197	.0079	35.3
35	.0129	0113	.0114	.185	.0069	36.3
36	.0127	.0109	.0108	.174	.0065	37.2
37	.0116	.0097	.0101	.164	.0061	38.1
38	.0125	.0101	.0093	.149	.0056	39.0
39	.0103	.0082	.0083	.130	.0049	39.9
40	.0097	.0074	.0070	.108	.0040	40.7
41	.0055	.0041	.0054	.087	.0033	41.6
42	.0060	.0043	.0042	.067	.0025	42.4
43	.0049	.0033	.0030	.050	.0019	43.3
44	.0029	.0019	.0020	.033	.0012	44.2
45	.0021	.0013	.0013	.020	.0007	45.1
46	.0014	.0008	.0008	.010	.0004	46.0
47	.0007	.0004	.0004	.005	.0002	47.0
48	.0005	.0003	.0003	.003	.0001	48
49	.0003	.0002	.0002	.001	.0000	49
50	.0003	.0001	.0001	.001	.0000	50
51	.0001	.0000	.0000	.000	.0000	51

* The oblique projection brings the maximum points into arbitrary agreement, the values for the ages indicated also being determined thereby. The rates for these ages are found from those of the nuptial curve by using the projection-ratio.

The difference between the nuptial and ex-nuptial probabilities of confinement are more comprehensively indicated by a decennial table. In Table CXII. hereunder these are given as the number of cases respectively occurring per 10,000 married and per 100,000 "never married" women. The rates, based upon the numbers of the "unmarried," are somewhat smaller.

TABLE CXII. Shewing the Probabilities of Nuptial and Ex-nuptial Confinement and their Ratio, for Five-Year Age-groups. Australia 1907-1914.

Age Groups.	No. of Married Women at Census 1911.	No. of Cases of Nuptial Confinement in 8 Years.	Probability of Maternity* during 1 Year.	No. of "Never Married" Women at Census 1911.	No. of Unmarried Women at Census 1911.	No. of Cases of Ex-nuptial Confinement	Probability of Maternity† during 1 Year.	Probability of Maternity during 1 Year.	Ratio of Probability of ex-nuptial to nuptial maternity. Based upon the Never Married.	Based upon the Un-Married.
11–14	19	34	2,226	168,778	168,778	152	11	11	.0005	.0005
15–19	8,637	33,245	4,791	214,875	214,905	14,889	862	862	.0180	.0180
20–24	65,506	208,667	3,962	152,967	153,514	21,695	1,765	1,759	.0445	.0444
25–29	109,832	267,886	3,036	78,036	79,918	9,696	1,546	1,510	.0509	.0497
30–34	112,532	204,093	2,257	44,341	47,903	4,822	1,353	1,253	.0600	.0555
35–39	104,825	134,481	1,597	29,953	35,888	2,909	1,208	1,009	.0757	.0632
40–44	94,917	53,143	697	21,483	30,325	1,040	602	427	.0865	.0613
45–49	82,263	5,280	80	15,006	27,172	131	108	60	.136	.075
50–54	60,939	60	1.2	9,734	23,463	7	9	3	.73	.025
55–60	38,905	4	.12	5,698	20.063	0	0	0	?	?

* Probability per annum per 10,000 married women of same age-group.
† Probability per annum per 100,000 "never married" women of same age-group.

18. **Theory of fertility, sterility and fecundity.**—The *fertility-ratio* or *probability of maternity in a unit of time* may be defined as the proportion of cases, which, subjected to a given degree of risk for a unit of time, result in maternity; and similarly, the *sterility ratio* or *probability of maternity* is the arithmetical complement of the probability; or calling these respectively p and q, $p + q = 1$. If instead of "a unit of time," we write "*various given periods of time*," we arrive at the conception of a varying *degree* of fertility or sterility, which for brevity, we may call the fertility, $\mathfrak{q}$, or the sterility, $\mathfrak{s}$. That is to say, instead of making a sharp *qualitative* cleavage between the fertile and the infertile or sterile, both are to be regarded as varying quantitatively. Any compilation shewing the frequency of cases of maternity according to duration of marriage reveals the propriety of this mode of envisaging the question. But we have seen that fertility decreases after a certain age, hence age must also be taken into account. Further, the "degree of risk" varies with the age of the husband. Hence, if x denote the age of the wife, y that of the husband, i the duration of the risk, we have :—

$$(558) \ldots\ldots \mathfrak{q} = f(x, y, i)\ ; \quad \text{and} \quad \mathfrak{s} = 1 - \mathfrak{q}$$

Fertility and sterility in the sense indicated are determined by the question of a single case of maternity. If instead of this we substitute "result in n cases of maternity," or "result in the bearing of n' children," we arrive at the quantitative conception of fecundity. It is not unlikely that the "degree of risk" varies with the number of previous births. If so, we must write (x, y, i, n) in this last equation.

If the total number of married women of age x be denoted by ${}_xM$, the duration of their marriage be denoted by a suffix i, the number of nulliparæ, primiparæ, and multiparæ up to n by the suffixes 1, 2, . . . n, then we can have compilations of the types

$$(559)\ldots\ldots {}_xM = {}_xM_0 + {}_xM_1 + \ldots {}_xM_i$$

$$(560)\ldots\ldots {}_xM = {}_xM'_0 + {}_xM'_1 + \ldots {}_xM'_n$$

that is, compilation according to age and duration of marriage, or according to age and "issue." It is at once evident that an exhaustive compilation according to x, y, i and n is out of the question, since the individual numbers in each "parcel" would be too small. Hence, serviceable tables must ignore some of the factors.

In some countries fertility probably varies but slightly with the age of the husband, and in all the distribution according to the age probably does not materially vary. Hence, by ignoring the issue, tables of "fertility and sterility" and of "fecundity" may take the following forms, the partial tables serving all general practical purposes :—

Tables of Fertility and Sterility (effect of "Previous Issue" being Ignored).

Arguments of complete tables.	Argument of partial tables.
(i.) Age of wife, with (ii.) age of husband. (iii.) Duration of marriage.	(i.) Age of wife only (*i.e.*, with husbands of all ages). (ii.) Duration of marriage.

The tables themselves should shew, for each combination of age and duration of marriage, the proportion of married women who have borne one child.

Tables of Fecundity (effect of "Previous Issue" being Ignored).

(i.), (ii.), and (iii.) as above.	(i.) and (ii.) as above.

The tables themselves should shew, for each combination of age and duration of marriage, the proportion of married women who have borne n children, where n is successively 0, 1, 2, 3, 4, etc., etc.

Such tables will need to be for small age-groups (say for single years), and for durations of marriage, which change by small amounts (say one year), inasmuch as the age and duration change together, and the effect of age is considerable.

19. **Past fecundity of an existing population.**—The past fecundity of any population as at a particular moment is given by a census, both according to "duration of existing marriage" and according to "age." The usual tabulation according to *existing* marriage ignores the fact that the record is incomplete, and that for deduction purposes a previous marriage may to some extent modify the fecundity. The results in Tables CXIII. and CXIV. hereunder are deduced from the Census tabulations by applying the method outlined in § 4, p. 300, to the crude results. The aggregates for the same "issue" are not, of course, in agreement since in the one case the numbers according to the issue from *existing marriages* are recorded, and in the other, the numbers according to age include all previous issue.

Numbers who bore 1, 2..n Children; also Proportion found to be Sterile.

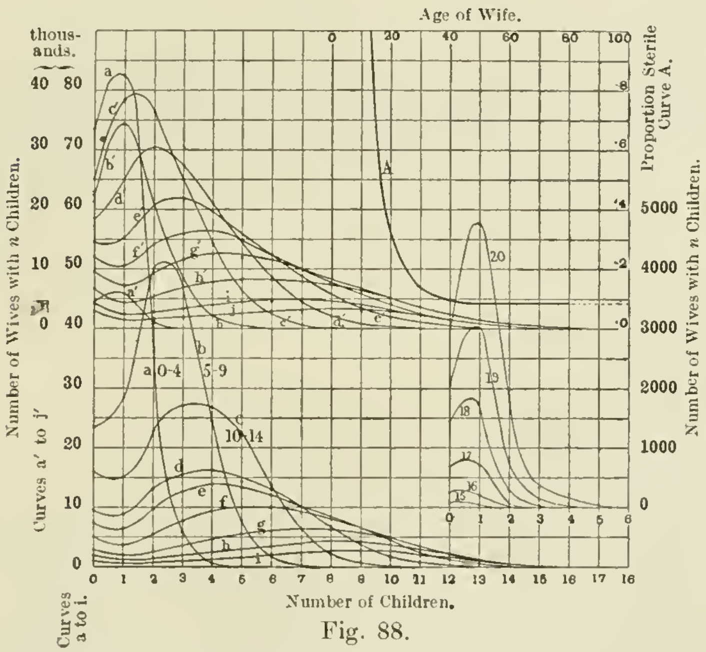

Fig. 88.

Curves a to i shew numbers who bore 0 to n children during durations of marriage 0 to 4, 5 to 9, 10 to 14, etc., see Table CXIII.

Curves a′ to j′ shew the numbers who bore 0 to n children according to age and without regard to duration of marriage; curve a′ denoting all under 20; curve b′ all aged 20 to 24 last birthday; curve c′ all aged 25 to 29, etc.; see Table CXIV.

Curves 15 to 20 shew numbers of wives who bore 0 to n children for ages 15 to 20 last birthday; see Table CXIV.

These curves are valid only for integral values of the abscissa (number of children).

Curve A shews the proportion of wives according to age, but of all durations of marriage, who proved sterile.

Table CXIII.—Shewing Issue of 1,000,000 Wives according to Duration of Existing

Duration Existing Marriage.	Number of Wives who had given Birth to Children to the Number of—									
	0.	1.	2.	3.	4.	5.	6.	7.	8.	9.
Under 5 years ..	73,765	82,436	37,904	5,874	469	23	..	..	..	..
5–9 years ..	23,504	28,564	50,165	47,053	24,421	7,800	1,776	344	87	7
10–14 ,, ..	16,031	15,059	22,961	27,141	26,897	22,421	13,774	6,325	2,265	718
15–19 ,, ..	9,586	8,821	13,150	15,427	16,200	14,542	13,072	10,191	6,814	3,767
20–24 ,, ..	7,374	6,465	9,714	12,603	13,916	13,276	12,066	10,253	8,336	6,602
25–29 ,, ..	5,082	3,806	5,450	7,701	9,413	10,078	10,095	9,162	8,043	6,656
30–34 ,, ..	2,947	2,036	2,669	3,566	4,684	5,581	5,977	6,336	6,223	5,734
35–39 ,, ..	1,904	1,212	1,436	1,921	2,600	3,080	3,478	3,864	4,354	4,101
40–44 ,, ..	1,055	600	778	948	1,206	1,582	2,024	2,330	2,511	2,618
45 and over ..	970	585	606	821	1,094	1,513	1,883	2,374	2,859	3,057
Totals for existing marriage ..	142,218	149,584	144,833	123,055	100,900	79,896	64,145	51,179	41,492	33,260
Total per million for all ages ..	123,995	146,153	145,107	124,239	103,088	82,140	67,029	53,803	44,026	35,392

* This does not include children by previous marriage, or ex-nuptial children ; it shews the relative frequency of issues of a given number according to " duration of marriage."

† The actual total number of wives was 733,773, of which 3747 gave no information either as to duration of marriage or as to number of children ; 12,073 gave no information as to number of children, but stated their age ; and 21,151 gave no information as to age, but stated the number of children. The 3747 were distributed proportionately to the partially specified totals, the two parts being 1362

Table CXIV.—Shewing Issue of 1,000,000 Wives according to Age, at

Age of Wives.	Number of Wives to whom had been Born Children to the Number of—										
	0	1.	2.	3.	4.	5.	6.	7.	8.	9.	10.
13	1	0	..	..	..	..	..	..	..	..	..
14	18	7	0	..	..	..	..	..	..	..	..
15	92	34	0	..	..	..	..	..	..	..	..
16	249	207	14	0	..	..	..	..	..	..	..
17	679	701	61	5	0	0	..	..	..	..	..
18	1,445	1,723	296	19	1	1	0	..	..	..	..
19	2,086	3,002	747	123	8	0	0	..	..	..	..
20	2,987	4,751	1,765	320	22	3	0	..	..	..	..
21-24	19,474	29,692	19,135	7,862	2,215	439	106	16	4	0	0
25-29	25,137	38,540	36,232	24,981	14,161	6,352	2,357	749	215	53	7
30-34	18,429	25,026	30,571	27,374	21,064	14,291	8,631	4,645	2,082	836	284
35-39	14,383	15,159	20,990	21,917	19,799	16,043	12,429	8,728	5,785	3,558	1,945
40-44	12,037	10,458	14,208	16,019	16,525	14,677	12,635	10,073	7,847	5,985	4,208
45-49	9,516	7,165	9,519	11,466	12,622	11,945	11,484	9,762	8,327	6,749	5,223
50-54	6,688	4,240	5,378	6,848	7,850	8,276	8,145	7,926	7,351	6,484	5,343
55-59	4,171	2,377	2,755	3,336	3,991	4,403	4,599	4,811	4,882	4,427	4,083
60-64	2,938	1,408	1,601	1,803	2,340	2,583	2,955	3,113	3,290	3,106	2,953
65-69	1,913	861	928	1,107	1,307	1,608	1,941	2,101	2,285	2,116	2,175
70-74	1,057	503	503	651	723	891	1,005	1,169	1,168	1,294	1,239
75-79	474	214	296	269	314	424	512	484	562	584	587
80-84	164	61	86	110	113	149	182	181	171	167	155
85-89	46	12	12	22	27	49	39	36	49	28	38
90-94	10	12	9	7	6	6	9	9	7	4	6
95-99	1	0	0	0	0	0	0	0	1	1	0
100-104	0	0	1	0	0	0	0	0	0	0	0
Totals	123,995	146,153	145,107	124,239	103,088	82,140	67,029	53,803	44,026	35,392	28,246
Totals per million for existing marriages	142,218	149,584	144,833	123,055	100,900	79,896	64,145	51,179	41,492	33,260	26,328

The actual total number of wives was 733,773, of which 343 gave no information as to age, or as to number of children ; 15,477 gave no information as to number of children, but stated their age ; 5432 gave no information as to age, but stated the number of children. The 343 were divided into two groups, viz., 254 and 89, these being distributed proportionately among the partially specified totals. The total additions thus become for the several ages and age-groups : 0, 0, 0, 5, 11, 28, 70, 167, 1228,

Marriage* at Census of 3rd April, 1911, Australia (Based upon 733,773 Wives.)†

Number of Wives who had given Birth to Children to the Number of—												Totals.
10.	11.	12.	13.	14.	15.	16.	17.	18.	19.	20.	21 and over.	
..	..	..	..	..	..	..	..	..	..	..	..	200,471
1	..	..	..	..	..	..	..	..	..	..	..	183,722
175	63	10	6	..	..	..	..	..	..	..	..	153,846
1,741	726	277	99	29	9	1	..	..	..	..	..	114,452
4,530	2,364	1,295	606	250	115	39	10	6	..	1	..	109,821
5,172	3,455	2,145	1,199	592	282	157	50	18	3	6	6	88,571
4,845	3,313	2,223	1,301	668	317	145	67	29	6	4	6	58,677
3,722	2,789	2,211	1,237	672	319	171	67	29	23	9	14	39,213
2,675	2,077	1,560	978	514	287	141	59	23	4	1	10	23,981
3,467	2,676	2,319	1,433	788	421	202	88	44	23	1	22	27,246
26,328	17,463	12,040	6,859	3,513	1,750	856	341	149	59	22	58	1,000,000
28,246	18,826	13,035	7,488	3,834	1,927	941	379	182	68	36	66	1,000,000

and 2385. The increased numbers thus become :—For Age-groups as indicated in table, 3358, 2903, 2074, 1371, 1210, 943, 591, 439, 230, 316; in all 13,435. For numbers of children as indicated in table, 2796, 2569, 2816, 2614, 2373, 2142, 1828, 1521, 1298, 1076, 876, 604, 493, 288, 125, 62, 24, 17, 6, 2, 1, and 5; in all 23,536. These aggregates of unspecified and partially specified were then distributed proportionately to the original numbers, see Vol. III., p. 1140-1, Census Report.

Census of 3rd April, 1911, Australia. (Based upon 733,773 Wives.)

Number of Wives to whom had been Born Children to the Number of—											Totals.
11.	12.	13.	14.	15.	16.	17.	18.	19.	20.	21 and over.	
..	..	..	..	..	..	..	..	..	..	..	1
..	..	..	..	..	..	..	..	..	..	..	25
..	..	..	..	..	..	..	..	..	..	..	126
..	..	..	..	..	..	..	..	..	..	..	470
..	..	..	..	..	..	..	..	..	..	..	1,446
..	..	..	..	..	..	..	..	..	..	..	3,485
..	..	..	..	..	..	..	..	..	..	..	5,966
..	..	..	..	..	..	..	..	..	..	..	9,848
0	0	..	..	..	..	..	..	..	..	..	78,943
5	3	0	0	0	0	..	..	..	..	..	148,792
97	31	7	3	3	1	0	..	..	..	..	153,375
907	380	160	67	16	3	1	0	0	0	0	142,270
2,437	1,443	766	328	178	78	20	13	0	1	3	129,939
3,458	2,241	1,287	685	327	156	73	26	8	8	5	112,052
3,699	2,611	1,482	754	356	196	67	33	8	8	13	83,756
3,021	2,166	1,313	669	335	171	74	36	21	7	8	51,656
2,205	1,759	1,000	548	298	144	70	33	6	7	12	34,172
1,592	1,305	762	410	236	113	51	21	11	4	8	22,855
904	656	463	238	114	58	13	13	8	..	11	12,681
357	347	194	105	53	14	4	7	6	1	1	5,809
116	76	47	22	10	4	6	..	..	..	3	1,823
24	16	6	4	1	3	0	..	..	..	1	413
4	1	1	0	0	..	..	..	..	..	1	92
0	0	0	1	0	..	..	..	..	..	0	4
0	0	0	0	..	..	..	..	..	..	..	1
18,826	13,035	7,488	3,834	1,927	941	379	182	68	36	66	1,000,000
17,463	12,040	6,859	3,513	1,750	856	341	149	59	22	58	1,000,000

2252, 2203, 2071, 1993, 1677, 1431, 888, 648, 476, 316, 157, 81, 23, 4, 0, 2; in all, 15,731; and for the numbers of children as indicated in the table, 591, 717, 730, 693, 602, 490, 415, 309, 277, 218, 181, 128, 76, 51, 18, 12, 7, 2, 3, 0, 0, 1; in all 5521. These aggregates for the unspecified, together with the partially-specified, were then distributed proportionately to the original numbers; see Vol. VIII. Census Report, pp. 1366-7.

The results given in Table CXIII. are shewn by curves (a) to (i) in Fig. 88 ; and those in Table CXIV. are shewn by the curves (a') to (j') in the same figure, the single year results of the latter table being marked 15, 16, ... 20. Interpolated curves would give the results for any other 5-year age or duration ranges.[1]

The curves of frequency of cases, according to number of issue, for the 5-year, or for the single-year age-groups, are of the same type, and are essentially dimorphic : strictly they give values only for integral values of the variable.[2] Thus they could no doubt be fairly well represented by curves of the type :—

$$(561)\ldots\ldots y = Ae^{-ax} + Bx^{b+cx}$$

in which x has the values 0, 1, 2, 3, ... etc.

20. **Fecundity during a given year.**—A different type of compilation is necessary to reveal what may be called the "*existing fecundity.*" The existing nuptial fecundity is shewn by the number of married women in each age-group, the number who failed to bear a child during the year, and the number who bore the nth child where $n = 1, 2, 3,$ . . . . etc. This is deduced from two sources, viz., (i.) from the Census record for the numbers of married women ; and (ii.) from the records of one year or for a series of years (1908-1914). The grand total of those who bore a child during the whole period of 7 years, if divided by 7.0666, gave a result substantially identical with that for the year 1911, which may be regarded as satisfactory.[3] This is seen from the close agreement of the numbers in the two upper portions of Table CXV. It is evident, therefore, that the vital statistics results for the Census year represent fairly satisfactorily the general case, and a 3 or 5-year result with the Census year as middle year would ordinarily be quite satisfactory.

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[1] It is clearly desirable that Census results should be compiled for single years, as soon as public appreciation of the value of a correct statement of age leads to accuracy.

[2] Statistical results furnish a number of examples of this character : for example the numbers of families living in houses with 1, 2, 3, . . . n rooms, etc.

[3] If the rate of change of the proportion married be supposed linear, the married female population at the Census is to the aggregate of married females as 1 : 7.1272. The ratio of the number of brides is 1 : 6.9473. The ratio of females is 1 : 7.1077, and of population 1 : 7.1150. It is obvious, therefore, that the ratio 7.0666 is very nearly correct.

TABLE CXV.—Shewing for various Age-groups and for all Durations of Marriage the Number who, during the year, bore the nth Child, where n = 0 to 10; and the Total of those who bore a Child later than the 10th. Australia, 1911 and 1908-1914.

Age of Mothers	Total Married Women *	No. who Bore a Child during the Year.†	No. who Bore no Child during the Year.	1st	2nd	3rd	4th	5th	Later than 5th.	6th	7th	8th	9th	10th	Later than 10th.	Order not Specified.
				NUMBER FOR WHICH THE CHILD BORN WAS THE—												
–19	8,716	4,146	‖4,570	3,456	619	53	4	0	0	0	0	0	0	0	0	14
20–24	65,959	25,957	40,002	13,039	7,717	3,642	1,085	246	62	50	8	4	0	0	0	166
25–29	110,591	33,817	76,774	9,271	8,672	7,109	4,727	2,419	1,554	1,093	336	86	29	8	2	65
30–34	113,310	25,682	87,628	3,632	4,327	4,522	4,328	3,501	5,342	2,527	1,565	745	317	131	57	30
35–40	105,550	16,839	88,711	1,279	1,539	1,997	2,277	2,243	7,476	2,143	1,848	1,383	970	591	541	28
40–44	95,573	6,763	88,810	303	316	405	531	722	4,479	740	777	771	706	607	878	7
45–	82,933	713	82,220	20	24	29	36	40	561	48	64	70	72	86	221	3
Totals	582,632	113,917	468,715	31,000	23,214	17,757	12,988	9,171	19,474	6,601	4,598	3,059	2,094	1,423	‖1,699	313

NUMBERS CORRESPONDING TO THE ABOVE BASED UPON THE TOTALS FOR THE PERIOD 1908-1914.

Age of Mothers	Total Married Women	No. who Bore a Child	No. who Bore no Child	1st	2nd	3rd	4th	5th	Later than 5th.	6th	7th	8th	9th	10th	Later than 10th.	Order not Specified.
–19	*8,716	§4,156	‖4,560	3,410	676	66	4	0	0	0	0	0	0	0	0	
20–24	65,959	26 277	39,682	13,248	8,043	3,578	1,102	246	60	48	10	2	0	0	0	
25–29	110,591	33,831	76,760	9,317	8,703	7,065	4,748	2,468	1,530	1,041	348	101	29	8	3	
30–34	113,310	25,639	87,671	3,592	4,317	4,624	4,281	3,504	5,321	2,523	1,529	757	326	123	63	
35–40	105,550	16,742	88,808	1,259	1,490	1,963	2,274	2,293	7,463	2,130	1,865	1,416	968	580	504	
40–44	95,573	6,609	88,964	288	312	418	547	666	4378	717	746	749	677	579	910	
45–	82,933	663	82,270	22	19	21	33	36	532	48	60	65	76	74	209	
Totals	582,632	113,917 ‡	468,715	31,136	23,560	17,735	12,989	9,213	19,284	6,507	4,558	3,090	2,076	1,364	1,689	

PROPORTIONS TO TOTALS OF SAME AGE; 1911 RESULTS.

Age of Mothers	Total Married Women	No. who Bore a Child	No. who Bore no Child	1st	2nd	3rd	4th	5th	Later than 5th.	6th	7th	8th	9th	10th	Later than 10th.	Order not Specified.
–19	100,000	47,568	52,432	39,651	7,102	618	46	0	0	0	0	0	0	0	0	161
20–24	100,000	39,353	60,647	19,768	11,700	5,522	1,645	373	94	76	12	6	6	0	6	251
25–29	100,000	30,579	69,421	8,384	7,842	6,428	4,274	2,187	1,405	988	304	78	26	7	2	59
30–34	100,000	22,665	77,335	3,205	3,819	3,991	3,820	3,090	4,714	2,230	1,381	657	280	116	50	26
35–40	100,000	15,953	84,047	1,212	1,458	1,892	2,157	2,125	7,083	2,030	1,751	1,310	919	560	513	26
40–44	100,000	7,076	92,924	317	331	424	556	755	4,686	774	813	807	739	635	918	7
45–	100,000	860	99,140	24	29	35	44	48	676	58	77	84	87	104	266	4

PROPORTIONS TO TOTALS OF SAME AGE; BASED UPON THE TOTALS FOR THE PERIOD 1908–1914.

Age of Mothers	Total Married Women	No. who Bore a Child	No. who Bore no Child	1st	2nd	3rd	4th	5th	Later than 5th.	6th	7th	8th	9th	10th	Later than 10th.	Order not Specified.
–19	100,000	47,682	52,318	39,123	7,756	757	46	0	0	0	0	0	0	0	0	
20–24	100,000	39,838	60,162	20,085	12,194	5,425	1,670	373	91	73	15	3	0	0	0	
25–29	100,000	30,591	69,409	8,425	7,870	6,388	4,293	2,232	1,383	941	315	91	26	7	3	
30–34	100,000	22,627	77,373	3,170	3,810	4,081	3,778	3,092	4,696	2,226	1,349	668	288	109	56	
35–39	100,000	15,862	84,139	1,193	1,412	1,860	2,154	2,172	7,071	2,018	1,767	1,342	917	550	477	
40–44	100,000	6,915	93,085	302	326	437	572	697	4,581	750	781	784	708	606	952	
45–	100,000	799	99,160	27	23	25	40	43	641	58	72	78	02	89	252	

* Adjusted numbers, see Census Report, Vol. II., p. 19, and also Vol. III., pp. 1136-7. The numbers given are the Census numbers adjusted and multiplied by a factor to make them agree with the mean female population of the year. † In cases where a woman bore twice in the same year, she has been counted twice. The results in this column are obtained from the vital statistics of the year 1911. ‡ The actual figures throughout have been multiplied by a factor (viz., 0.141509 = 1 ÷ 7.0666), so as to make this total, 113,917, to agree with the total above; hence, if the distribution for 1911 were identical with that of the seven-year period 1908-1914, the figures in the several columns would be identical. They are approximately so. § The whole of the numbers in the column are those for 1908-1914, multiplied by 0.141509. ‖ These numbers are obtained by subtracting the totals of those who bore children from the total number of married women.

TABLE CXVI.—Shewing the Number of Married Women at each Age, the Number of Cases of Maternity, and the Number for all Durations of Marriage, who had not given Birth to a Child. Australia 1907-1914.

Age last Birthday.	No. of Married Women at Census 1911.†	Average Annual No. Cases of Maternity, 1907-14.	No. of Wives at Census 1911 who had given Birth to no Children.†	Range in Years of Durations of Marriage. (up to)	Proportion of Married Women who had not given Birth to a Child. Crude.	Smoothed.	Age last Birthday.	No. of Married Women at Census 1911.	No. of Wives at Census 1911 who had given Birth to no Children.†	Proportion of Married Women who had not given Birth to a Child. Crude.	Smoothed.
(i.)	(ii.)	(iii.)	(iv.)	(v.)	(vi.)	(vii.)	(i.)	(ii.)	(iv.)	(vi.)	(vii.)
13	1	0.5	1	1*	1.0000	1.0000	..	..	..	..	..
14	18	3.7	13	2	.7222	.8140	..	..	..	..	..
	19	**4.2**	**14**	..	**.7368**	..	..	..	..	..	..
15	93	21.2	67	3	.7204	.6530	55	9,468	769	..	.0817
16	349	141.9	183	4	.5244	.5330	56	8,557	678	..	.0815
17	1,145	494.7	498	5	.4349	.4450	57	7,675	581	..	.0814
18	2,551	1,219	1,061	6	.4159	.3820	58	6,912	531	..	.0813
19	4,499	2,261	1,531	7	.3403	.3403	59	6,293	501	..	.0814
	8,637	**4,137.8**	**3,340**	..	**.3867**	..	..	**38,905**	**3,060**	**.0788**	..
20	6,933	3,150	2,192	8	.3162	.3075	60	5,746	479	..	.0815
21	10,100	4,423	2,772	9	.2744	.2815	61	5,277	458	..	.0816
22	13,047	5,428	3,422	10	.2622	.2580	62	4,871	435	..	.0820
23	16,521	6,306	3,973	11	.2405	.2365	63	4,505	412	..	.0823
24	18,905	6,669	4,123	12	.2181	.2165	64	4,161	382	..	.0827
	65,506	**25,976**	**16,482**	..	**.2516**	..	..	**24,560**	**2,166**	**.0882**	..
25	20,683	6,811	4,123	13*	.1993	.1990	65	3,829	353	..	.0837
26	21,620	6,903	3,958	14	.1831	.1825	66	3,502	319	..	.0842
27	22,180	6,751	3,678	15	.1658	.1670	67	3,194	283	..	.0848
28	22,584	6,691	3,448	16	.1527	.1524	68	2,880	247	..	.0854
29	22,765	6,192	3,238	17	.1422	.1424	69	2,621	211	..	.0861
	109,832	**33,348**	**18,445**	..	**.1679**	..	..	**16,026**	**1,413**	**.0882**	..
30	22,784	6,042	3,034	18*	.1332	.1339	70	2,365	190	..	.0868
31	22,726	5,065	2,849	19	.1254	.1266	71	2,099	168	..	.0876
32	22,542	5,240	2,684	20	.1191	.1203	72	1,867	146	..	.0885
33	22,421	4,722	2,540	21	.1133	.1147	73	1,652	129	..	.0896
34	22,059	4,338	2,416	22	.1095	.1101	74	1,444	115	..	.0908
	112,532	**25,407**	**13,523**	..	**.1202**	..	..	**9,427**	**748**	**.0793**	..
35	21,700	3,958	2,299	23	.1059	.1062	75	1,224	96	..	.0921
36	21,350	3,721	2,195	24	.1028	.1029	76	1,004	82	..	.0934
37	21,000	3,315	2,101	25	.1000	.1000	77	818	70	..	Uncertain
38	20,560	3,118	2,017	26	.0981	.0979	78	650	59	..	
39	20,215	2,629	1,942	27	.0961	.0959	79	510	48	..	
	104,825	**16,741**	**10,554**	..	**.1007**	..	..	**4,208**	**355**	**.0844**	
40	19,851	2,148	1,880	28	.0947	.0942	80	397	38	..	
41	19,457	1,548	1,823	29	.0936	.0927	81	317	30	..	
42	19,026	1,386	1,766	30	.0928	.0913	82	241	23	..	
43	18,543	939	1,710	31	.0922	.0900	83	184	17	..	
44	18,040	595	1,653	32	.0916	.0888	84	140	13	..	
	94,917	**6,616**	**8,832**	..	**.0930**	..	..	**1,279**	**121**	**.0946**	
45	17,554	346	1,577	33	.0898	.0877	85	105	10	..	
46	17,064	174.2	1,494	34	.0876	.0868	86	80	8	..	
47	16,554	85.6	1,403	35	.0847	.0860	87	56	6	..	
48	15,975	38.7	1,306	36	.0817	.0852	88	35	5	..	
49	15,216	13.2	1,203	37	.0791	.0845	89	24	4	..	
	82,363	**657.7**	**6,983**	..	**.0848**	..	..	**300**	**33**	**1.100**	
50	14,303	4.2	1,116	38	.0780	.0837	90	20	3	..	
51	13,162	1.5	1,049	39	.0797	.0832	91	16	2	..	
52	12,088	0.9	981	40	.0812	.0827	92	12	1	..	
53	11,100	0.6	914	41	.0823	.0823	93	9	0	..	
54	10,286	0.1	847	42	.0823	.0819	94	7	0	..	
	60,939	**7.3**	**4,907**	..	**.0805**	..	..	**64**	**6**	**.0937**	
							95-100	21	0		

* Actually extends to about 1 year greater than shewn. † Graduated.

21. **Number of married women without children, all durations of marriage.**—The relative numbers of married women of each age, and for all durations of marriage, who are without children, are readily determinable by means of a Census. That for 1911 gave the results shewn in Table CXVI. above. The smoothed results in column (vii.) of the table are shewn by curve A on Fig. 88. The ratio very rapidly falls to the value of about one-fourth, which is attained during age 22; one-eighth

is reached during age 31 ; one-tenth during age 37 ; and the minimum during age 58, which age is, of course, somewhat uncertain. After the age of that minimum the results are very uncertain. Apparently the curve will require several terms of the type Ae^{-ax} to empirically represent it, thus the ratio being denoted by σ, and the age ξ being reckoned from say 12 or 13, the ratio will be of the form :—

$$(562)\ldots\ldots\sigma = A + Be^{-b\xi}\ C + e^{-c\xi} + \ldots . + K\xi^{k}$$

22. **Sterility-ratios according to age and duration of marriage.**—The effect of the age of the husband being ignored, the number of cases of sterility, (or more strictly of childlessness,)[1] according to duration of marriage, for women of different ages in Australia was found from the Census of 1911 to be as shewn in the following table :—

[1] Physiological sterility is the condition, not merely of childlessness, but of childlessness due either to failure to conceive, or to retain the fertilised ovum the full time. The data of ordinary statistics cannot conclusively establish the frequency of physiological sterility, since what is given are merely measures of childlessness.

A number of instances are given in the "Handbuch der Medizinischen Statistik," by Friedrich Prinzing, Dr. Med., 1906, Cap. III. ; "Die sterilen und kinderlosen Ehen," pp. 30-40.

The following estimations of sterility may be mentioned :—*

Authority.	No. of Marriages under Observation.	Elapsed Period after Marriage.	No. of Sterile Cases.	Ratio.
Dresden Returns ..	27,911	5 years	672	0.02407
Dresden Returns ..	27,911	10 years & more	134	0.00480
Austerlitz, Prag, 1891-1900	3,920	Not stated	295	0.0753
Hofmeier	2,220	Not stated	?	0.147
Lier and Ascher	2,500	Not stated	?	0.090
Huizinga (Groningen) ..	1,180	Not stated	?	0.115
Verrijn Stuart, Netherlands	9,443	16 to 21 years	?	0.131
Do., poorer classes ..	?	Not stated	?	†0.141 ‡0.110
Do., middle classes ..	?	Not stated	?	†0.162 ‡0.109
Do., well-to-do classes	?	Not stated	?	†0.160 ‡0.126

† Town. ‡ Country.

*Other results are :—Spencer, Wells & Sims (Great Britain), 0.125 ; Duncan (Glasgow and Edinburgh), 0.163 ; Ansell, 1919 cases, Married Women, 0.079 ; A Swedish County, 0.100 ; Massachusetts, 1885, 0.176 ; Women over 50, 0.119.

The whole of the above statements are, of course, defective, inasmuch as sterility is a function both of duration of marriage as well as of age, etc.

Table CXVII.—Sterility according to Age and Duration of Existing Marriage. Australia, 3rd April, 1911 (Census).

Age of Wives at Time of Census	DURATION OF EXISTING MARRIAGE. Under 5 Years. Childless Mothers.	Total Mothers.	Sterility Rate.	5 to 10 Years. Childless Mothers.	Total Mothers.	Sterility Rate.	10 to 15 Years. Childless Mothers.	Total Mothers.	Sterility Rate.	15 to 20 Years. Childless Mothers.	Total Mothers.	Sterility Rate.	20 to 25 Years. Childless Mothers.	Total Mothers.	Sterility Rate.
Under *14	1	1	1.000	..	..	..	..	..	..	..	..	..	..	..	..
*14	13	18	.722	..	..	..	..	..	..	..	..	..	..	..	..
*15	67	92	.728	..	..	..	..	..	..	..	..	..	..	..	..
*16	179	338	.530	..	..	..	..	..	..	..	..	..	..	..	..
*17	490	1,044	.469	..	..	..	..	..	..	..	..	..	..	..	..
*18	1,042	2,512	.415	..	..	..	..	..	..	..	..	..	..	..	..
*19	1,496	4,270	.350	..	..	..	..	..	..	..	..	..	..	..	..
*20	2,114	6,693	.316	7	261	.027	..	..	..	..	..	..	..	..	..
21–24	13,378	43,424	.308	474	11,926	.040	..	..	..	..	..	..	..	..	..
25–29	14,724	45,673	.322	3,004	47,789	.063	346	10,587	.033	1	21	.048	..	..	..
30–34	7,398	19,739	.375	3,998	38,675	.103	2,079	40,121	.052	262	8,594	.030	2	21	.095
35–39	4,099	8,118	.505	2,889	16,992	.170	2,693	32,715	.082	1,348	31,792	.042	279	9,324	.030
40–44	2,597	3,575	.726	2,120	6,731	.315	2,096	14,568	.144	1,723	24,408	.071	1,250	32,477	.038
45–49	1,753	1,869	.938	1,712	2,749	.623	1,517	5,280	.287	1,199	9,253	.130	1,365	22,758	.060
50–54	893	894	.999	1,108	1,187	.933	1,089	1,865	.584	803	3,230	.249	803	7,780	.103
55–59	431	431	1.000	530	531	.998	701	779	.900	566	1,088	520	530	2,265	.234
60–64	247	247	1.000	255	255	1.000	332	332	1.000	420	447	.940	411	791	.520
65–69	140	140	1.000	117	117	1.000	173	173	1.000	198	199	.995	305	337	.905
70–74	64	64	1.000	74	74	1.000	91	91	1.000	100	100	1.000	127	128	.992
75–79	20	20	1.000	28	28	1.000	37	37	1.000	38	38	1.000	53	53	1.000
80–84	3	3	1.000	7	7	1.000	11	11	1.000	4	4	1.000	17	17	1.000
85–89	1	1	1.000	2	2	1.000	3	3	1.000	2	2	1.000	4	4	1.000

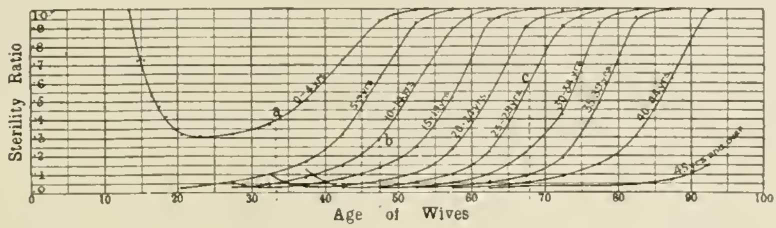
Age of Wives at Time of Census	DURATION OF EXISTING MARRIAGE. 25 to 30 Years. Childless Mothers.	Total Mothers.	Sterility Rate.	30 to 35 Years. Childless Mothers.	Total Mothers.	Sterility Rate.	35 to 40 Years. Childless Mothers.	Total Mothers.	Sterility Rate.	40 to 45 Years. Childless Mothers.	Total Mothers.	Sterility Rate.	Over 45 Years. Childless Mothers.	Total Mothers.	Sterility Rate.
35–39	1	8	.125	..	..	..	..	..	..	..	..	..	..	..	..
40–44	221	8,075	.027	..	..	..	..	..	..	..	..	..	..	..	..
45–49	1,005	28,666	.035	165	6,556	.025	..	..	..	..	..	..	..	..	..
50–54	955	17,237	.055	663	20,004	.033	121	5,081	.024	..	..	..	..	..	..
55–59	456	4.762	.096	511	9,574	.053	432	12,606	.034	91	3,087	.029	..	..	..
60–64	346	1,590	.218	284	2,946	.096	388	6,437	.060	250	7,789	.032	59	2,209	.027
65–69	292	522	.559	213	978	.218	189	2,120	.089	226	4,083	.055	214	6,630	.032
70–74	188	206	.913	125	295	.424	106	569	.186	101	1,150	.088	221	5,660	.039
75–79	72	72	1.000	68	73	.932	59	121	.488	46	305	.151	116	2,994	.039
80–84	15	15	1.000	23	23	1.000	25	26	.962	18	57	.316	46	945	.049
85–89	2	2	1.000	5	5	1.000	9	9	1.000	5	8	.625	14	200	.070
90–94	..	..	..	..	..	..	1	1	1.000	1	1	1.000	6	41	.146
95–99	..	..	..	1	1	1.000	..	..	..	..	..	..	..	..	..

* The results are from Census Report III., p. 1136. The general results are obtained from an unpublished series of compilations according to age-groups, and duration-of-marriage groups. In neither case were the "unspecified" distributed; such distribution, however, can affect the results only very slightly.

An examination of the results given in the table shews that initially the sterility-ratio decreases; it attains a minimum, and then increases; see particularly the duration of marriage 0 to 4 years (*i.e.*, under 5 years). The initial fall may be regarded as the normal decrease of childlessness with increase of the duration of the risk. From the minimum onward, however, the curve shews the true measure of sterility for a given duration of marriage, and for any age terminating the given duration of marriage.

The curves on Fig. 89 are the *sterility-ratios according to age*, each curve denoting a separate range of duration of marriage. By projection[1] Fig. 90, shewing the *curves of equal sterility*, is derived. From these, the correlative durations of marriage and ages, corresponding to any degree of sterility, can be at once seen. The dots give the positions as determined from the data,[2] the curves throughout are smoothed.

Fig. 89.



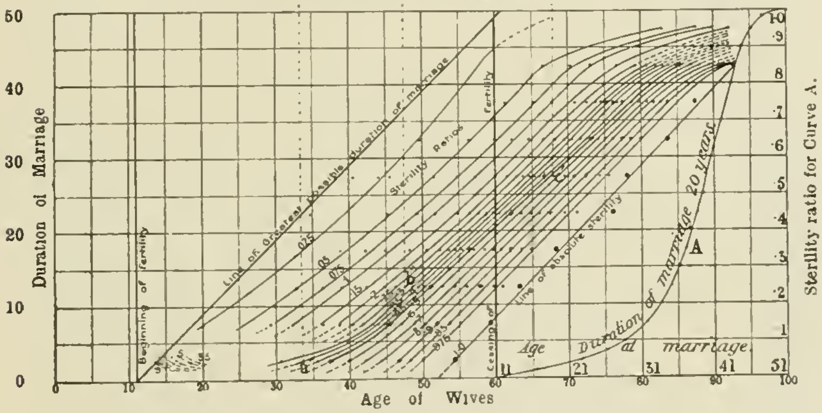


Fig. 90.

In Fig. 89 the ordinates to the curves denote the degrees of sterility: the abscissæ denote the age corresponding to the duration of marriage shewn on any curve in question.

In Fig. 90, the intersections of the curves with the lines of equal sterility on Fig. 89, are projected, to the ordinates-line corresponding to the mean of the range of durations, viz., 2.5, 7.5, 12.5, etc. years. Smoothed curves have then been drawn shewing the probable position of the curves of equal sterility.

Curve A in Fig. 90 denotes the sterility-ratio according to age at marriage where the duration of marriage is 20 years.

On Fig. 90 they represent the projected results, and the lines drawn among them, the smoothed general results deduced therefrom. Thus the

[1] It has been assumed that the group-results for the ranges 0-5, 5-10, 10-15, etc., are sensibly correct for the durations 2.5, 7.5, 12.5, etc., as is evident from Fig. 90. This is not quite exact; the error is not large, however, and the inherent limitations of the determination of the ratio render the measure of uncertainty of but little moment.

[2] The three broken lines crossing from Fig. 89 to Fig. 90, indicate the scheme of projection. Thus, the point b, viz., the intersection of the curve assumed to represent a sterility of 0.3 for a duration of marriage of 12.5 years, is found in the graph (plan), Fig. 90, as the point b′, viz., on the line parallel to the axis of age at the distance (ordinate) therefrom 12.5, and similarly for point a and c and a′ and c′.

new curves so obtained represent completely the sterility-ratios according to age taken in conjunction with past duration of marriage.[1]

It is obvious that tables shewing average sterility can be constructed

(i.) according to *age at marriage* and time since elapsed ; and

(ii.) according to *age attained* after the given interval between it and marriage.

As, however, the one differs from the other merely by the whole amount of the duration, it is immaterial in which form they are set out. In the following table (CXVIII.) the former method is adopted ; Figs. 89 and 90, however, give the age attained after a given duration of marriage.[2]

TABLE CXVIII.—Shewing for various Ages and Durations of Marriage the Degree of Sterility experienced. Australia, 1911.

Sterility-Ratio.	Corresponding Durations of Marriage (in Years).								
	5	10	15	20	25	30	35	40	45
	When the Age at Marriage is :—*								
.025	..	13.8	15.5	16.6	17.1	17.1	16.7†	16.0†	15.1†
.050	..	19.3	21.3	22.9	23.7	24.3	24.3	24.4	27.6
.075	..	23.1	24.9	26.3	27.2	27.8	28.0	28.1	31.9
.100	..	25.8	27.6	28.8	29.6	30.1	30.3	30.9	34.1
.150	..	29.3	30.7	31.8	32.4	32.6	32.5	33.4	37.9
.200	..	31.6	32.7	33.8	34.1	34.3	34.5	35.4	
.250	..	33.4	34.3	35.1	35.4	35.7	35.9	36.8	
.300	..	34.9	35.5	36.2	36.6	37.0	37.2	37.9	
.350	34.0	35.8	36.5	37.1	37.5	37.9	38.0	38.6	
.400	35.1	36.7	37.2	37.8	38.3	38.7	38.8	39.6	
.450	36.1	37.5	38.1	38.7	39.2	39.7	40.0	40.8	
.500	37.1	38.7	38.8	39.5	39.8	40.4	40.9	41.7	
.600	39.0	39.8	40.1	40.7	41.0	41.4	41.7	42.6	
.700	40.8	41.4	41.6	42.1	42.2	42.5	42.8	43.7	
.800	42.5	43.0	43.1	43.3	43.5	43.7	43.7	44.6	
.900	44.5	44.5	44.6	44.7	44.0	44.0	45.0	46.0	
.950	45.9	46.1	46.1	46.2	46.3	46.3	46.3	47.0	
.975	47.6	47.7	47.7	47.8	47.7	47.6†	47.6†	48.0†	
1.000	51.6†	51.5†	51.4†	51.3†	51.2†	51.1†	50.9†	50.7†	

* The table is thus interpreted :—Reading horizontally, if the age at marriage was say 16.6 years, and the duration of marriage was 20 years, 0.025 would be the proportion without children. Similarly if the age at marriage was 17.1 years, and the duration of marriage was either 25 years or 30 years, or reading vertically, for the duration of marriage of 15 years, if the age at marriage were 15.5, then 0.025 would be sterile ; if the age were 21.3, then 0.050 would be sterile ; and so on.

† The apparent anomaly in these results may possibly be explained by the more fertile not living sufficiently long to be included in the category of those whose duration of marriage attained the number of years indicated.

The sterility-ratios given in the table for durations of marriage 0-5, do not accord very closely with those deduced by the method of Part

[1] Strictly these curves represent the mean of 5-year groups, both as regards duration of marriage and age. The corrections to make them instantaneous results, however, are small.

[2] Data have not been compiled which would enable these results to be worked out with very great precision. For this it would of course be necessary to compile according to single years both as regards age and duration of marriage ; and give results according to " age at marriage" and " duration of marriage" instead of existing age.

XIII., §§ 11-13, pp. 245 to 250. The probability of a birth, and that of childlessness should together equal unity: For 0-6 years the agreement, however, is closer; see Fig. 71, p. 249, or the values given in Table LXXV., p. 247. As, however, the results for the shorter durations are necessarily somewhat uncertain, these differences are not remarkable. It may be pointed out the results indicated in Table LXXI., p. 238, shew that for the age 51 the probability of a birth is 1.17 per thousand, hence the final value should probably be 0.999, rather than 1.000. But tables of this kind are, of course, probably never reliable to this order of precision.

23. **Curves of sterility according to duration of marriage.**—The sterility-ratios determined from the age of the married woman only, are based upon the *assumption that fertility is independent of the age of the husband*: this is shewn hereafter not to be the case. Or we may regard the results as true for the average condition (*i.e.*, the condition including husbands of all ages). Continuing this assumption and taking the curve for a duration of marriage of 20 years, it is found that the proportion sterile who are married at the ages 11, 12, . . . 51 respectively are as shewn in Fig. 90, Curve A. The ordinate at age 11 is not necessarily zero, but owing to the fact that marriages at that age usually arise from special circumstances, the value of the sterility-ratio is practically zero.[1] The curve has a point of inflexion, for marriages at about age 40, (*i.e.*, $d^2y / dx^2 = 0$ for $x = 40$), and the sterility-ratio changes most rapidly at about age 28 (*i.e.*, $d^3y / dx^3 = 0$ for $x = 28$). The curves of sterility can be obtained by plotting the ages in the vertical columns in Table CXVIII., as abscissæ, and the value of the observed sterility as an ordinate. For every given duration of marriage there will be a different curve.

24. **Fecundity according to age and duration of marriage: various distributions and ratios.**—As already pointed out, fecundity is a function of the age of the husband and of the wife, as well, of course, as of the duration of marriage. It has been shewn herein also, for various durations of marriage, that on the average (*i.e.*, the results being for husbands of all ages combined), and for those only who come under observation in cases of birth, the number of children borne, according to duration of marriage (i), is about $\frac{2}{3} + \frac{4}{11} i$; see formula (523) of Part XIII., §§ 34, 35, and Table XC., pp. 279-283. The surface of representation of this is, for the most part, sensibly a plane. It defines the polygenesic[2] distribution, see p. 285; and thus may be called the *polygenesic surface*. In the case of this distribution differences of age have much less influence, if any, than differences in duration of marriage. It is important to bear in mind, however, that this distribution, as above stated, applies only to a limited

[1] That is the marriages are what have been (somewhat ill-advisedly) called prejudiced"—and do not represent the average liability of becoming fertile.

[2] The word "polygenesis" has been used to indicate the origination of a race arising from several independent ancestors or germs. The above use will, however, lead to no confusion, and is consistent with the general mode of word construction. The word polyphorous (from πολύφόρος = bearing many) is used hereinafter for a different function.

number of married women, viz., *those whose total fecundity happens to come under review through repeated child-bearing.* In Part XIII., § 36, p. 285, the *total number of children borne* by married women of given limits of age and duration of marriage has been called the "general genesic," or "fecundity" distribution. For many purposes, however, it is desirable to know the *number of mothers* (z) instead of the number of children (say, $z' = kz$, $k = 0, 1, 2 \ldots n$) being the number borne by each woman). It is also preferable to relate the number of married women to the *exact* number, k, of children borne by each. Let, therefore, ${}_0m$, ${}_1m$, ${}_2m \ldots {}_nm$ denote the number of married women who bore $0, 1, 2 \ldots n$ children respectively, the range of whose ages are between x_0 and x_1, x_1 and x_2, etc., and the range of whose durations-of-marriage are t_0 and t_1, t_1 and t_2, etc., the ages and duration limits, however, being quite independent. Then the various quantities of importance may be embraced by the following distributions, which will hereinafter be defined, viz.:—(i.) The age-genesic distribution, (ii.) the durational genesic distribution, (iii.) the age-fecundity distribution, (iv.) the durational fecundity distribution, (v.) the age-polyphorous distribution, (vi.) the durational polyphorous distribution, (vii.) the duration-and-age-fecundity distribution, (viii.) the age-and-duration fecundity distribution, (ix.) the duration-and-age polyphorous distribution, and (x.) the age-and-duration polyphorous distribution. The *ages* may be those at the moment of enumeration or at the moment of marriage: for given purposes either may be required.

These distributions are most clearly defined by means of a symbolic table, Table CXIX., shewing the two types of possible compilation of the results exhibiting the degree of fecundity characteristic of a community. With the aid of this table the various types of distribution—essentially ratios—are readily symbolically represented.

TABLE CXIX.—Scheme of Compilation according (i.) to Age, and (ii.) to Duration of Marriage, exhibiting the Characteristics of the Fecundity of a Population.

Either (i.) Age-Group,	or (ii.) Duration-Group.	Total Number of children borne by *each* married woman where the age is given, or during the *existing marriage* where the duration is given.							Totals of (horizontal) rows.
		0	1	2	..	k	..	n	
x_0 to x_1	t_0 to t_1	${}_0m_1$	${}_1m_1$	${}_2m_1$	..	${}_km_1$	..	${}_nm_1$	M_1
x_1 to x_2	t_1 to t_2	${}_0m_2$	${}_1m_2$	${}_2m_2$	..	${}_km_2$	..	${}_nm_2$	M_2
..	..	..	..	..	..	..	..	..	..
..	..	..	..	..	..	..	..	..	..
x_{p-1} to x_p	t_{p-1} to t_p	${}_0m_p$	${}_1m_p$	${}_2m_p$	..	${}_km_p$	..	${}_nm_p$	M_p
..	..	..	..	..	..	..	..	..	..
..	..	..	..	..	..	..	..	..	..
x_{s-1} to x_s	t_{s-1} to t_s	${}_0m_s$	${}_1m_s$	${}_2m_s$	..	${}_km_s$	..	${}_nm_s$	M_s
Totals of (vert.) columns		${}_0M$	${}_1M$	${}_2M$	..	${}_kM$	..	${}_nM$	M

To distinguish clearly between (i.) and (ii.), the m and M quantities are not accented for the former, and are accented (m' and M') for the latter. The surface, the z co-ordinate to which is the height above the xk, or the xt plane, as the case may be, is :—

$$(563) \ldots\ldots z = f_1\,(x, k)\ ; \ \text{or}\ z = f_2\,(t, k),$$

in which, of course, k is a variable.

The following table, viz., CXX., gives symbolically the definition of each distribution.

TABLE CXX.—Types of Distribution.

DOUBLE ENTRY DATA.

Age Distribution.		Group Formulæ.			Durational Distribution.
(i.) Age genesic	=	$z_{ag} = {}_k m_p \,/\, M$	${}_k m'_p \,/\, M' = z'_{dg}$	=	(ii.) Durational genesic
(iii.) Age fecundity	=	$z_{af} = {}_k m_p \,/\, {}_k M$	${}_k m'_p \,/\, {}_k M' = z'_{df}$	=	(iv.) Durational fecundity
(v.) Age polyphorous ..	=	$z_{ap} = {}_k m_p \,/\, M_p$	${}_k m'_p \,/\, M'_p = z'_{dp}$	=	(vi.) Durational polyphorous

The equations of the continuous surfaces for the above are $z = f\,(x, k)$

Equations of the continuous surfaces for tho above are $z = f\,(t, k)$.

TRIPLE ENTRY DATA.

A table required for each range of duration. A table required for each age-group.

Age Distributions.		Group Formulæ.			Durational Distributions.
(vii.) Duration and age fecundity	=	$z_{d.af} = {}_k m''_p \,/\, {}_k M''$	${}_k m'''_p \,/\, {}_k M''' = z_{a.df}$	=	(viii.) Age and durational fecundity
(ix.) Duration and age polyphorous	=	$z_{d.ap} = {}_k m''_p \,/\, M''_p$	${}_k m'''_p \,/\, {}_p M''' = z_{a.dp}$	=	(x.) Age and durational polyphorous

The equation of the continuous surface for each range of duration is $z = f\,(x,k)$

Tho equation of the continuous surface for each range of duration is $z=f(t,k)$

25. **The age-genesic distribution.**—This distribution furnishes at once the means for determining how a given total of married women may be "partitioned" according to (i.) age, and (ii.) the number of children borne by them. These ratios, multiplied by 1,000,000, are given in Table CXIV., pp. 322-3, for various age-limits. The ignored elements are the durations of marriage and the ages of the husbands.

26. **The durational genesic distribution.**—This distribution similarly furnishes the basis for ascertaining how a given total of married women may be subdivided according to (i.) duration of existing marriage, and (ii.)

number of children borne by them. The ratios multiplied by 1,000,000 are given in Table CXIII., pp. 322-3. The ignored elements are the ages both of the wives and their husbands.

27. **The age-fecundity distribution.**—This distribution represents the relative numbers, according to age, of married women who bore a given number of children: thus it enables the relative frequency according to age of those who bore any given number of children to be compared, as between one community and another, a fact which will be immediately obvious from the table hereunder. The ignored elements are the durations of marriage and the ages of the husbands.

TABLE CXXI.—Shewing, for Wives of all Durations of Marriage combined, and for each Total Number of Children borne by them (*i.e.*, 1, 2, 3 . . . . to *n*), the Proportion Contributed by each Age-group indicated. Australia, Census 3rd April, 1911. Age-Fecundity Distribution.

Ages of Wives at Census	No. of Wives	Ratio to Total	Result as by Table CXIII.†	Ratio of the Number in a given Age-group to the Total of all Ages, of those who bore *k* Children, where *k* is successively 0, 1, 2 . . . . *n*.																					
				0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	over 20
13	1	00000	00000	0000																					
14	18	00002	00002	0001	0000																				
15	92	00013	00013	0006	0002																				
16	345	00047	00047	0017	0014	0000																			
17	1,061	00145	00145	0048	0047	0004	0000																		
18	2,557	00348	00348	0101	0115	0020	0000																		
19	4,376	00596	00597	0146	0201	0052	0010	0001																	
20	7,224	00985	00985	0209	0318	0122	0026	0002	0000																
21–24	57,896	07890	07894	1378	1991	1313	0630	0215	0054	0016	0001	0000													
25–29	109138	14874	14879	*1802	*2596	*2495	2008	1377	0782	0357	0142	0050	0013	0002	0003	0000									
30–34	112523	15335	15337	1376	1720	2085	*2200	*2048	1754	1316	0881	0478	0240	0107	0052	0023	0008	0007	0000	0002					
35–39	104619	14258	14228	1146	1049	1456	1766	1922	*1959	1871	1636	1372	1025	0699	0492	0305	0219	0175	0009	0003					
40–44	95,392	13000	12994	1026	0742	0981	1285	1597	1785	*1894	*1888	1793	1724	1526	1319	1117	1043	0893	0094	0080	0408	0841	0000	0625	0556
45–49	82,237	11207	11205	0896	0510	0658	0922	1212	1449	1697	1816	*1888	*1911	1864	1876	1745	1728	1787	0177	0171	1878	1308	1190	3125	0833
50–54	61,447	08374	08376	0669	0304	0378	0549	0770	1003	1199	1467	1659	1840	*1874	*1952	*2019	*2005	*1973	*0189	*0205	*1918	*2057	1429	1875	*1944
55–59	37,900	05165	05166	0442	0171	0193	0274	0381	0533	0683	0886	1090	1241	1441	1534	1645	1778	1744	0170	0191	1918	1776	*2857	*2500	1667
60–64	25,065	03416	03417	0314	0100	0113	0151	0229	0318	0440	0565	0726	0857	1045	1150	1349	1284	1448	0152	0147	1838	1776	0952	1875	1667
65–69	16,640	02268	02285	0215	0063	0065	0094	0131	0175	0280	0370	0506	0585	0761	0840	0972	1033	1069	0123	0118	1470	1215	1906	0000	1111
70–74	9,297	01267	01268	0125	0035	0037	0051	0070	0108	0140	0215	0265	0355	0422	0467	0499	0605	0582	0057	0064	0327	0654	1190	0000	2222
75–79	4,254	00580	00581	0058	0016	0021	0023	0031	0052	0076	0090	0121	0154	0197	0814	0253	0243	0260	0024	0016	0122	0373	0476	0000	
80–105	1,691	00230	00233	0025	0006	0007	0011	0014	0028	0031	0043	0052	0055	0062	0071	0073	0054	0062	0005	0003	0121	0000			
13–105	733773	1.00	1.00	104761	109720	106195	90218	73962	58482	47045	37540	30537	24399	19317	12805	8841	5023	2575	1280	625	245	107	42	16	36
Totals		1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00

Note.—The figures marked with an asterisk are the maxima in the vertical columns and those underlined are the maxima in the horizontal lines.
† The figures though very approximate to those in the column to the left are obtained from a wholly different distribution of unspecified and partially specified cases.
The figures in the body of the table are, of course, decimals. They are not deduced from those given in Table CXIV., pp. 322-3, but from the results of a more detailed distribution of the unspecified quantities for various age and duration-of-marriage groups, see Table CXXIII., p. 338-9 later.

Fig. 91 shews the characteristics of the age-fecundity surface, the age being that at the time of the Census. If compiled according to the "age at marriage" the form of contours is, of course, materially changed.

Age-Fecundity Surface.

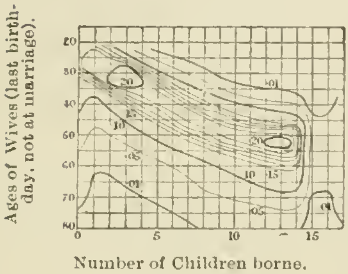


Fig. 91.

28. **The durational fecundity distribution.**—This distribution is exactly analogous to that preceding, the arguments in the table being limits of duration of marriage and the number of children borne. The values could be obtained *roughly* from the data in Table CXIII., by dividing the numbers of wives who bore a given number of children by the corresponding total (*i.e.*, of the vertical columns, see pp. 322-3). More *accurately* it could be found from the data given in Table CXXIII. hereinafter (pp. 338-9), the results for all ages being added together for the required numbers.

The ignored elements are the ages of the wives and of the husbands. If instead of being made out for *all* ages, durational fecundity surfaces are determined for various age-groups, their characteristics will not markedly differ, as might be inferred from Table CXVIII., p. 330 hereinbefore.

29. **The age-polyphorous distribution.**—The data which give the age-fecundity distribution by dividing the tabular numbers by the totals according to the number of children borne, give also the age-polyphorous distribution if divided by the totals of the respective age-groups, see Table CXXII. hereunder, in which the required ratios are given. The distribution thus shews the relative frequency with which married women in any given age-group bear 0, 1, 2 . . . etc., children.

The ignored elements are the duration of marriage and the age of the husbands.

Age-Polyphorous Surface.

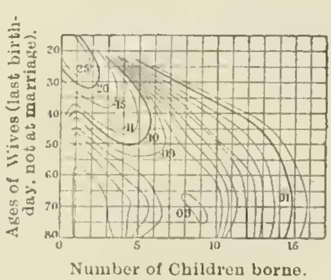


Fig. 92.

Fig. 92 shews the characterisics of the age-polyphorous surface, the age being that at the time of the Census. If compiled according to the "age at marriage" the form of the contours is, of, course, materially changed.

TABLE CXXII.—Shewing, for all Durations of Marriage combined, the Relative Numbers of Married Women of given Age-groups who bore 0, 2, 3 . . . to n Children. Australia, Census of 3rd April, 1911. Age-polyphorous Distribution.

Ages of Wives.	No. of Wives	Ratio of the Number who bore the kth Child to the total Married Women of the Age-groups indicated, where k =																						
		0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	Over 20	Total
13	1	1.000	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	1.0
14	18	7200	2800	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	1.0
15	92	7280	2720	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	1.0
16	345	5300	4410	0290	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	1.0
17	1061	4690	*4850	0420	0040	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	1.0
18	2557	4149	*4939	0849	0055	0004	0004	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	1.0
19	4376	3497	*5034	1252	0203	0014	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	1.0
20	7,224	3034	*4828	1789	0324	0022	0003	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	1.0
21–24	57,896	2495	*3772	2408	0980	0275	0055	0013	0002	00003	..	..	..	..	..	..	..	..	..	..	..	..	..	1.0
25–29	109138	1730	*2610	2428	1660	0933	0419	0153	0049	0014	00003	00005	00004	00001	..	..	..	..	..	..	..	..	..	1.0
30–34	112523	1281	1678	*1967	1764	1346	0912	0550	0294	0130	0052	0018	00055	00018	00003	00002	00001	00001	..	..	..	..	..	1.0
35–39	104619	1147	1100	1478	*1522	1359	1095	0842	0587	0400	0240	0129	0060	0026	0010	0004	0001	..	..	..	..	..	..	1.0
40–44	95,392	1126	0853	1093	1215	*1238	1095	0934	0743	0574	0440	0309	0177	0104	0055	0024	0013	0005	0001	0001	..	..	..	1.0
45–49	82,237	1142	0680	0849	1010	*1091	1030	0971	0829	0700	0567	0438	0292	0188	0106	0056	0028	0013	0006	0002	0001	0001	..	1.0
50–54	61,447	1140	0543	0652	0806	0927	*0956	0918	0896	0825	0730	0589	0407	0290	0164	0083	0039	0021	0008	0004	0001	0000	0001	1.0
55–59	37,900	1222	0495	0540	0653	0743	0823	0847	*0878	0878	0799	0734	0539	0384	0236	0118	0058	0031	0012	0005	0003	0001	0001	1.0
60–64	25,065	1312	0437	0479	0542	0677	0742	0828	0845	*0884	0834	0805	0588	0475	0257	0149	0078	0037	0020	0007	0001	0001	0002	1.0
65–69	16,640	1353	0415	0417	0508	0581	0617	0788	0837	*0928	0858	0883	0647	0516	0312	0165	0094	0044	0022	0008	0005	0000	0002	1.0
70–74	9,297	1408	0417	0419	0498	0559	0682	0713	0869	0869	*0930	0878	0644	0474	0327	0161	0079	0043	0009	0008	0005	0000	0008	1.0
75–79	4,254	1425	0416	0524	0487	0531	0712	0825	0790	0872	0881	*0893	0555	0527	0287	0157	0073	0024	0007	0009	0005	..	..	1.0
80–105	1,691	1532	0373	0473	0597	0609	0875	0905	*0958	0934	0798	0710	0538	0385	0160	0095	0035	0012	0012	..	..	..	..	1.0
13–105 Nos.	733773	104761	109720	106195 *	90218	73962	58482	47045	37540	30537	24399	19317	12805	8841	5023	2575	1280	625	245	107	42	16	36	1.0
Ratio	100000	14277	14953	14472	12295	10080	07970	06411	05116	04162	03325	02633	01745	01205	00685	00351	00174	00085	00033	00015	00006	00002	00005	1.0
Result as by CXIII.†		14222	14958	14483	12306	10090	07990	06415	05118	04149	03326	02632	01746	01204	00686	00351	00175	00086	00034	00015	00006	00002	00006	1.0

Note.—The figures marked with an asterisk are the maxima in the horizontal lines, and those underlined are the maxima in the vertical columns, excepting in the case of column 0, where .1126 is the minimum.

† The figures though very approximate to the line above are given by a wholly different distribution of unspecified and partially specified cases.

The figures in the body of the table are, of course, decimals. They are not deduced from those given in Table CXIV., pp. 322-3, but from the results of a more detailed distribution of the unspecified quantities for various age and duration-of-marriage groups, see Table CXXIII., pp. 338-9 later.

30. **The durational polyphorous distribution.**—The data from which the durational fecundity is derived furnish also the numbers required for the computation of the durational polyphorous distribution, viz., that which shews for given durations of marriage, or between given limits of duration of marriage, the relative frequency with which given numbers of children are borne. The ignored elements are the ages of the wives and of their husbands. This table has not been computed, but the necessary data are given in Table CXXIII. hereinafter.

31. **Fecundity distributions according to age, duration of marriage and number of children borne.**—The fecundity distribution tables, so far, are of the type $z = f(x, y)$, but if age, duration of marriage and number of children borne, be simultaneously taken into account, then the distribution-frequency is of the type $z = f(w, x, y)$, and cannot be represented by a single three-dimensional graph, for example, height contours upon a plane. It is necessary in fact to have a graph for each value of w adopted in the tabulations.

The exigencies of tabulation, of course, also require that a separate table of the values of z shall be given for each value of one co-ordinate (say w), for the values given by double entry of the other two (say x and y).

In Table CXXIII., hereunder, the results are tabulated for single years of age from 13 to 20, (last birthday), for the ages 21 to 24, and then for every five year age-group onward. The table gives, for existing marriage, the number of wives, of various ages and durations of marriage, who failed to give birth to children, or who gave birth to 1, 2, 3, etc.

In the tables as originally compiled, there was a considerable number of unspecified cases, viz., the following :—

Class (i.), the larger class, in which the ages were specified.

Class (ii.), a relatively small class, in which the ages were *not* specified.

In each of these were three sub-classes as follow, viz.:—

(*a*) in which the duration of marriage was not specified;

(*b*) in which the number of children was not specified;

(*c*) in which neither the duration of marriage nor the number of children was specified.

It was consequently necessary to effect a distribution in order to get anything like the most probable results.[1]

The method of distribution was that outlined in § 4, Table XCVII., and formulæ (543) to (547). That is to say, sub-class (*c*) was first distributed proportionately among sub-classes (*a*) and (*b*), and sub-classes (*a*) and (*b*) of Class (i.) were distributed proportionately among the fully specified cases. In Class (ii.) the corrected sub-classes (*a*) and (*b*) were then proportionately distributed among the fully specified corrected groups of Class (i.). The details of the distribution shewed that the result was very satisfactory judged by the regularity of the ratios (see § 5 hereinbefore).

[1] The method of adopting the fully specified cases as characteristic of the whole, involves merely multiplying each by the ratio of the totals. An examination of actual results shewed that recourse to this procedure was unsatisfactory. It rejects part of the evidence available. To distribute the partially specified cases is, therefore, much to be preferred.

TABLE CXXIII.—Shewing, for Various Durations of existing Marriage, the Number of Wives in Various Age-groups who bore k Childre[n] where k = 0, 1, 2, etc. Australian Census, 3rd April, 1911.

Number of Wives to whom had been born Children to the Number of :—

Duration of Marriage.	Age* 13	Age 14.			Age 15.			Age 16.				Age 17.					Age 18.						
	0	0	1	Total	0	1	Total	0	1	2	Total	0	1	2	3	Total	0	1	2	3	4	5	Total
0–5	1	13	5	18	67	25	92	183	152	10	345	498	514	45	4	1,061	1,061	1,263	216	14	1	1	2,5
5–10	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	1	..	..	..	
Totals	1	13	5	18	67	25	92	183	152	10	345	498	514	45	4	1,061	1,061	1,263	217	14	1	1	2,5

	Age 19.						Age 20.							Age 21-24.									
	0	1	2	3	4	Total.	0	1	2	3	4	5	Total.	0	1	2	3	4	5	6	7	8	Tota
0–5	1,530	2,203	548	84	3	4,368	2,185	3,445	1,178	140	5	0	6,953	13,947	20,116	9,708	1,499	104	4	0	0	0	45,3
5–10	..	..	..	5	3	8	7	43	114	94	11	2	271	493	1,725	4,232	4,175	1,482	314	73	10	..	12,5
10–15	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	1	5	1	4	1	2	
Totals	1,530	2,203	548	89	6	4,376	2,192	3,488	1,292	234	16	2	7,224	14,440	21,841	13,940	5,675	1,591	319	77	11	2	57,8

*25–29	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	Over 20	Tota
0–5	15,392	20,412	10,234	1,632	144	5	..	..	..	..	..	..	..	..	..	.	..	..	..	..	..	..	47,8
5–10	3,127	7,205	14,778	14,400	7,689	2,358	500	78	17	1	..	..	..	..	..	..	..	..	..	..	..	..	50,1
10–15	360	865	1,480	2,082	2,350	2,212	1,171	453	133	31	4	4	..	..	..	..	..	..	..	..	..	..	11,1
15–20	1	0	1	5	0	1	4	2	4	1	1	0	1	..	..	..	..	..	..	..	..	..	
Totals	18,880	28,482	26,493	18,119	10,183	4,576	1,675	533	154	33	5	4	1	..	..	..	..	..	..	..	..	..	109,1
*30–34																							
0–5	7,788	8,174	4,099	675	55	1	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	20,7
5–10	4,185	6,678	11,083	10,308	5,759	1,988	466	97	17	2	..	..	..	..	..	..	..	..	..	..	..	..	40,5
10–15	2,166	3,459	6,051	7,827	8,093	7,022	4,372	2,087	727	204	49	13	5	2	..	..	..	..	..	..	..	..	42,
15–20	273	565	906	1,037	1,241	1,244	1,348	1,122	710	377	155	52	15	2	2	1	1	..	..	..	..	..	9,
20–25	2	0	1	2	1	2	3	0	5	2	2	2	..	..	..	..	..	..	..	..	..	..	
Totals	14,414	18,876	22,140	19,849	15,149	10,257	6,189	3,306	1,459	585	206	67	20	4	2	1	1	..	..	..	..	..	112,
*35–39																							
0–5	4,395	2,767	1,307	203	24	5	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	8,7
5–10	3,060	3,060	4,503	4,068	2,228	786	203	60	24	1	1	..	..	..	..	..	..	..	..	..	..	..	17,
10–15	2,832	3,210	5,454	6,288	6,087	4,943	3,240	1,511	585	207	57	24	3	2	..	..	..	..	..	..	..	..	34,4
15–20	1,422	2,034	3,458	4,424	4,782	4,596	4,297	3,546	2,596	1,393	666	281	106	39	8	1	..	..	..	..	..	..	33,
20–25	292	435	739	944	1,092	1,128	1,062	1,019	985	898	626	324	161	68	37	11	2	1	..	..	..	..	9,
25–30	1	0	1	0	0	0	1	2	0	1	0	1	0	1	..	..	..	..	..	..	..	..	
Totals	12,002	11,506	15,462	15,927	14,213	11,458	8,803	6,138	4,190	2,500	1,350	630	270	110	45	12	2	1	..	..	..	..	104,
*40–44																							
0–5	2,811	787	244	23	4	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	3,
5–10	2,300	1,523	1,530	1,068	571	232	53	7	3	1	..	..	..	..	..	..	..	..	..	..	..	..	7,
10–15	2,236	1,974	2,607	2,626	2,435	1,821	1,046	486	177	71	22	5	..	..	..	..	..	..	..	..	..	..	15,
15–20	1,836	1,999	3,069	3,654	3,859	3,359	2,931	2,227	1,442	819	387	177	65	27	10	4	..	..	..	..	..	..	25,
20–25	1,323	1,570	2,587	3,634	4,156	4,165	3,984	3,531	3,034	2,541	1,814	957	559	264	110	56	17	3	3	..	..	..	34,
20–35	233	287	385	585	782	864	890	835	819	775	721	550	363	232	109	60	33	7	6	0	1	2	8,
30–35	0	0	0	0	2	0	4	1	1	0	4	0	1	1	1	..	..	..	..	..	..	..	
Totals	10,739	8,140	10,422	11,590	11,809	10,441	8,908	7,087	5,476	4,207	2,948	1,689	988	524	230	120	50	10	9	0	1	2	95,3
*45–49																							
0–5	1,901	112	11	1	1	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	2,
5–10	1,882	624	309	139	46	21	5	1	2	..	..	..	..	..	..	..	..	..	..	..	..	..	3,
10–15	1,646	1,105	1,001	854	553	305	168	63	29	9	3	2	..	..	..	..	..	..	..	..	..	..	5,7
15–20	1,281	1,055	1,479	1,567	1,516	1,195	826	466	277	132	59	16	12	3	1	17	7	3	1	0	1	..	9,9
20–25	1,444	1,478	2,312	2,973	3,364	3,130	2,870	2,297	1,666	1,163	749	383	195	88	31	122	68	22	6	2	3	2	24,3
25–30	1,061	1,053	1,631	2,441	3,050	3,276	3,515	3,250	3,018	2,652	2,100	1,459	913	508	263	88	32	21	7	3	1	1	30,3
30–35	175	170	242	339	437	545	601	738	772	706	690	542	422	268	165	0	0	..	..	..	..	..	6,8
35–40	0	1	0	0	1	0	0	2	0	1	0	0	1	1	..	..	..	..	..	..	..	..	
Totals	9,390	5,598	6,985	8,314	8,968	8,472	7,985	6,817	5,764	4,663	3,601	2,402	1,543	868	460	227	107	46	14	5	5	3	82,2
*50–54																							
0–5	969	1	..	...	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	
5–10	1,221	68	13	4	2	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	1,3
10–15	1,224	451	208	126	45	25	11	4	4	1	..	..	..	..	..	..	..	..	..	..	..	..	2,
15–20	877	600	619	565	406	219	119	51	30	20	7	2	5	..	..	..	..	..	..	..	..	..	3,5
20–25	862	748	1,055	1,289	1,291	1,098	782	552	352	166	83	44	23	11	4	2	1	..	..	..	..	..	8,3
25–30	1,019	769	1,237	1,742	2,144	2,329	2,244	2,145	1,761	1,264	847	454	268	110	49	22	13	6	1	..	..	..	18,4
30–35	704	590	749	1,076	1,580	1,893	2,156	2,275	2,380	2,414	2,049	1,439	977	546	265	117	64	25	14	1	2	3	21,3
35–40	129	111	128	151	229	304	331	477	540	624	634	560	509	339	189	99	49	16	17	5	1	3	5,4
40–45	..	..	..	2	..	..	..	1	..	..	..	..	2	1	1	..	1	..	..	..	..	1	
Totals	7,005	3,338	4,009	4,955	5,697	5,868	5,643	5,505	5,067	4,489	3,620	2,499	1,784	1,007	508	240	128	47	22	6	3	7	61,4

* Ages at date of Census.

ABLE CXXIII.—Shewing, for Various Durations of existing Marriage, the Number of Wives in various Age-groups who bore k Children, where k=0, 1, 2, etc. Australian Census, of 3rd April, 1911.—*Cont.*

of Marriage.	Number of Wives to whom had been born Children to the Number of :—																						
	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	Over 20	Total.
5–59																							
0–5	466	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	466
5–10	578	1	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	579
0–15	770	60	21	3	1	0	1	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	856
5–20	630	285	174	75	37	8	4	1	0	1	..	..	..	..	..	..	..	..	..	..	..	..	1,215
0–25	575	388	387	390	292	184	112	73	32	26	5	5	2	1	..	..	..	..	..	..	..	..	2,472
5–30	493	398	488	625	721	762	661	433	264	143	93	48	16	11	9	1	1	..	..	..	..	..	5,167
0–35	544	369	510	751	889	1138	1,176	1,286	1,174	927	707	402	215	118	53	23	10	2	..	..	..	..	10,294
5–40	463	318	398	558	784	898	1,069	1,311	1,583	1,600	1,515	1,161	873	509	261	111	64	26	9	10	4	4	13,529
0–45	113	57	69	71	92	128	188	222	275	331	462	425	348	254	126	83	44	19	10	2	..	2	3,321
5–	..	..	..	..	..	..	..	..	..	..	1	..	..	..	..	..	..	..	..	..	..	..	1
tals	4,632	1,876	2,047	2,473	2,816	3,118	3,211	3,326	3,328	3,028	2,783	2,641	1,454	893	449	218	119	47	19	12	4	6	37,900
0–64																							
0–15	905	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	905
5–20	477	22	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	499
0–25	459	190	130	74	22	5	4	1	..	..	..	..	..	..	..	..	..	..	..	..	..	..	885
5–30	380	231	265	270	248	168	97	45	29	9	5	4	..	..	..	..	..	..	..	..	..	..	1,751
0–35	312	263	267	320	428	445	417	330	211	142	86	35	11	5	3	1	..	..	..	..	..	..	3.216
5–40	422	235	302	397	551	690	829	802	888	687	524	304	220	56	38	17	12	6	4	1	1	2	6,988
0–45	270	170	207	253	381	468	620	796	887	1,019	1,088	832	642	401	207	105	45	21	7	1	1	3	8,424
5–	64	44	29	45	67	84	108	145	261	234	315	297	319	183	125	72	35	18	8	2	1	1	2,397
tals	3,289	1,095	1,200	1,359	1,697	1,860	2,075	2,119	2,216	2,091	2,018	1,472	1,192	645	373	195	92	45	19	4	3	6	25,065
5–69																							
0–15	456	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	456
5–20	213	1	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	214
0–25	340	30	4	1	..	1	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	376
5–30	328	115	73	50	19	4	4	1	..	..	..	..	..	..	..	..	..	..	..	..	..	..	594
0–35	234	139	180	157	139	102	55	39	26	13	6	2	3	..	..	..	..	..	..	..	..	..	1,095
5–40	204	154	161	233	278	339	335	241	178	103	56	15	16	3	2	4	..	..	..	..	..	..	2,322
0–45	245	117	159	214	278	272	485	554	591	509	399	246	146	69	43	22	11	3	..	..	..	..	4,363
5–	231	134	116	191	253	308	433	557	749	803	1,009	813	694	447	230	131	63	33	13	8	0	4	7,220
tals	2,251	690	693	846	967	1,026	1,312	1,392	1,544	1,428	1,470	1,076	859	519	275	157	74	36	13	8	0	4	16,640
0–74																							
0–20	351	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	351
20–25	139	1	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	140
25–30	205	18	1	1	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	225
30–35	145	75	74	24	19	3	3	2	1	1	..	..	..	..	..	..	..	..	..	..	..	..	347
35–40	119	84	84	116	105	72	29	32	14	3	1	1	..	..	..	..	..	..	..	..	..	..	660
40–45	110	71	98	126	133	191	187	141	100	73	31	18	6	6	0	1	1	..	..	..	..	..	1,293
45–	240	139	133	196	263	367	444	633	693	788	784	579	435	298	150	72	39	8	7	5	0	8	6,281
otals	1,309	388	390	463	520	633	663	808	808	865	816	598	441	304	150	73	40	8	7	5	0	8	9,297
75–79																							
0-30	278	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	278
30–35	79	4	0	0	1	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	84
35–40	70	32	26	5	6	2	0	1	0	2	..	..	..	..	..	..	..	..	..	..	..	..	144
40–45	50	38	63	62	38	35	37	20	8	6	0	1	0	2	0	0	1	..	..	..	..	..	361
45–	129	103	134	140	181	266	314	315	363	367	380	235	224	120	67	31	9	3	4	2	..	..	3,387
otals	606	177	223	207	226	303	351	336	371	375	380	236	224	122	67	31	10	3	4	2	..	..	4,254
0–105																							
0–35	117	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	117
35–40	38	1	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	39
40–45	27	17	11	11	5	2	1	1	1	..	..	..	..	..	..	..	..	..	..	..	..	..	76
45–	77	45	69	90	98	146	152	161	157	135	120	91	65	27	16	6	2	2	..	..	..	..	1,459
otals	259	63	80	101	103	148	153	162	158	135	120	91	65	27	16	6	2	2	..	..	..	..	1,691
Totals Ages 13–105 and all Durations of Marriage.	104,761	109,720	106,195	96,218	73,962	58,482	47,045	37,540	30,535	24,399	19,317	12,805	8,841	5,023	2,575	1,280	625	245	107	42	16	36	733,773

* Ages at date of Census.

From the data furnished, distributions (vii.) to (x.) can readily be computed.

32. **The duration and age-fecundity distributions.**—For a series of duration-of-marriage-groups these distributions are obtained by computing, for successive age-groups and for each number of children borne, the relative frequency of the mothers within the indicated age-limits who bore a given number of children to the total mothers of all ages (which are included) bearing the same number of children. These results may be obtained by a re-arrangement of the data in Table CXXIII., pp. 338-9. The distribution is (vii.) of Table CXX., p. 333.

The ignored element is only the age of husbands.

33. **The duration and age-polyphorous distributions.**—These, for a series of duration-of-marriage groups, are obtained by computing for a series of age-groups the relative frequency of the mothers within the age-group who bore a given number of children to the total of all mothers in the same age-group (*i.e.*, who bore 0 to n children). The results may be obtained by the same re-arrangement as is required for the distribution referred to in § 32, the present distribution being (ix.) in Table CXX., p. 333. The ignored element is, again, the age of the husbands.

34. **The age and durational fecundity distributions.**—By dividing in each age-group the number of mothers who bore any given number of children, and whose duration of marriage was between given limits, by the total number of mothers who bore the same number of children (*i.e.*, for all durations of marriage in the age-group in question), the ratios in Table CXXIV. hereinafter are obtained. Each series of ratios is the age and durational fecundity distribution for the fundamental age-group. This case is (viii.) in Table CXX., p. 333. The only ignored element is the age of the husbands.

35. **The age and durational polyphorous distributions.**—As in the case of the distributions immediately preceding Table CXXIII., pp. 338-9 furnishes the required data. The series of divisors in each age-group are the totals for the indicated limits of duration of marriage. Thus for married women of a given age and a given duration of marriage, the relative frequency of giving birth to 0, 1, 2 . . . n children are obtained, and these are shewn in Table CXXV. below. This case is (x.) in Table CXX., p. 333, and the only ignored element is again the age of the husbands.

TABLE CXXIV.—Shewing, for a Series of Limits of Duration of Existing Marriage, and according to the Age groups given in the Table, the Ratios of Married Mothers who bore k (where k = 0, 1, 2 . . . . 20, and " over 20") Children, to the Total Number who, for all Durations of Marriage, Bore that Number. Census 3rd April, 1911. Australia. Duration and Age Fecundity Distribution.

Duration of Marriage.	Proportion of the Number of Women who, within the Indicated Limit of Duration of Marriage, Bore k Children to the Total Number of Married Women who Bore k Children, where k =																					
	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	over 20
0–5	1.000	1.000	9988	9533	5714	1.000																
5–10	..	..	0012	0467	4286																	
Nos.	3,353	4,162	820	107	7	1																
*13–19	1.00	1.00	1.00	1.00	1.00	1.00																
0–5	9699	9308	7147	2774	0678	0125																
5–10	0301	0692	2853	7225	9291	9844	9480	9091	1.00													
10–15	..	..	..	0001	0031	0031	0520	0909														
Nos.	16,632	25,329	15,232	5,909	1,607	321	77	11	2													
*20–24	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00													
0–5	8153	7166	3863	0901	0141	0011																
5–10	1656	2530	5578	7947	7551	5153	2985	1463	1104	0303												
10–15	0191	0304	0559	1149	2308	4834	6991	8500	8637	9394	.9000	1.00										
15–20	..	..	..	0003	0000	0002	0024	0037	0259	0303	.1000	..	1.00									
Nos.	18,880	28,482	26,493	18,119	10,183	4,576	1,675	533	154	33	5	4	1									
*25–29	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00									
0–5	5404	4330	1852	0341	0036	0001																
5–10	2903	3538	5006	5193	3802	1938	0753	0293	0117	0034												
10–15	1503	1833	2733	3943	5342	6846	7064	6313	4983	3488	2379	0194	2500	5000								
15–20	0189	0299	0409	0522	0820	1213	2178	3394	4866	6444	7524	0776	7500	5000	1.00	1.00	1.00					
20–25	0001	0000	0000	0001	0000	0002	0005	0000	0034	0034	0097	0030	0000									
Nos.	14,414	18,876	22,140	19,849	15,149	10,257	6,189	3,306	1,459	585	206	67	20	4	2	1	1					
*30–34	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00					
0–5	3662	2405	0845	0127	0016	0004																
5–10	2550	2659	2913	2554	1568	0686	0231	0098	0057	0004	0007											
10–15	2360	2790	3528	3948	4283	4314	3682	2462	1396	0808	0423	0381	0111	0182								
15–20	1185	1768	2236	2778	3365	4012	4881	5777	6196	5572	4933	4460	3926	3546	1777	0834						
20–25	0243	0378	0478	0593	0768	0984	1206	1660	2351	3592	4637	5143	5963	6182	8223	9166	1.00	1.00				
25–30	..	..	..	..	..	..	..	0003	0000	0004	0000	0016	0000	0090								
Nos.	12,002	11,506	15,462	15,927	14,213	11,458	8803,	6,138	4,190	2,500	1,350	630	270	110	45	12	2	1				
*35–39	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00				
0–5	2617	0966	0234	0020	0003																	
5–10	2142	1871	1468	0921	0484	0222	0060	0010	0005	0002												
10–15	2082	2425	2502	2266	2062	1744	1174	0686	0323	0169	0075	0030										
15–20	1710	2456	2945	3153	3268	3217	3291	3143	2633	1947	1313	1048	0658	0515	0435	0333						
20–25	1232	1929	2482	3135	3519	3989	4472	4982	5541	6040	6152	5666	5658	5038	4783	4667	3400	3000	3333			
25–30	0217	0353	0369	0505	0662	0828	0999	1178	1496	1842	2446	3256	3674	4428	4739	5000	6600	7000	6667	0000	1.00	1.00
30–35	..	..	..	..	0002	0000	0004	0001	0002	0000	0014	0000	0010	0019	0043							
Nos.	10,739	8,140	10,422	11,590	11,809	10,441	8,908	7,087	5,476	4,207	2,948	1,689	988	524	230	120	50	10	9	0	1	2
*40–44	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	..	1.00	1.00
0–5	2025	0200	0016	0001	0001																	
5–10	2004	1115	0442	0167	0052	0025	0006	0001	0003													
10–15	1753	1974	1434	1027	0617	0360	0210	0092	0050	0019	0008	0008										
15–20	1364	1885	2117	1885	1690	1410	1034	0684	0482	0283	0164	0067	0078	0034	0022	0749	0651	0652	0714	0000	2000	
20–25	1538	2640	3310	3576	3751	3695	3594	3370	2890	2494	2080	1595	1264	1014	0674	5374	6355	4783	4286	4000	6000	6667
25–30	1130	1881	2335	2936	3401	3867	4398	4767	5236	5688	5832	6074	5917	5853	5717	3877	2991	4565	5000	6000	2000	3333
30–35	0186	0304	0346	0408	0487	0433	0758	1083	1339	1514	1916	2256	2735	3088	3587	0000						
35–40	..	0001	0000	0000	0001	0000	0000	0003	0000	0002	0000	0000	0006	0011	0060							
Nos.	9,390	5,598	6,985	8,314	8,968	8,472	7,985	6,817	5,764	4,663	3,601	2,402	1,543	868	460	227	107	46	14	5	6	3
*45–49	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
0–5	1383	0003																				
5–10	1743	0204	0032	0008	0004																	
10–15	1747	1351	0519	0254	0079	0043	0019	0007	0008	0002												
15–20	1252	1797	1544	1140	0713	0373	0211	0093	0059	0014	0019	0008	0028									
20–25	1231	2211	2632	2601	2266	1871	1386	1003	0695	0370	0229	0176	0129	0110	0079	0083	0078					
25–30	1455	2304	3086	3516	3763	3969	3977	3896	3475	2816	2340	1817	1503	1092	0964	0917	1016	1276	0455			
30–35	1005	1768	1868	2172	2773	3226	3820	4133	4697	5378	5660	5758	5476	5422	5217	4875	5000	5320	6363	1667	6667	4285
35–40	0184	0332	0319	0305	0402	0518	0587	0866	1066	1390	1752	2241	2853	3366	3720	4125	3828	3104	3182	8333	3333	4285
40–45	0000	0000	0000	0004	0000	0000	0000	0002	0000	0000	0000	0000	0011	0010	0020	0000	0078	0000	0000	0000	0000	1430
Nos.	7,905	3,338	4,009	4,955	5,697	5,868	5,643	5,505	5,067	4,489	3,620	2,49[illegible]	1,784	1,007	568	240	128	47	22	6	3	7
*50–54	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00

TABLE CXXIV.—Shewing, for a Series of Limits of Duration of Existing Marriage, and according to the Age groups given in the Table, the Ratios of Married Mothers who bore k (where k = 0, 1, 2 . . . . 20, and "over 20") Children to the Total Number who, for all Durations of Marriage, bore that Number. Census, 3rd April, 1911. Australia. Duration and Age Fecundity Distribution—*continued*.

Duration of Marriage.	Proportion of the Number of Women who, within the Indicated Limit of Duration of Marriage, Bore k Children to the Total Number of Married Women who Bore k Children, where k =																				
	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	over 20
0–5	1006																					
5–10	1248	0005	0000																			
10–15	1662	0320	0103	0012	0004	0000	0003															
15–20	1360	1519	0850	0303	0131	0026	0012	0003	0000	0003												
20–25	1241	2068	1891	1577	1037	0590	0349	0219	0096	0086	0018	0024	0014	0011								
25–30	1064	2122	2384	2527	2560	2444	2059	1302	0793	0472	0334	0235	0110	0123	0200	0046	0084					
30–35	1174	1967	2491	3037	3157	3650	3662	3867	3528	3062	2540	1971	1479	1322	1181	1055	0841	0426				
35–40	1000	1695	1944	2256	2784	2280	3329	3942	4757	5284	5444	5688	6004	5700	5813	5092	5378	5532	4737	0833	1.00	6667
40–45	0244	0304	0337	0288	0327	0410	0586	0667	0826	1033	1660	2082	2393	2844	2806	3807	3697	4042	5263	0167	0000	3333
†45	..	..	..	..	..	..	..	..	..	.	0004	0000	0000	0000								
Nos. *55–59	4,632 1.00	1,876 1.00	2,047 1.00	2,473 1.00	2,816 1.00	3,118 1.00	3,211 1.00	3,326 1.00	3,328 1.00	3,028 1.00	2,783 1.00	2,041 1.00	1,454 1.00	893 1.00	449 1.00	218 1.00	119 1.00	47 1.00	19 1.00	12 1.00	4 1.00	6 1.00
0–15	2752																					
15–19	1450	0200																				
20–25	1396	1735	1083	0544	0130	0627	0019	0066														
25–30	1155	2110	2208	1987	1461	0903	0467	0212	0131	0043	0025	0027										
30–35	0949	1854	2225	2355	2522	2392	2010	1557	0952	0679	0426	0238	0092	0078	0086	0051						
35–40	1283	2146	2517	2921	3247	3710	3996	3785	4007	3286	2597	2665	1846	0868	1019	0872	1304	1333	2105	2500	3333	3333
40–45	0820	1553	1725	1862	2245	2516	2988	3656	4003	4873	5391	5652	5386	6217	5550	5385	4892	4667	3684	2500	3333	5000
†45	0195	0402	0242	0331	0395	0452	0520	0684	0907	1119	1561	2018	2676	2837	3351	3692	3804	4000	4211	5000	3334	1667
Nos. *50–64	3,289 1.00	1,095 1.00	1,200 1.00	1,359 1.00	1,697 1.00	1,860 1.00	2,675 1.00	2,119 1.00	2,216 1.00	2,091 1.00	2,018 1.00	1,472 1.00	1,192 1.00	645 1.00	373 1.00	195 1.00	92 1.00	45 1.00	19 1.00	4 1.00	3 1.00	6 1.00
0–15	2026																					
15–29	0946	0014																				
20–20	1510	0435	0058	0012	0000	0010																
25–30	1457	1667	1053	0530	0196	0039	0030	0007														
30–35	1041	2014	2598	1856	1438	0994	0420	0280	0168	0091	0041	0019	0035									
35–40	0906	2232	2323	2754	2875	3304	2553	1731	1153	0721	0381	0139	0186	0058	0073	0255						
40–45	1088	1696	2294	2530	2875	2651	3697	3980	3828	3565	2714	2286	1700	1323	1564	1401	1486	0833	1.00	1.00	0000	1.00
†45	1026	1942	1674	2258	2616	3002	3300	4002	4851	5623	6864	7556	8079	8613	8363	8344	8514	9167				
Nos. *65–69	2,251 1.00	690 1.00	693 1.00	846 1.00	967 1.00	1,826 1.00	1,312 1.00	1,392 1.00	1,544 1.00	1,428 1.00	1,470 1.00	1,676 1.00	859 1.00	519 1.00	275 1.00	157 1.00	74 1.00	36 1.00	13 1.00	8 1.00	0 1.00	4 1.00
0–20	2681																					
20–25	1062	0026																				
25–30	1566	0464	0026	0022																		
30–35	1108	1933	1897	0518	0365	0047	0045	0025	0012	0011												
35–40	0910	2165	2154	2505	2019	1137	0437	0396	0173	0035	0012	0017										
40–45	0840	1830	2513	2722	2558	3017	2821	1745	1238	0844	0380	0301	0136	0197	0000	1370	0250					
†45	1833	3582	3410	4233	5058	5799	6697	7834	8577	9110	9608	9682	9864	9803	1.00	8630	9750	1.00	1.00	1.00	0000	1.00
Nos. *70–74	1,309 1.00	388 1.00	390 1.00	463 1.00	520 1.00	633 1.00	663 1.00	808 1.00	808 1.00	865 1.00	816 1.00	598 1.00	441 1.00	304 1.00	150 1.00	73 1.00	40 1.00	8 1.00	7 1.00	5 1.00	0 1.00	8 1.00
0–30	4587																					
30–35	1304	0226	0000	0000	0044																	
35–40	1155	1808	1166	0241	0265	0066	0000	0030	0000	0053												
40–45	0825	2147	2825	2996	1681	1155	1054	0595	0216	0160	0000	0043	0000	0164	0000	0000	1000					
†45	2129	5819	6009	6763	8010	8779	8946	9375	9784	9787	1.00	9957	1.00	9936	1.00	1.00	9000	1.00	1.00	1.00		
Nos. *75–79	606 1.00	177 1.00	223 1.00	207 1.00	226 1.00	303 1.00	351 1.00	336 1.00	371 1.00	375 1.00	380 1.00	236 1.00	224 1.00	122 1.00	67 1.00	31 1.00	10 1.00	3 1.00	4 1.00	2 1.00		
0–35	4517																					
35–40	1467	0016																				
40–45	1043	0270	1375	1089	0486	0132	0065	0062	0063													
†45	2973	0714	8625	8911	9514	9968	9935	9938	9937	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00				
Nos. *80–105	259 1.00	63 1.00	80 1.00	101 1.00	103 1.00	148 1.00	153 1.00	162 1.00	158 1.00	135 1.00	120 1.00	91 1.00	65 1.00	27 1.00	16 1.00	6 1.00	2 1.00	2 1.00				
Grand Total	104761	109720	106195	90,218	73,962	58,482	47045	37540	30535	24399	19317	12805	8,841	5,023	2,575	1,280	625	245	107	42	16	36

* Ages. † 45 and over.

TABLE CXXV.—Shewing, for Various Durations of existing Marriage, the Proportion of Women of Various Groups of Ages, who bore 1, 2, 3 . . . *n* Children, the Total for each Age-group between the Limits of Duration of Marriage being Unity. Australia, Census of 3rd April, 1911. Duration and Age-polyphorous Distribution.

Duration of Marriage and Age.	Proportion of the Total of Women within the Indicated Limit of Duration of existing Marriage who bore Children to the Number of k, in which k =																						T'tals	Total No. for the Duration.
	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	over 20		
0–5	3972	4931	0970	0121	0005	0001	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	1.00	8,441
5–10	..	..	1111	5556	3333	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	1.00	9
*13–19	3968	4925	0970	0128	0008	0001	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	1.00	8,450
0–5	3083	4502	2080	0313	0022	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	1.00	52,331
5–10	0391	1384	3401	3341	1169	0247	0057	0008	0002	..	..	..	..	..	..	..	..	..	..	..	..	..	1.00	12.777
10–15	..	..	..	0833	4168	0833	3333	0833	..	..	..	..	..	..	..	..	..	..	..	..	..	..	1.00	12
*20–24	2554	3890	2339	0908	0247	0049	0012	0001	0000	..	..	..	..	..	..	..	..	..	..	..	..	..	1.00	65,120
0–5	3219	4269	2140	0341	0030	0001	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	1.00	47,819
5–10	0624	1437	2947	2871	1533	0470	0100	0015	0003	0000	..	..	..	..	..	..	..	..	..	..	..	..	1.00	50,153
10–15	0323	0776	1328	1868	2109	1985	1051	0406	0119	0027	0004	0004	..	..	..	..	..	..	..	..	..	..	1.00	11,145
15–20	0476	0000	0476	2380	0000	0476	1906	0952	1906	0476	0476	0000	0476	..	..	..	..	..	..	..	..	..	1.00	21
*25–29	1730	2610	2428	1660	0933	0419	0153	0049	0014	0003	00005	00004	00001	..	..	..	..	..	..	..	..	..	1.00	109,138
0–5	3745	3931	1972	0325	0025	0001	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	1.00	20,792
5–10	1031	1645	2732	2540	1419	0490	0115	0024	0004	0000	..	..	..	..	..	..	..	..	..	..	..	..	1.00	40,587
10–15	0515	0822	1438	1860	1924	1670	1039	0496	0172	0049	0010	0003	0001	0001	..	..	..	..	..	..	..	..	1.00	42,077
15–20	0302	0620	1000	1146	1370	1374	1489	1239	0784	0416	0171	0057	0016	0002	0002	0001	0001	..	..	..	..	..	1.00	9,051
20–25	0909	0000	0454	0909	0454	0909	1365	0000	2273	0909	0909	0909	..	..	..	..	..	..	..	..	..	..	1.00	22
*30–35	1281	1678	1967	1764	1346	0912	0550	0294	0130	0052	0018	00055	00018	00003	00002	00001	00001	..	..	..	..	..	1.00	112,525
0–5	5051	3180	1502	0233	0028	0006	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	1.00	8,701
5–10	1701	1700	2502	2262	1238	0437	0113	0033	0013	0001	0000	..	..	..	..	..	..	..	..	..	..	..	1.00	17,994
10–15	0822	0932	1583	1826	1767	1435	0941	0349	0170	0060	0016	0007	0002	0000	..	..	..	..	..	..	..	..	1.00	34,443
15–20	0423	0605	1027	1315	1420	1366	1307	1054	0772	0414	0138	0084	0032	0010	0002	0001	..	..	..	..	..	..	1.00	33,649
20–25	0297	0443	0751	0961	1112	1148	1081	1037	1003	0914	0636	0329	0164	0069	0038	0011	0004	0002	..	..	..	..	1.00	9,824
25–30	1250	0000	1250	0000	0000	0000	1250	2500	0000	1250	0000	1250	0000	1250	..	..	..	..	..	..	..	..	1.00	8
*35–40	1147	1100	1478	1522	1359	1095	0842	0587	0400	0240	0129	0060	0026	0010	0004	0001	0000	0000	..	..	..	..	1.00	104,619
0–5	7265	2034	0632	0059	0010	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	1.00	3,869
5–10	3156	2090	2099	1466	0783	0318	0073	1010	0004	0001	..	..	..	..	..	..	..	..	..	..	..	..	1.00	7,288
10–15	1443	1274	1682	1692	1570	1174	0675	0313	0114	0046	0014	0003	..	..	..	..	..	..	..	..	..	..	1.00	15,506
15–20	0710	0773	1187	1413	1492	1299	1133	0861	0557	0317	0150	0068	0025	0010	0004	0001	..	..	..	..	..	..	1.00	25,865
20–25	0386	0458	0754	1058	1212	1214	1161	1030	0884	0740	0529	0279	0163	0077	0032	0016	0005	0001	0001	..	..	..	1.00	34,308
25–30	0273	0336	0451	0685	0916	1012	1042	0978	0959	0908	0844	0644	0125	0271	0128	0070	0039	0009	0007	0000	0001	0002	1.00	8,539
30–35	0000	0000	0000	0000	1334	0000	2667	0667	0667	0000	2667	0000	0667	0667	0667	..	..	..	..	..	..	..	1.00	15
*40–44	1126	0853	1093	1215	1238	1095	0934	0743	0574	0440	0309	0177	0104	0055	0024	0013	0005	0001	0001	0000	0625	0556	1.00	95,390
0–5	9383	0553	0054	0005	0005	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	1.00	2,026
5–10	6213	2060	1021	0459	0152	0069	0016	0003	0007	..	..	..	..	..	..	..	..	..	..	..	..	..	1.00	3,029
10–15	2869	1926	1745	1487	0964	0532	0293	0110	0050	0016	0005	0003	..	..	..	..	..	..	..	..	..	..	1.00	5,738
15–20	1292	1064	1492	1582	1529	1205	0833	0470	0279	0133	0060	0016	0012	0003	0001	0017	0007	0003	0001	0000	0001	..	1.00	9,914
20–25	0593	0606	0949	1220	1380	1285	1178	0943	0684	0477	0307	0157	0080	0036	0013	0050	0028	0009	0002	0001	0001	0001	1.00	24,368
25–30	0350	0347	0538	0805	1005	1080	1157	1071	0995	0874	0692	0482	0301	0167	0087	0029	0010	0007	0002	0001	0000	0000	1.00	30,339
30–35	0257	0249	0355	0497	0641	0800	0888	1083	1133	1036	1012	0795	0619	0393	0242	..	..	..	..	..	..	..	1.00	6,816
35–40	0000	1429	0000	0000	1428	0000	0000	2857	0000	1429	0000	0000	1429	1428	..	..	..	..	..	..	..	..	1.00	7
*45–49	1142	0680	0849	1010	1091	1030	0971	0829	0700	0567	0438	0292	0188	0106	0056	0028	0013	0006	0002	0001	0001	0000	1.00	82,237
0–5	9890	0010	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	1.00	970
5–10	9335	0520	0099	0031	0015	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	1,308
10–15	5831	2149	0991	0600	0214	0119	0052	0019	0020	0005	..	..	..	..	..	..	..	..	..	..	..	..	1.00	2,009
15–20	2491	1705	1759	1605	1153	0622	0338	0145	0085	0057	0020	0006	0014	..	..	..	..	..	..	..	..	..	1.00	3,520
20–25	1031	0894	1261	1452	1544	1313	0935	0660	0421	0198	0099	0053	0028	0013	0005	0002	0001	..	..	..	..	..	1.00	8,363
25–30	0553	0417	0672	0946	1164	1264	1218	1164	0956	0686	0460	0246	0145	0060	0027	0012	0007	0003	00005	..	..	..	1.00	18,424
30–35	0330	0277	0351	0505	0741	0888	1012	1067	1117	1133	0960	0675	0458	0256	0124	0055	0030	0012	0007	00004	0001	0001	1.00	21,319
35–40	0237	0204	0236	0278	0421	0559	0610	0878	0994	1148	1166	1030	0937	0624	0348	0182	0090	0029	0013	0009	0002	0005	1.00	5,435
40–45	..	..	..	2222	..	..	..	1111	..	..	..	..	2223	1111	1111	..	1111	..	..	..	..	1111	1.00	9
*50–59	1140	0543	0652	0806	0927	0956	0918	0896	0825	0730	0589	0407	0290	0164	0083	0039	0021	0008	0004	0001	0000	0001	1.00	61,447

TABLE CXXV.—Shewing for Various Durations of Existing Marriage the Proportion of Women of Various Groups of Ages, who Bore 1, 2, 3 . . . n Children, the Total for each Age-group between the Limits of Duration of Marriage being Unity. Australia, Census of 3rd April, 1911. Duration and Age-polyphorous Distribution—*continued.*

Duration of Marriage. and Age.	Proportion of the Total of Women within the Indicated Limit of Duration of Existing Marriage who Bore Children to the Number of k, in which k =																							Total No. for the Duration.
	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	over 20	T'tals	
0–5	1000	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	1.00	466
5–10	9983	0017	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	1.00	579
10–15	8995	0701	0245	0035	0012	0000	0012	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	1.00	856
15–20	5185	2346	1432	0617	0305	0066	0033	0008	0000	0008	0000	..	..	..	..	..	..	..	..	..	..	..	1.00	1,215
20–25	2327	1570	1565	1578	1182	0744	0453	0295	0129	0105	0020	0020	0008	0004	..	..	..	..	..	..	..	..	1.00	2,472
25–30	0954	0770	0944	1210	1395	1475	1279	0838	0512	0277	0180	0093	0031	0021	0017	0002	0002	..	..	..	..	..	1.00	5,167
30–35	0528	0358	0495	0730	0864	1105	1143	1250	1140	0900	0687	0391	0209	0115	0051	0022	0010	0002	..	..	..	..	1,00	10,294
35–40	0342	0236	0294	0412	0579	0664	0791	0979	1157	1183	1120	0858	0646	0376	0193	0082	0048	0020	0007	0007	0003	0003	1.00	13,529
40–45	0340	0172	0208	0214	0277	0385	0566	0668	0828	0997	1391	1280	1048	0765	0380	0250	0132	0057	0030	0006	0000	0006	1.00	3,321
45 and over	..	..	..	..	..	..	..	..	..	..	1.000	..	..	..	..	..	..	..	..	..	..	..	1.00	1
*54–59	1222	0495	0540	0653	0743	0823	0847	0878	0878	0799	0734	0539	0384	0236	0118	0058	0031	0012	0005	0003	0001	0001	1.00	37,900
0–15	1.00	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	1.00	905
15–20	9559	0441	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	1.00	499
20–25	5186	2147	1469	0836	0249	0057	0045	0011	..	..	..	..	..	..	..	..	..	..	..	..	..	..	1.00	885
25–30	2170	1319	1513	1542	1416	0959	0554	0257	0166	0052	0029	0023	..	..	..	..	..	..	..	..	..	..	1.00	1,751
30–35	0970	0631	0830	0995	1331	1384	1297	1026	0656	0442	0267	0109	0034	0016	0009	0003	..	..	..	..	..	..	1.00	3,216
35–40	0604	0336	0432	0568	0789	0988	1186	1148	1271	0983	0750	0435	0315	0080	0054	0024	0017	0009	0006	0001	0001	0003	1.00	6,898
40–45	0320	0202	0246	0300	0452	0556	0736	0945	1053	1209	1292	0988	0762	0476	0246	0125	0053	0025	0008	0001	0001	0004	1.00	8,424
45 and over	0267	0184	0121	0188	0280	0350	0451	0605	0839	0976	1314	1239	1331	0764	0521	0300	0146	0075	0033	0008	0004	0004	1 00	2,397
*60–64	1312	0437	0479	0542	0677	0742	0828	0845	0884	0834	0805	0588	0475	0257	0149	0078	0037	0020	0007	0001	0001	0002	1.00	25,065
0–15	1.00	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	1.00	456
15–20	9953	0047	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	1.00	214
20–25	9043	0798	0106	0027	0000	0026	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	1.00	376
25–30	5522	1936	1229	0842	0320	0067	0067	0017	..	..	..	..	..	..	..	..	..	..	..	..	..	..	1.00	594
30–35	2137	1269	1644	1434	1269	0932	0502	0356	0238	0119	0055	0018	0027	..	..	..	..	..	..	..	..	..	1.00	1,095
35–40	0379	0663	0693	1003	1197	1460	1443	1038	0767	0443	0241	0065	0069	0013	0009	0017	..	..	..	..	..	..	1.00	2,322
40–45	0562	0268	0364	0490	0637	0623	1112	1270	1354	1167	0915	0564	0335	0152	0099	0050	0025	0007	..	..	..	..	1.00	4,363
45 and over	0320	0185	0161	0265	0350	0427	0600	0772	1037	1112	1397	1126	0961	0619	0319	0181	0087	0046	0018	0011	0000	0006	1.00	7,220
*65–69	1353	0415	0417	0508	0581	0617	0788	0837	0928	0858	0883	0647	0516	0312	0165	0094	0044	0022	0008	0005	0000	0002	1.00	16,640
0–20	1.00	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	1.00	351
20–25	9929	0071	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	1.00	140
25–30	9112	0800	0044	0044	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	1.00	225
30–35	4179	2161	2133	0692	0547	0087	0086	0058	0029	0028	..	..	..	..	..	..	..	..	..	..	..	..	1.00	347
35–40	1803	1273	1273	1758	1591	1091	0433	0485	0212	0045	0015	0015	..	..	..	..	..	..	..	..	..	..	1.00	660
40–45	0852	0549	0758	0974	0129	1427	1446	1090	0773	0565	0240	0139	0046	0047	0000	0008	0007	..	..	..	..	..	1.00	1,293
45 and over	0382	0221	0212	0312	0419	0587	0707	1008	1103	1255	1248	0922	0692	0474	0239	0115	0062	0013	0011	0008	0000	0013	1.00	6,281
*70–74	1408	0417	0419	0498	0559	0682	0713	0869	0869	0930	0878	0644	0474	0327	0161	0079	0043	0009	0008	0005	0000	0008	1.60	9,297
0–30	1.00	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	1.00	278
30 35	9405	0476	0000	0000	0119	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	1.00	84
35–40	4861	2222	1806	0347	0417	0139	0000	0069	0000	0139	..	..	..	..	..	..	..	..	..	..	..	..	1.00	144
40–45	1385	1053	1745	1717	1052	0975	1025	0554	0222	0166	0000	0028	0000	0055	0000	0000	0028	..	..	..	..	..	1.00	361
45 and over	0381	0304	0396	0413	0534	0785	0927	0930	1072	1084	1122	0694	0661	0354	0198	0092	0027	0009	0012	0006	..	..	1.00	3,387
*75–79	1425	0416	0524	0487	0531	0712	0825	0790	0872	0881	0893	0555	0527	0287	0157	0073	0024	0007	0009	0005	..	..	1.00	4,254
0–35	1.00	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	1.00	117
35–40	9744	0256	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	1.00	39
40–45	3553	2237	1447	1447	0658	0263	0132	0132	0131	..	..	..	..	..	..	..	..	..	..	..	..	..	1.00	76
45 and over	0528	0308	0473	0617	0672	1001	1042	1103	1076	0925	0822	0624	0446	0185	0110	0041	0014	0012	..	..	..	..	1.00	1,459
80–105	1532	0373	0473	0597	0609	0875	0905	0958	0934	0798	0710	0538	0384	0160	0095	0035	0012	0012	..	..	..	..	1.00	1,691

* Totals for ages indicated. Ages at the time of the Census.

36. **Fecundity-distributions according to age at marriage.**—By subdivision, according to duration of marriage, of the numbers in Table CXXIII., pp. 338-9, and subsequent rearrangement, tables can be prepared giving very approximately the distributions corresponding to the ages at marriage[1]. As this involves the relative numbers marrying at successive ages, it is essential to know the frequency of marriage at given ages. This is furnished by Table LIV., p. 190[2]. The results are as follow :—

TABLE CXXVI.—Shewing the Relative Number of Marriages according to Ages of Brides. Australia, 1907-1914,[3] and the Average Age for each Year Group.

Alleged age (last birthday) ..	12	13	14	15	16	17	18	19	20	21	22	23	24	25
Mean age ..	12.66	13.66	14.67	15.67	16.61	15.57	18.54	19.52	20.52	21.49	22.49	23.49	24.49	25.49
No. of marriages per 1,000,000	3	13	242	1,620	7,992	22,885	43,889	* 64,027	* 81,033	* 90,337	92,609	87,491	79,199	68,610

Alleged age (last birthday, ..	26	27	28	29	30	31	32	33	34	35	36	37	38	39
Mean age ..	26.49	27.48	28.48	29.48	30.48	31.49	32.49	33.49	34.49	35.49	36.49	37.49	38.49	39.49
No. of Marriages per 1,000,000	58,749	48,897	40,286	33,259	26,627	21,480	16,927	14,553	† 11,548	† 10,451	† 9,415	† 8,444	† 7,540	† 6,703

Alleged age (last (birthday) ..	40	41	42	43	44	45	46	47	48	49	50	51	52	53 to 95
Mean age ..	40.49	41.49	42.49	43.49	44.49	45.49	46.49	47.49	48.49	49.49	50.49	51.49	52.49	..
No. of marriages per 1,000,000	† 5,931	† 5,225	† 4,584	† 4,003	† 3,481	† 3,014	† 2,598	† 2,230	† 1,906	† 1,623	† 1,375	† 1,160	† 975	7,064

* Smoothed for misstatement of age. † Smoothed to a curve by a multiplier changing regularly, viz., the reciprocals of 1.105, 1.110, 1.115, etc.

The preceding table shews that, from the age 18 onwards, the average age is, sensibly, the age last birthday plus one half-year, and no serious error will result if it be so taken even for the ages earlier than 18. Hence a correction can be readily made for the effect of mortality, and a synthetic table prepared in the following way :—

Let a, b, etc., denote the marriages at ages (last birthday) x_1, x_2, etc., reduced for a half-years' mortality ; a', b', etc., these reduced for one and a half years' mortality ; a'', b'', etc., the same reduced for two and

[1] Original compilation according to age at marriage is, of course, the best method of obtaining the proper numbers.

[2] This gives 8 years' experience in Australia of the frequency of marriage at different ages, the total cases being 301,918.

[3] These numbers are deduced from those shewn on pp. 190-191 by distributing the 111 unspecified cases.

a half years' mortality, the mortality being both of husbands and wives,[1] and so on. Then, ignoring migration, the numbers according to age, as, at a census, and for a given duration of marriage, will be as shewn in the following table, viz. :—

TABLE CXXVII.—Scheme of Compilation of Numbers according to Duration of Marriage.

Durations of Marriage.	Age at Census.							
	x_1	x_2	x_3	x_4	x_5	x_6	x_7	etc.
0 – 1 ..	a	b	c	d	e	f	g	etc.
1 – 2 ..		a'	b'	c'	d'	e'	f'	etc.
2 – 3 ..			a''	b''	c''	d''	e''	etc.
3 – 4 ..				a'''	b'''	c'''	d'''	etc.
4 – 5 ..					a^{iv}	b^{iv}	c^{iv}	etc.

The total numbers of married women for durations of 0 to 5 years, 5 to 10 years, etc., are consequently :—

$$(564)\ldots {}_0M_5 = (a) + (a' + b) + (a'' + b' + c) + (a''' + b'' + c' + d) + (a^{\text{iv}} + b''' + c'' + d' + e) + (b^{\text{iv}} + c''' + \ldots + f) + \text{etc.}$$

$$(565)\ldots {}_5M_{10} = (a^{\text{v}}) + (a^{\text{vi}} + b^{\text{v}}) + (a^{\text{vii}} + b^{\text{vi}} + c^{\text{v}}) + \text{etc.}$$

$$(566)\ldots {}_{10}M_{15} = (a^{\text{x}}) + (a^{\text{xi}} + b^{\text{x}}) + (a^{\text{xii}} + b^{\text{xi}} + c^{\text{x}}) + \text{etc.; etc.; etc.}$$

It is obvious that a synthetic table can be prepared by means of which the partition can be effected of a group of married women between given limits of age and duration of marriage : in this way the mean age of any element may also be readily ascertained. Obviously the successive quantities vertically are, with sufficient precision for the purpose in view, respectively—in actuarial notation :—

$$m_x(1-\tfrac{1}{2}q'_x)\,;\; m_{x-1}(1-\tfrac{1}{2}q'_{x-1}).\,p'_{x-1}\,.\,p'_x\,;\; m_{x-2}(1-\tfrac{1}{2}q'_{x-2}).\,p'_{x-2}\,.\,p'_{x-1}\,.\,p'_x$$

m denoting the number of marriages, according to the age of the woman,

[1] For rigorous results the fact must be taken into account that the death of husbands also removes the women from the category "married." Hence the correction for mortality includes the probable number of deaths of wives, and of husbands, diminished, however, by the joint deaths, which are counted, of course, once only.

q'_x the probability of a woman[1] of age x either dying or becoming a widow within one year, and p'_x the probability of living in wifehood one year.[2]

Adopting the roughly approximate method we obtain from the data in Table LIV., pp. 190-1,[3] the figures shewn in Table CXXVIII. hereunder.

TABLE CXXVIII.—Shewing Example of Computation of Distribution of Numbers according to Age at and Duration of Marriage.

Duration of m'rriage	Ages at Census.															
	12	13	14	12–14	15	16	17	18	19	15–19	20	21	22	23	24	20–24
0–1	1	4	73	78	489	2,409	6,898	13,227	18,084	41,107	20,198	32,616	27,898	26,351	23,854	130,917
1–2	..	1	4	5	73	488	2,404	6,878	13,190	23,033	18,028	20,132	32,502	27,795	26,249	124,706
2–3	..	..	1	1	4	73	488	2,398	6,858	9,821	13,149	17,969	20,062	32,382	27,687	111,249
3–4	..	..	..	0	1	4	73	486	2,391	2,955	6,837	13,106	17,906	19,988	32,256	90,093
4–5	..	..	..	0	0	1	4	73	485	563	2,384	6,815	13,060	17,840	19,910	60,009
0–5	1	5	78	84	567	2,975	9,867	23,062	41 008	77,479	60,596	90,638	111,428	124,356	129,956	516,974
5–6	..	..	..	..	..	..	1	4	73	78	483	2,376	6,791	13,012	17,771	40,433
6–7	..	..	..	..	..	..	..	1	4	5	72	481	2,368	6,766	12,962	22,649
7–8	..	..	..	..	..	..	..	..	1	1	4	72	479	2,359	6,740	9,654
8–9	..	..	..	..	..	..	..	..	..	0	1	4	72	477	2,350	2,904
9–10	..	..	..	..	..	..	..	..	..	0	..	1	4	72	475	552
5–10	..	..	..	..	..	..	1	5	78	84	560	2,934	9,714	22,686	40,298	76,192

In the above results it is obvious that the age at marriage is at once approximately, though not exactly, obtained for each sub-group by subtracting the "duration" from the "age." The general result may be represented as follows :—

Let s, with appropriate suffixes, denote an element of S, the total between given limits of age and duration of a series of groups of s : then

$$(567) \ldots\ldots {}_{x,t}S_{x',t'} = \begin{array}{l} s_1 + s_2 + \ldots . \text{ etc.;} \\ + \\ s'_1 + s'_2 + \ldots . \text{ etc.} \\ + \\ \text{etc.} + \text{etc.} + \ldots . \text{ etc.;} \end{array}$$

[1] If husbands and wives were of the same age the probability of mortality which takes both into account would be approximately $1 - \frac{1}{2}q_m - \frac{1}{2}q_f + \frac{1}{4}q_m q_f$ for one half-year, the suffixes m and f denoting male and female respectively. Corresponding changes must also be made in the p_x factors.

[2] For greater rigour account must be taken of the exact interval; the half-year and year is not exactly correct, because the distribution is not uniform. This refinement, however, is not called for, because migration and other irregularities prejudice the data to a much greater extent.

[3] Similarly those given in Table CXXVI. could be used, and would perhaps be more reliable as they are smoothed results. The table includes only allowances for deaths of wives: the deaths of husbands have been omitted from consideration.

x and x' denoting the age limits, and t and t' the duration limits. Consequently if G be any given total of a series of groups, and g be the value of any component group, its approximate value is given by[1]

$$(568)\ldots\ldots g = G \,.\, s \,/\, S$$

the suffixes being the same for g and s.

For greater precision the values of s must be taken in Table CXXVIII. as modified not only by death but also by migration. In this way tables compiled according to the ages as at the Census can be reconstructed to furnish results according to the ages at marriage. The recasting of the ages may be effected as follows :—

Let x_c and x_e denote respectively the ages at which fertility commences (say 11 or 12), and ends (say 58 or 59) ; t_a and t_b the limits of any *duration of marriage* adopted in compilation, x_1 and x_2 being also *any age limits* adopted, as at the *moment to which the compilation refers* (the Census) ; then the whole *range of ages, x', at marriage* is given by :—

$$(569)\ldots\ldots x'_1 \text{ to } x'_2 \equiv (x_1 - t_b) \text{ to } (x_2 - t_a)$$

because on the inferior side an age will be included less than the lower age limit by the whole amount of the longer term of the duration, and on the superior side an age which is less only by the shorter term of the duration.

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[1] The group syntheses (a) in Table CXXVIII. further extended ; (b) those obtained by taking no account of deaths, and (c) those given by the Census are respectively as a, b, and c hereunder, a and b being reduced so as to give the same *total* as at the Census.

Owing to Census defects the Census results (c) cannot be regarded as absolutely correct; and owing to migration effects the synthesis results (a) or (b) will, of course, materially differ from the Census. It is evident, however, that the general correspondence between the Census and the synthetic results is sufficiently well established over a wide range of durations and ages, and that the correspondence furnishes a sufficient reason for relying upon the subdivision of the group-totals into their elements, especially for the earlier ages, and lesser durations of marriage. For age 20, viz., the age at which the misstatements are known to be large, the results are by (a) 7158 ; 66 : (b) 7157 ; 67 : (c) Census, 6953 ; 271. Corrections for mortality are probably an unnecessary refinement.

Duration.	Ages.					Totals.
	11–14	15–19	20–24	25–29	30–34	
			Synthesis (a)			
0–5	8	7,523	50,199	49,515	20,894	128,139
5–10	..	8	7,398	49,166	48,291	104,863
10–15	..	..	8	7,239	47,925	55,172
15–20	..	..	..	8	7,052	7,060
20–25	..	..	..	..	8	8
	8	7,531	57,605	105,927	124,171	295,242
			Synthesis (b)			
0–5	9	7,342	49,224	48,750	20,637	125,967
5–10	..	9	7,347	49,224	48,750	105,330
10–15	..	..	9	7,347	49,224	56,580
15–20	..	..	..	9	7,347	7,356
20–25	..	..	..	..	9	9
	9	7,356	56,580	105,330	125,967	295,242
			Synthesis (c)			
0–5	19	8,422	52,331	47,819	20,792	129,383
5–10	..	9	12,777	50,158	40.587	103,531
10–15	..	..	12	11,145	42,077	53,234
15–20	..	..	..	21	9,051	9,072
20–25	..	..	..	..	22	22
GTtl.	19	8,431	65,120	109,143	112,529	295,242

Hence it follows that if the age-groups and durations both change by a constant, the range of ages at marriage will be always as above (569). Since ages outside the limits x_c and x_e have no significance as regards fertility they may be ignored and consequently the earliest age of marriage may be taken say, as 11, and the latest say, as 58.

If $x'_2 = x'_1 + 1$ and $t_b + t_a + 1$, the range of ages extends over two years. The subdivision by applying the synthetic results can consequently give only approximate results and cannot sensibly attain to the accuracy of "direct compilation according to the age at marriage."

37. **Complete tables of fecundity.**—Still disregarding the age of husbands, complete tables of fecundity are based, as in the case of sterility, on the age at—and duration of—marriage. They give the proportions of those married at each age who bear 0, 1, 2, 3, etc., children, after the lapse of given durations of marriage. They are most serviceable if developed in the following way, viz. :—

Arguments for each Table.	*Body of Table.*
Age *at marriage*; and duration of marriage (for the child-bearing period only).	Proportion—for each increase of 1 year of age, and for each increase of 1 year in the duration of marriage—of the grand total of married mothers who bear 0 children (Sterility table); who bear 1 child; who bear 2 children; etc., etc.

From such a table as the above the derivative tables, previously indicated, can be readily prepared. Tables compiled on the basis of age at marriage could be distinguished as *gamogenesic*, etc., see p. 285. Fig. 93, and the notes thereto, exhibit in perspective the nature of the gamogenesic surfaces representing the proportion of wives who, having married at a given age, have borne during a duration of marriage of t years k children.

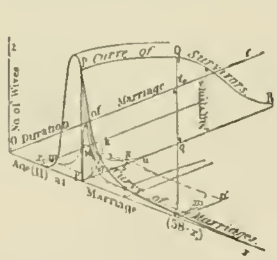

Fig. 93.

38. **Digenesic surfaces and diisogenic contours.**—If the husband's age be not ignored fecundity relations become greatly increased in complexity. For example, instead of a maternity rate or a birth-rate according to the age of wife, we have a series for each age of the

husbands ; the compilation-table becomes one of double entry, and the various fertility and fecundity-relations become correspondingly multiplied. If the ages of husbands and wives constitute respectively the abscissæ and ordinates of verticals, the heights of which represent the particular *birth-rates*, *maternity-rates* or else that characterise the combinations of ages in question, the surface defined by the totality of the verticals may be called a *digenesic surface*. That is to say, a vertical z of a digenesic surface is represented by :—

$$(570) \ldots\ldots z = \beta \,.\, F(x, y) = \beta_{xy}, \text{ or } \mathfrak{p}_{xy}$$

where β_{xy} is the birth-rate (or $\mathfrak{p}_{xy}$ is the maternity rate) for the group of wives of the ages y to $y + dy$, the ages of whose husbands lie between x and $x+dx$. To avoid circumlocution let the case be restricted to the consideration of birth-rate only. The curves $z =$ constant, or lines of equal birth-rate on this surface are *diisogenic contours* (they have been called by Körösi and Galton[1] *isogens*). Any series of ages x, y, x', y', x'', y'', etc., for which β_{xy} is constant may be called the *diisogenous ages*. The system of orthogonal trajectories which define the lines of the most rapid increase or decrease of birth-rate for any points through which they pass, may be called the meridians of these points.

The diisogeny of communities has not yet been generally investigated. Körösi has examined the question for the population of Budapest. For Australia the results are given hereinafter, and differ materially from the results for Budapest.

39. **Diisogenic graphs and their significance.**—Owing partly to paucity in the number of instances when they are distributed into small age-groups, coupled with the fact that even "physiological fecundity" is probably by no means uniform in the human race, and the further fact that the intentional restriction of fecundity is operative in widely different degrees, the crude data, distributed say in year-groups, do not give very definite indications of the exact position of the contours, though they reveal unmistakeably that the birth-rate is not only profoundly affected

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[1] See, "An estimate of the degrees of legitimate natality as derived from a table of natality compiled by the author from his observations made at Budapest." By Joseph Körösi, Phil. Trans., Vol. 186, Pt. II., pp. 781-875, 1896.

"Isogens," by Francis Galton, Proc. Roy. Soc., Lond., Vol. 55, p. 18.

The question had engaged the attention of a large number of persons, for example, A. N. Kiaer, 1875. Stieda, R. Boeckh, Bertillon, Neefe, and others.

by the age of the mother but is also by no means unaffected by the age of the father. It is also evident that, for a considerable range of single-year age-groups, the relation that β is constant when $x + my$ is constant, is approximately true, a point to which we shall recur; see (576) hereinafter. Thus the problem to be solved is that of determining, from the somewhat irregular surface indicated by the crude data, the more regular surface which ideally defines the general characteristic of birth-rate as related to the ages of the husband and wife, viz., x, y. The magnitude of the accidental differences between single-year age-groups is so large that meticulous precision is out of the question. Hence, using limited ranges of age we may proceed as follows in order to "smooth" the surface.

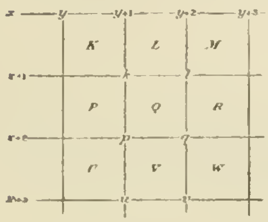

Fig. 94.

Let K, L, M, etc., Fig. 94 denote crude values of the birth-rates for the age-groups x to $x+1$, y to $y+1$, etc. Then *approximately*

$$(571)\ldots k = \tfrac{1}{4}(K+L+P+Q);\ l = \tfrac{1}{4}(L+M+Q+R);\ \text{etc., etc.}$$

If this does not give a sufficiently smooth surface we can reconstitute a smoothed value of Q, Q' say, from k, l, p, and q, thus :—

$$(572)\ldots Q' = \tfrac{1}{4}(k+l+p+q) = \tfrac{1}{16}\{4Q+2(L+P+R+V)+(K+M+U+W)\}$$

In this last the weight assigned to the values L, P, etc., and K, M, etc., vary reciprocally as the square of the distance to the centre of the group-square from the centre of Q. If the results are extremely irregular it may even be preferable to adopt :—

$$(573)\ldots\ldots Q' = \tfrac{1}{9}(K+L+M+P+Q+R+U+V+W)$$

instead of the preceding formula. The smoothed values being to hand, the contours may readily be drawn. When deemed necessary small corrections can first be applied to the heights for any systematic error introduced by the process of smoothing.

Since the group heights are too small for the central value when the surface is convex upwards, too great when it is concave upwards, a limitation which is accentuated when the mean of a number of heights is formed, as in (571) to (573). This error is analogous to that dealt with in Part IX., §§ 8 and 9, formulæ (311) to (323), and Table XV. The corrections may be ascertained as soon as the surface is approximately determined.

If z_k . . . . . . to z_w are the vertical heights of the centres of the squares in Fig. 94, for the average height z_0 for the whole area of 4 squares embraced between the lines joining the points K, M, W, U, would be rigorously

$$(574)\ldots.z_0 = \frac{1}{36}\{16z_q+4(z_l+z_r+z_v+z_p) + (z_k+z_m+z_w+z_u)\}$$

provided the sections of the surface are curves of the third or a lesser degree ; or, if the four component surfaces K, L, Q, P, etc., were "ruled surfaces," the height z_0 would be, also quite rigorously :—

$$(575)\ldots.z_0 = \frac{1}{16}\{4z_q+2(z_l+z_r+z_v+z_p) + (z_k+z_m+z_w+z_u)\}$$

If the external factors, therefore, are made unity in (574) and (575) the internal will be, respectively, $\frac{4}{9}$, $\frac{1}{9}$, $\frac{1}{36}$, and $\frac{1}{4}$, $\frac{1}{8}$, and $\frac{1}{16}$. It is evident from these results that the elimination of systematic error involves in all cases the assignment of a high "weight" to the central value. But it is equally certain that if the central values be considered liable to deviations from the general trend of the surface, which, compared with the systematic errors introduced are small, we may practically reach a better result by employing (571) or even (573).[1]

Another and more satisfactory method of obtaining values of β_{xy} is to smooth the series of the values of the type K, P, U, etc. ; *i.e.*, with y constant ; and independently those at right angles thereto, viz., K, L, M, etc., *i.e.*, with x constant. The means of the two results for each point are then adopted as a first smoothing, and the process repeated as often as is found necessary. This leads to more rigorous results, but can be readily employed only when the original results do not deviate largely from the general trend of the surface.

40. **Diisogens, their trajectories and tangents.**—The general nature of surfaces such as are here under consideration has been indicated in Pt. XII., §§ 21 and 22, pp. 201-203, and the fundamental formulæ of orthogonal trajectories have been given. The system of contours upon such surfaces (diisogens) probably do not conform to any simple geometrical specification ; the present imperfect data certainly do not point to their representation by any system of curves of a simple character, though the settlement of this question must remain for more extended investigation and more accurate data. At any point (x, y) whatsoever, dy / dx furnishes the relation by means of which the birth-rate equivalence

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[1] The question of the adjustment of such values, has been systematically treated by E. Blaschke, Ph. D., see his "Methoden der Ausgleichung von Wahrscheinlichkeiten," Wien, 1893. See also Phil. Trans., Vol. 186, II., pp. 870-5, 1895. See also Part XII. herein, § 39, pp. 230-2.

of pairs of ages may be expressed in the form $K\beta = x + G\Delta y =$ a constant. For we shall have, for the direction of the tangent to a diisogen, $dy/dx = \tan\theta = 1/G$ say. Hence it follows that

$$(576)\ldots x - \Delta y \cot\theta = x - \Delta y \frac{dx}{dy} = C; \text{ that is } x + G\Delta y = C$$

If k be the reciprocal of K then kC will be the constant value of the birth-rate for the diisogen in question. Ordinarily dx/dy is negative.

Parallelism of the tangents of diisogens to the x axis would imply that the increase of the age of the *husband* had no influence whatever on the birth-rate, while the parallelism of the tangents to the y-axis would denote that the age of the *wife* had no influence. If, therefore, the age of the wife has, in general, the preponderating influence, the diisogens must make a smaller angle with the x-axis than with the y-axis. If the diisogens are inclined 45° to each axis, then the birth-rate is constant when $x + y$ is constant.[1]

41. **Digenesic age-equivalence in two populations.**—As already shewn, the diisogens or their orthogonal trajectories determine the correlative changes in the ages of husbands and wives which give equivalence of birth-rate, *i.e.*, *diisogeny*. The *diisogenic factor G* in formula (576) for any pair of ages (*i.e.*, of husband and wife) is the coefficient which must be multiplied into the age of the wife so that the product, plus the age of the husband, will be continually proportional to the birth-rate. It holds, of course, only for a moderate range of age-differences about the point for which it is ascertained. Thus the expressions :—

$$(577)\ldots\ldots x - y\frac{dx}{dy} = \text{constant}\ ;\ x\frac{dy}{dx} - y = \text{constant},$$

apply only to a limited region. For two populations the differential coefficients are not identical. Hence, *for a given difference of age in the wife*, the equivalent difference of age in the husband is not the same. The factor to make one equal the other may be called the *masculine factor of age-equivalence, E*. Similarly the factor to make the difference in the wives' age equal, *for a given difference in the age of husband*, may be called the *feminine factor of equivalence, E'*. Suffixes can be used to denote the ages (of husband and wife) to which these factors exactly apply.

[1] Roughly speaking this represents the general character of the relation indicated (on Table 3, facing p. 852, Phil. Trans., Vol. 186, Pt. II.), by Körösi. Thus, for quite a large range of ages, the birth-rate would appear, according to that authority, to depend merely upon the sum of the ages of husband and wife, and not upon their individual ages. This condition may be called *equilateral diisogeny*, and is probably not a general condition.

Let δy denote any small difference in the age of wives at the point x, y, common to the populations A and B, the tangents to the diisogens making the angles θ_a and θ_b respectively, with the x-axis. Then since $\delta x_a = \delta y \cot \theta_a$ and $\delta x_b = \delta y \cot \theta_b$, we have

$$(578)\ldots\ldots E = \frac{\delta x_b}{\delta x_a} = \frac{\delta y \cot \theta_b}{\delta y \cot \theta_a} = \frac{G_b}{G_a} = \frac{\tan \theta_a}{\tan \theta_b} = \frac{dF_a(x)/dx}{dF_b(x)/dx}$$

Similarly—

$$(579)\ldots\ldots E' = \frac{\delta y_b}{\delta y_a} = \frac{\delta x \tan \theta_b}{\delta x \tan \theta_a} = \frac{G_a}{G_b} = \frac{1}{E}$$

that is, the masculine and feminine factors of age-equivalence are reciprocals.

42. **Birthrate-equivalences for given age-differences.**—The factors of age-equivalence merely disclose the equivalent differences of age for two populations for a given age-difference in either sex, but not the birth rate equivalence. This latter depends not only upon the direction of the tangents to the graphs of the diisogens in plan (*i.e.*, upon the tan gents to their horizontal projections), but also to the angle of slope ψ of the orthogonal trajectories. The tangent to any point Q, on a trajectory will be required. The angle it makes with the z-axis will be ζ so that $\zeta + \psi = 90°$. The following procedure will always be abundantly accurate for determining the age-equivalence and digenesic efficiency for any point Q the co-ordinates of which in plan are x_q, y_q. Let P′ P P″, Q′ Q Q″, and R′ R R″ in Fig. 95 be three diisogens (the values of which are known), crossed by the orthogonal trajectory P, Q, R, which in general is, of course, a curve of double curvature (tortuous curve). Let this trajectory be projected orthogonally on to the horizontal plane X O Y passing through P : this projection is the broken line P q r, the projections of short stretches of the diisogens being similarly the broken lines q′ q q″ and r′ r r″ ; P′ P P″ is itself in the plane of projection.

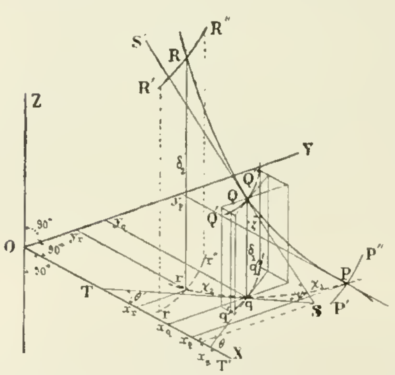

Fig. 95.

Let the curved line Pq be denoted by X_1, and the curved line Pr, of which Pq forms part, by X_2, measured along the curve ; and let also the difference of birth-rates for P and Q (*i.e.*, Qq) be denoted by δ_1, and the

difference for P and R be denoted by δ_2 ; then we may assume that the curved triangle, P Q R r q P in relation to lengths along the curved axis P q r, is, with sufficient precision, given by $\delta = b\chi + c\chi^2$, and therefore that the tangent at the point Q is $d\delta/d\chi = b + 2c\chi$. Thus we shall have :—

$$(580)..\tan \psi = \frac{1}{\chi_2 - \chi_1} \left\{ \delta_1 \frac{\chi_2}{\chi_1} - \delta_2 \frac{\chi_1}{\chi_2} + 2\chi_1 \left(\frac{\delta_2}{\chi_2} - \frac{\delta_1}{\chi_1} \right) \right\}$$

When $\delta_2 = 2\delta_1$, this becomes

$$(581)......\tan \psi = \frac{\delta_1}{\chi_1} \cdot \frac{\chi_2^2 - 2\chi_2\chi_1 + 2\chi_1^2}{\chi_2 (\chi_2 - \chi_1)}$$

and when, in addition, $\chi_2 = 2\chi_1$, the expression becomes, of course,

$$(582)......\tan \psi = \delta_1 / \chi_1 = \delta_2 / \chi_2$$

The direction of this line of slope tangentially passing through Q, and making the angle ψ with the horizontal plane, is shewn by the projection S q T, which is tangential to p q r, passing tangentially through the point q. It, of course, makes the angle θ' with the O X axis. Consequently the angles θ' and ψ, or their complements θ and ζ, give all the necessary relations required.

Since the line, Q q, in the figure $= \delta_1 = \beta_q - \beta_p$, viz., the difference of birth-rates indicated by the diisogens at P and Q, the horizontal equivalent thereof, Sq $= s$, say, measured in the direction of the tangent to the orthogonal trajectory at Q is :—

$$(583)......s = \delta_1 \cot \psi = (\beta_q - \beta_p) \tan \zeta$$

since $\zeta + \psi = 90°$. Thus, in plan, the rate of change of the birth-rate at any point x, y on a diisogen can be ascertained from the position of the diisogens on either side, and the position of the orthogonal trajectory through the point. Thus the age-equivalence of this difference of birth-rate is to be found by *dividing* by the sine and cosine of the angle which the orthogonal trajectory makes with the co-ordinate axes, θ and θ' ; their sum, $\theta + \theta' = 90°$. Consequently the masculine birth-rate-equivalence, H say, for wives of the one age, is :—

$$(584)......H = (\beta_q - \beta_p) \tan \zeta \operatorname{cosec} \theta$$

since $1 / \sin \theta = \operatorname{cosec} \theta$, and the feminine birthrate-equivalence H', for husbands of the one age, is

$$(585)......H' = (\beta_q - \beta_p) \tan \zeta \sec \theta$$

We thus have, from these two equations, for two populations, A and B for any common small difference of birth-rate, the ratio :—

$$(586)\ldots\ldots\quad \frac{H_b}{H_a} = \frac{\tan\zeta_b \operatorname{cosec}\theta_b}{\tan\zeta_a \operatorname{cosec}\theta_a}\,;\ \text{and}\ \frac{H'_b}{H'_a} = \frac{\tan\zeta_b \sec\theta_b}{\tan\zeta_a \sec\theta_a}$$

These relations, however, can be determined very readily from appropriate graphs of the populations.

43. **Diisogeny in Australia.**—Diisogeny is doubtless best exhibited by the maternity rates, not the birth-rates, the ratios to be ascertained being the proportions which the number of cases of maternity bear to the number of women at risk in any age-group with husbands of any age-group.

In order to ascertain the nuptial maternity rates of Australia according to *pairs of ages*, the nuptial cases of maternity have been taken out for the seven years 1908 to 1914 inclusive, that is, for the Census year 1911, and for the three years before and after that year. In order to relate these cases of maternity in age-groups to the numbers of married couples in the same age-groups at the Census, they have been divided, not by 7, but by a number which gave the true average, viz., 7.13143.[1] The results thus obtained are shewn by the uppermost of the figures in Table CXXIX. hereunder. Thus the results used are equivalent to a total 5,232,988 married women, among whom maternity was experienced 814,617 times. This gives an annual maternity rate of 0.15567. But of this number of married women, 7.6368 per cent. were 60 years of age and over, and 12.7667 per cent. were 55 years of age and over, so that about 87 per cent. were of child-bearing age. Hence the birth-rate for married

[1] This figure was ascertained in the following way :—The number of females in the years 1908 to 1914 inclusive were multiplied by a linear changing ratio (determined from the intercensal period 1901-1911) in order to obtain the numbers of married women during the years in question, the results being as hereunder :—

1908	..	2.018706	×	.33355	=	677,377
1909	..	2.058512	×	.33818	=	696,148
1910	..	2.103318	×	.34081	=	716,832
1911	..	2.156781	×	.34344	=	740,725
1912	..	2.224484	×	.34607	=	769,827
1913	..	2.301011	×	.34870	=	802,363
1914	..	2.361643	×	.35133	=	829,716

$$\frac{\text{Total population}}{\text{Census population}} \equiv \frac{31{,}697{,}287}{4{,}455{,}005} = 7.11498$$

$$\frac{\text{All females, 7 years}}{\text{Census females}} \equiv \frac{15{,}224{,}455}{2{,}141{,}970} = 7.10769$$

$$\frac{\text{Total married females, 7 years}}{\text{Census year (whole)}} \equiv \frac{5{,}232{,}988}{734{,}226} = 7.12722$$

$$\frac{\text{Total No. of married women in 7 years}}{\text{No. at Census date}}\ \frac{5{,}232{,}988}{733{,}773} \ldots\ldots \equiv 7.13162$$

$$\frac{\text{Total births in 7 years}}{\text{Births in Census year}} \ldots\ldots \equiv 7.12758$$

This was found to agree with other deductions as to the number of years, viz., $7 + \epsilon$ where ϵ was a small fraction (as shewn above) varying between 0.10769 to 0.13162. The actual division used was 7.13143, the reciprocal of which is 0.140224. This, multiplied into the births during the 7 years, gave the uppermost figure shewn in the table.

women of 13 to 54 years of age inclusive was 0.17845, or for women of 13 to 59 years of age inclusive, 0.16854. Körösi's results were 46,926 children from 71,800 married couples, in 4 years, that is 0.16339 per annum.[1]

The numbers of husbands and of wives recorded in the Australian Census of 3rd April, 1911, were not equal. It was deemed probable that the number of wives recorded would be the best basis for determining the distribution according to the age of the married women at the Census: in this way the numbers exposed to risk are ascertained in each age-group. The adjusted distribution[2] gives the numbers which constitute the denominators of the ratios.

In general there is a considerable number of cases for each pair of age-groups adopted; the table discloses the number. It is evident, however, that in extreme instances the numbers are small, and the maternity rates consequently ill-determined.[3] They may be regarded, however, as well ascertained where the number of mothers has been shewn in heavy figures.

The age-distribution as at the Census probably differs but little from the average distribution over the 7 years, which yielded the births: hence the ratios ascertained may be accepted as very closely representing the true amounts. The results are shewn in Table CXXIX. hereunder.

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[1] The average crude birth-rate for Australia for 1908-14 was .02745, and for Hungary for 1908-12, 0.3632. Apparently the Budapest maternity-rate is not larger than that of Australia.

[2] The following is a conspectus of the data:—

		Unspecified as regards Wife's Age.	Husbands whose Wives were with them.	Husbands whose Wives were Absent.	Total Husbands
Unspecified as regards		4,108	620,846	11,084	
husband's age ..	2,368	506	2,874	1,045	3,919
Wives whose husbands were with them ..	619,106	4,614	623,720	112,129	735,849
Wives whose husbands were absent ..	108,892	1,161	110,053	..	..
Total Wives ..	..	5,775	733,773	..	..

The adjustment was effected as follows:—The 506 doubly unspecified cases were divided into 185 and 321, that is in the proportion of each to their sum, and those were distributed proportionally among the wives and husbands unspecified. Next the 1161 wives, unspecified as regards age, were distributed proportionally among the 108,892 whose ages were given, thus making up the total 110,053. A like proceeding was followed in the case of the 1045 husbands, unspecified as regards age, so as to make the total 112,129. The individual totals were then reduced by multiplying throughout by 0.981485, so as to form the same aggregate 110,053 as in the case of the wives. One half of each was then distributed proportionally to the individual original numbers, thus making the grand total 733,773. See Table I., pp. 1106-7, Vol. III., Census Report.

[3] In general, tables prepared in this manner have the advantage that it may at once be seen whether any change of a ratio, necessary to make it conform to a general law, is probable or otherwise. A result like that shewn for ages of husbands 17, and of wives 16, viz., 1.55, is of course not impossible, but it would not be true for a very large number of cases.

TABLE CXXIX.—Shewing the Average Number of Cases of Maternity per annum for Married Couples of various Pairs of Ages, based upon 814,617 Cases of Maternity and 5,232,988 Couples; the Number of Couples to whom they may be referred, and the Corresponding Birth Rates for the Groups, in Australia during the 7 Years, 1908-1914. Diisogenic Distribution (for Maternity).

Ages of Husbands.	AGES OF WIVES.																								Total Cases of Maternity.	Total Number of Wives.
	13	14	15	16	17	18	19	20	21–24	25–29	30–34	35–39	40–44	45–49	50–54	55–59	60–64	65–69	70–74	75–79	80–84	85–89	90–94	95–99		
14	..	..	0.0	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..
	..	..	2	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	2
	..	..	0.0	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..
15	..	..	..	0.0	..	0.0	..	0.1	0.1	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	0.2	..
	..	..	..	1	..	1	..	2	?	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	4
	..	..	..	0.0	..	0.0	..	0.05	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..
16	..	..	0.0	0.8	1.0	0.1	0.4	0.1	0.6	0.1	..	..	..	..	..	..	..	..	..	..	..	..	..	..	3.1	..
	..	..	3	4	2	3	2	1	1	1	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	17
	..	..	0.0	0.2	0.5	0.03	0.2	0.1	0.6	0.1	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..
17	..	0.4	0.3	2.1	3.9	3.1	3.1	1.8	1.4	2.0	0.1	..	..	..	..	..	..	..	..	..	..	..	..	..	18.2	..
	..	..	1	2	8	6	10	6	15	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	48
	..	?	0.3	1.55	0.49	0.52	0.31	0.30	0.09	?	?	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..
18	..	0.1	1.0	7.0	15.8	21.9	15.8	8.4	10.1	0.4	0.1	..	..	..	..	..	..	..	..	..	..	..	..	..	80.6	..
	..	..	4	10	31	57	39	20	38	6	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	203
	..	..	0.25	0.70	0.51	0.38	0.41	0.42	0.28	0.07	?	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..
19	0.3	0.0	2.4	13.6	30.7	66.9	71.5	43.0	53.6	7.3	0.7	0.0	..	..	..	..	..	..	..	..	..	..	..	..	290.0	..
	?	?	?	14	67	111	98	73	126	29	8	3	..	..	..	..	..	..	..	..	..	..	..	..	..	529
	..	..	..	0.971	0.458	0.602	0.729	0.589	0.425	0.251	0.09	0.0	..	..	..	..	..	..	..	..	..	..	..	..	..	..
20	0.0	0.0	1.4	13.6	44.2	94.8	142.9	134.8	185.1	24.7	3.2	0.8	0.0	..	..	..	..	..	..	..	..	..	..	..	645.5	..
	1	4	16	29	86	180	255	302	434	75	12	3	3	..	..	..	..	..	..	..	..	..	..	..	..	1,400
	0.0	0.0	0.09	0.468	0.513	0.526	0.560	0.446	0.426	0.329	0.27	0.27	0.0	..	..	..	..	..	..	..	..	..	..	..	..	..
21	0.1	2.4	8.3	62.3	228.1	578.1	1,048.9	1,338.2	6,067.8	1,747.9	180.7	27.9	4.3	0.3	0.0	..	..	..	..	..	..	..	..	..	11295.3	..
to	?	3	35	126	451	1,099	1,916	2,889	13,541	5,028	762	154	42	20	5	..	..	..	..	..	..	..	..	..	..	26,071
24	..	0.8	0.237	0.494	0.506	0.526	0.547	0.463	0.447	0.347	0.237	0.181	0.10	0.015	0.0	..	..	..	..	..	..	..	..	..	..	..
25	0.1	0.4	5.2	29.7	124.1	361.1	706.7	1,126.1	10,446.9	13,001.3	2,321.9	302.6	30.9	1.8	0.0	0.0	..	..	..	..	..	..	..	..	28458.8	..
to	..	1	24	99	301	725	1,413	2,567	26,001	38,398	8,387	1,435	275	81	32	8	1	1	..	..	..	..	..	..	..	79,749
29	?	0.4	0.216	0.300	0.301	0.498	0.429	0.477	0.401	0.338	0.276	0.210	0.112	0.022	0.0	0.0	..	..	..	..	..	..	..	..	..	..
30	..	0.1	1.4	8.8	32.7	90.2	193.9	374.1	4,383.8	11,421.4	9,338.4	1,532.5	137.4	4.6	0.3	0.0	..	..	..	..	..	..	..	..	27519.6	..
to	..	..	5	37	68	232	441	904	11,814	38,291	38,816	8,672	1,635	378	91	22	8	4	1	..	..	..	..	..	..	101,415
34	..	?	0.28	0.238	0.495	0.388	0.439	0.413	0.371	0.298	0.243	0.176	0.083	0.0122	0.0033	0.0	..	..	..	..	..	..	..	..	..	..
35	..	..	0.8	3.4	10.2	30.4	64.6	113.6	1,381.8	5,153.9	8,024.6	5,974.8	630.1	15.3	0.4	0.0	..	..	..	..	..	..	..	..	21403.9	..
to	..	..	4	6	31	93	138	281	3,897	17,480	36,510	34,426	7,844	1,476	282	67	12	9	2	1	..	..	..	..	..	102,559
39	..	..	0.2	0.57	0.329	0.327	0.467	0.404	0.354	0.294	0.220	0.173	0.0803	0.00103	0.0014	0.0	..	..	..	..	..	..	..	..	..	..

Ages.	13	14	15	16	17	18	19	20	21–24	25–29	30–34	35–39	40–44	45–49	50–54	55–59	60–64	65–69	70–74	75–79	80–84	85–89	90–94	95–99	Totals	Totals
40	..	..	0.4	0.3	4.1	6.9	20.8	35.6	450.7	1,745.5	3,876.4	5,302.7	2,509.2	61.7	0.4	0.0	..	..	..	..	..	..	..	..	14014.7	..
to	..	..	3	5	6	20	61	88	1,396	6,529	18,471	34,517	32,562	7,358	1,450	340	70	27	7	3	..	..	..	..	..	102,903
44	..	..	0.13	0.06	0.08	0.345	0.341	0.404	0.324	0.267	0.2098	0.1536	0.0775	0.00838	0.00028	0.0	..	..	..	..	..	..	..	..	..	..
45	..	0.1	0.1	0.7	1.5	3.6	7.9	10.5	155.9	612.3	1,322.3	2,587.4	2,160.3	297.1	0.7	0.0	..	..	..	..	..	..	..	..	7160.4	..
to	..	..	1	..	5	10	16	44	507	2,381	6,864	17,242	32,140	29,881	6,525	1,100	259	73	12	7	..	..	..	..	..	97,087
49	..	..	0.1	?	0.3	0.36	0.493	0.238	0.307	0.257	0.1922	0.1500	0.0672	0.00995	0.00011	0.0	..	..	..	..	..	..	..	..	..	..
50	..	..	..	0.4	0.8	1.9	2.4	4.5	47.8	177.9	401.6	727.1	859.7	187.3	3.5	0.0	..	..	..	..	..	..	..	..	2414.9	..
to	..	..	..	3	1	4	11	10	179	798	2,289	5,675	14,352	26,833	22,679	4,456	876	241	39	10	3	..	..	..	..	78,458
54	..	..	..	0.13	0.8	0.47	0.22	0.45	0.267	0.2229	0.1754	0.1281	0.0599	0.00698	0.00015	0.0	..	..	..	..	..	..	..	..	..	..
55	..	..	..	..	..	0.0	0.8	1.3	12.2	53.5	102.8	203.0	198.8	60.9	15.4	0.0	..	..	..	..	..	..	..	..	648.7	..
to	..	..	..	..	..	4	5	5	65	247	680	1,771	4,192	10,395	17,599	12,995	2,708	521	121	33	9	..	..	..	..	51,350
59	..	..	..	..	..	0.0	0.16	0.26	0.187	0.216	0.1511	0.1146	0.0474	0.00585	0.00087	0.0	..	..	..	..	..	..	..	..	..	..
60	..	..	..	..	0.1	0.1	0.3	0.1	5.7	17.8	38.1	54.1	59.7	11.5	0.7	0.0	..	..	..	..	..	..	..	..	188.2	..
to	..	..	..	..	..	1	2	1	33	100	265	596	1,438	3,453	7,583	10,590	8,319	1,834	378	65	15	1	2	..	..	34,677
64	..	..	..	..	?	0.1	0.15	0.1	0.172	0.178	0.143	0.0907	0.0415	0.00333	0.00009	0.0	..	..	..	..	..	..	..	..	..	..
65	..	..	..	..	..	..	..	0.7	2.2	7.6	14.8	17.6	16.7	5.3	0.0	0.0	..	..	..	..	..	..	..	..	64.9	..
to	..	..	..	..	..	..	..	..	17	45	115	307	626	1,494	3,150	5,192	7,466	5,989	1,281	234	35	8	..	..	..	25,869
69	..	..	..	..	..	..	..	?	0.129	0.106	0.128	0.057	0.0260	0.0037	0.0	0.0	..	..	..	..	..	..	..	..	..	..
70	..	..	..	..	..	..	..	..	0.3	1.1	1.8	3.9	3.1	2.0	0.0	0.0	..	..	..	..	..	..	..	..	12.2	..
to	..	..	..	..	..	..	..	..	6	19	43	133	238	542	1,151	2,014	3,411	4,700	3,541	801	98	14	6	1	..	16 717
74	..	..	..	..	..	..	..	..	0.05	0.058	0.042	0.029	0.013	0.004	0.0	0.0	..	..	..	..	..	..	..	..	..	..
75	..	..	..	..	..	0.0	0.0	0.0	0.3	0.4	1.9	1.8	2.1	1.1	0.0	0.0	..	..	..	..	..	..	..	..	7.6	..
to	..	..	..	..	..	1	0	1	2	16	21	42	98	202	395	632	1,277	2,332	2,593	1,786	332	36	8	..	..	9,774
79	..	..	..	..	..	0.0	0.0	0.0	0.15	0.025	0.090	0.043	0.021	0.005	0.0	0.0	..	..	..	..	..	..	..	..	..	..
80	..	..	..	..	..	0.0	0.0	0.0	0.0	0.1	0.7	0.7	0.4	0.1	0.0	0.0	..	..	..	..	..	..	..	..	2.0	..
to	..	..	..	..	..	1	0	0	1	2	11	18	32	61	117	176	307	587	908	891	549	82	26	..	..	3,829
84	..	..	..	..	..	0.0	0.0	0.0	0.0	0.05	0.63	0.39	0.013	0.002	0.0	0.0	..	..	..	..	..	..	..	..	..	..
85	..	..	..	..	..	..	..	..	0.0	0.1	0.0	0.1	0.0	0.0	0.0	0.0	..	..	..	..	..	..	..	..	0.2	..
to	..	..	..	..	..	..	..	..	1	0	1	5	7	12	21	42	67	117	158	221	156	89	7	..	..	904
89	..	..	..	..	..	..	..	..	0.0	?	0.0	0.02	0.0	0.0	0.0	0.0	..	..	..	..	..	..	..	..	..	..
90	..	..	..	..	..	..	..	..	..	0.0	0.0	0.0	0.0	0.0	0.0	0.0	..	..	..	..	..	..	..	..	0.0	..
to	..	..	..	..	..	..	..	..	..	1	0	0	1	5	1	3	8	32	34	29	34	20	5	..	..	173
94	..	..	..	..	..	..	..	..	..	0.0	0.0	0.0	0.0	0.0	0.0	0.0	..	..	..	..	..	..	..	..	..	..
95	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	0.0	..	..	..	..	..	..	..	..	..	..
to	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	1	1	5	5	5	6	5	..	..	..	28
99	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	0.0	..	..	..	..	..	..	..	..	..	..
100	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	0.0	..	..	..	..	..	..	..	..	..	..
and	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	4	0	3	..	..	..	..	..	..	..	7
over	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	0.0	..	..	..	..	..	..	..	..	..	..
T'tls	0.5	3.5	21.3	142.7	497.2	1259.1	2280.0	3192.9	23206.3	33975.3	25630.1	16737.0	6612.7	649.0	21.4											
	1	8	98	336	1,055	2,548	4,407	7,194	58,062	109,446	113,275	104,998	95,485	82.098	61,082	37,642	24,850	16,474	9,080	4,080	1,237	255	54	1	114229.0	733,773

NOTE.—The uppermost number denotes the average number of births deduced from 7 years' results. The middle number is the number of married women in the age-group at the Census of 1911, viz., the middle year of the 7 years referred to. The lowest number is the maternity rate deduced by dividing the uppermost by the middle number.

Diisogenic Surface. Australia.

Ages of Wives $= y$.

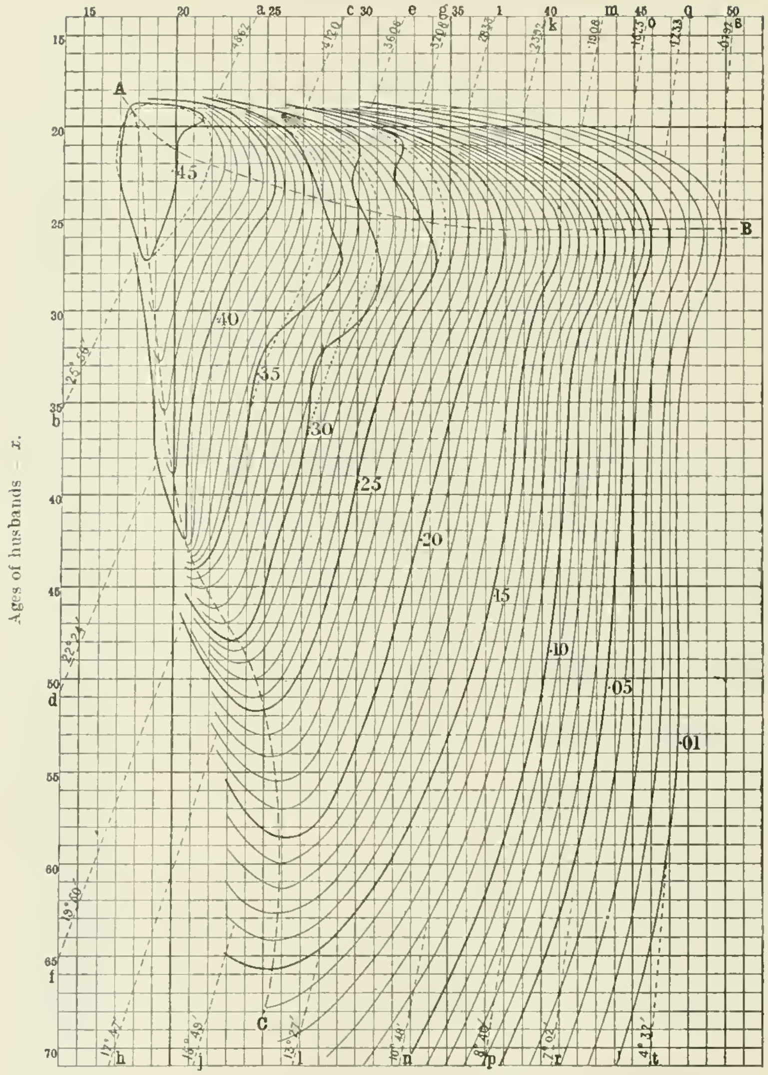

Fig. 96.

The maternity rates shewn, denote the ratio of the average number of cases of maternity occurring to the number of wives at risk, whose ages are shewn at the top line of the figure, their husbands' ages being shewn at the left hand side of the figure. The dotted lines roughly represent the major part of the surface between the principal meridians AB and AC.

The results tabulated above, and slightly smoothed, are shewn on Fig. 96, in which the heavy lines, viz., the birth-rates ·05, ·10, ·15, ... ·45, are first ascertained from the tabular results, and then smoothed. The thin lines, shewing differences of ·01 in the maternity rate, are then drawn in (having regard to second differences). The diisogens 0·45, 0·35,

0·30, and 0·25 exhibit some peculiarity. This is probably not due to physiological differences in fertility; the dotted lines are believed to better represent the character of the physiological law (modified, of course, by the incidence of social traditions). The broken line A B is one of the principal meridians of the surface, and denotes the ages where small differences in the ages of the *husbands* have no effect on the fertility. The broken line A C is the other principal meridian, and denotes the ages where (*very*) small differences in the age of the *wives* have no effect. As the higher ages—during the fertile period of woman's life—are reached, the age of the husband has apparently very little influence at least from 35 to 65 years of age. The graph, Fig. 96, however, requires no interpretation.

Diisogenic Surface. Budapest.

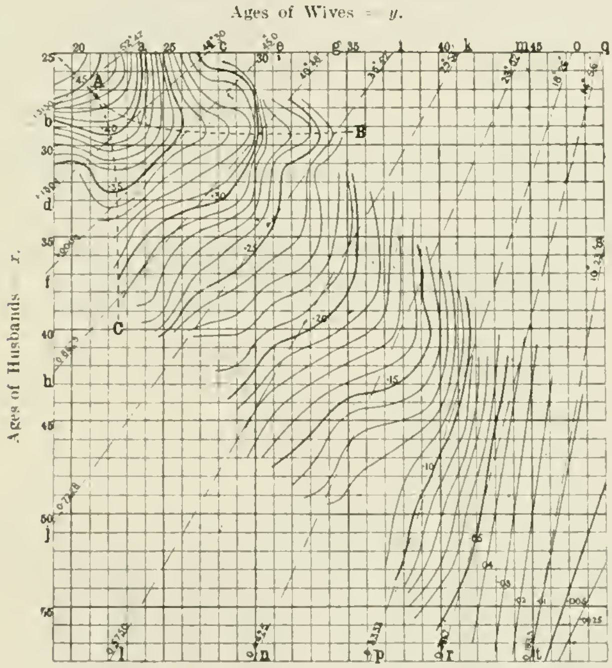

Fig. 97.

44. **Diisogeny generally.**—Körösi's results[1] for Budapest (as re-reduced by me) are shewn on Fig. 97. The results for individual ages shew great irregularity, but were computed as indicated hereinbefore, §§ 38 and 39. The irregularities doubtless would disappear with larger numbers. It will be seen in Fig. 97 that at age 36 for husbands and 30 for wives, the direction of the diisogen is inclined 45° to the axes x and y for a considerable length thereof and that, for an extended range of ages of husband and wife, the relation roughly holds that the birth-rate is

[1] Phil. Trans., Vol. 186, Pt. II., pp. 781-875.

constant when the *sum* of the ages of husband and wife are constant, and further that it *decreases* with *increase* in this sum.[1] The statement acquires greater generality, however, if put in another way, viz. :—

For ages greater than that of the maximum fertility of women and for those combinations of ages of husband and wife which are most common, the fertility-ratio may be regarded as represented—very roughly of course—by straight lines : that is to say, x and y being respectively the ages of husband and wife at the time of the birth, the fertility-ratio is constant when $kx + y$ is constant. These constant values are typically represented by the lines ab, cd, ef, etc., on Fig. 97, and by a′b′, c′d′, e′f′, etc., on Fig. 96. The pairs of ages, x and y, which give identical fertility-ratios, may be called *corresponding age-pairs.* They do not, of course, actually lie on straight lines, as is evident from either Fig. 96 or Fig. 97. Moreover *the fertility-ratio* (and thus the value of k) *diminishes with increase of the sum of the corresponding age pairs* (the age of maximum value having been passed). Obviously, also, k differs for various populations.[2] A rough general comparison of the Budapest (Körösi's) results for Budapest, and those for Australia is best indicated by shewing the position of the lines of "corresponding age-pairs" according to the value of the fertility-ratio (birth-rate).

TABLE CXXX.—Comparison of Approximate Lines of Equal Fertility according to Pairs of Ages ; Australia and Budapest.

Fertility or Maternity Ratio, viz., ratio of Births (or Cases of Maternity) to Married Women at risk.	AUSTRALIA (Maternity rates). Intersection Point of Lines of Equal Fertility. $x = -50$; $y = +55\frac{1}{2}$					BUDAPEST (Birth rate). Intersection Point of Lines of Equal Fertility. $x = \pm 0$; $y = +55\frac{1}{2}$					Reference letters.
	Angle = θ		Tangent = k		When $x = 0$ $y =$	Angle = θ		Tangent = k		When $x = 0$ $y =$	
	Obsd.	Calc.	Obsd.	Calc.		Obsd.	Calc.	Obsd.	Calc.		
	° ′	° ′			Years	° ′	° ′			Years.	
000 ..	..	4.0	..	..	..	..	9.0	..	..	..	..
001 ..	4.32	4.28	.0792	.0781	51.54	10.23	9.58	.1833	.1757	55.5	s t
0.05 ..	7.2	6.21	.1233	.1113	49.34	14.56	13.53	.2667	.2472	55.5	q r
0.10 ..	8.40	8.42	.1525	.1530	47.88	18.26	18.46	.3333	.3398	55.5	o.p
0.15 ..	10.48	11.3	.1908	.1953	45.96	23.52	23.39	.4425	.4379	55.5	m n
0.20 ..	13.27	13.24	.2392	.2382	43.54	29.54	28.32	.5750	.5437	55.5	k t
0.25 ..	15.49	15.45	.2833	.2820	41.34	35.52	33.25	.7228	.6598	55.5	i j
0.30 ..	17.47	18.6	.3208	.3269	39.46	40.48	38.18	.8632	.7898	55.5	g h
0.35 ..	19.50	20.27	.3608	.3729	37.47	45.0	43.11	1.0000	.9385	55.5	e f
0.40 ..	22.24	22.48	.4120	.4204	34.90	48.30	48.4	1.1301	1.1132	55.5	c d
0.45 ..	25.56	25.9	.4862	.4695	31.19	52.42	52.57	1.3127	1.3240	55.5	a b
0.50 ..	?	..	?	..	?	?	..	?	..	?	

The two systems of lines are:—

For Australia ; $k_b\,(x + 50) + y = 55.5$; and

For Budapest ; $k_b\,(x + o) + y = 55.5$.

[1] Or $x + Ky =$ constant, see (576). Galton and Körösi's suggestion that $k = 1$ is an extremely improbable one, and is not borne out by the data, as Körösi's own results shew.

[2] As already mentioned, the number of cases of maternity reviewed by Körösi was 46,926, occurring in the years 1889, 1890, 1891 and 1892, and these were attributable to 71,800 families according to the Census of 1st January, 1891, giving a rate of 16.339 per 100 families per annum over all. See p. 790, op. cit. The number is, of course, insufficient to determine the surface with great accuracy.

It would appear that the directions of these lines vary about equally for equal changes of rate since they are given by the formulæ.

$$(587)..\theta = 4^\circ + 47^\circ r, \text{ for Australia }; \ \theta = 9^\circ + 97\tfrac{2}{3}{}^\circ r \text{ for Budapest.}$$

These values are approximately correct for the regions within lines AB and AC on Figs. 96 and 97, but not outside those regions.

45. **Multiple Diisogeny.**—The equal frequency of twins, or of triplets, etc., according to pairs of ages may be called *multiple diisogeny*, the series of ages giving equal frequency being in this case also known as "corresponding pairs." The *twin digenous surface, triplet digenous surface*, etc., are the surfaces defined by the terminals of the z co-ordinates corresponding to the frequency of twins, triplets, etc., the x and y co-ordinates representing as before the ages of the husband and wife respectively.

In order that the results may be unequivocal, the ratios to be used should be those of the number of births "of at least $n + 1$ children," to the number of births "of at least n children." That is, the ratio of twins should be to the cases of maternity; the ratio of triplets should be to the cases where there were at least twins; of quadruplets to at least triplets, and so on. Suppose in a population P there were :—

A′	cases of maternity	in which only	*one*	child was born ;
B′	,,	,,	*two*	children were born ;
C′	,,	,,	*three*	,, ,, .

and let $A' + B' + C' + \text{etc.} = M$, the total cases of maternity. Then the maternity ratio for the population is :—

$$(588)......m_1 = (A' + B' + C' + \text{etc.}) / P = M/P.$$

The twin ratio, so taken as to include all mothers who had at least two children at a birth, is :—

$$(589)......m_2 = (B' + C' + \text{etc.}) / M = B/M.$$

The triplet ratio, or that based on all mothers who had at least three children at a birth, is :—

$$(590)......m_3 = (C' + \text{etc.}) / (B' + C' + \text{etc.}) = C/B,$$

and so on.

In this system we have :—

$$(591) \ldots . m_1 = M/P \; ; \; m_1 m_2 = B/P \; ; \; m_1 m_2 m_3 = C/P \; ; \text{ etc. ;}$$

that is, the population multiplied by the product of the ratios $m_1 . m_2 \ldots m_n$ gives the number of women bearing at least n children. The ratio m_n is thus the relative frequency with which a woman—who in any childbirth has given birth to n children—will have given birth to the $(n + 1)$ child on the same occasion.

For the 9 years, 1907 to 1915, in Australia there were in all 1,042,588 cases of maternity ; 10,630 cases of twins and triplets, and 100 cases of triplets : that is 1,031,858 single births, 10,530 cases of twins, and 100 cases of triplets. The ratios and their degree of fluctuation are shewn in the following table :—

Table CXXXI.—Shewing Frequency of Occurrence of Twins and Triplets (Nuptial and ex-Nuptial Cases combined). Australia, 1907-1915.

Year 1900.*	Cases of Maternity.	Twins including Triplets, etc.	Triplets.	Ratio of Twins to Cases of Maternity m_2	Ratio of Triplets to Cases of Twins. m_3	Ratio of Twins, 5 Year Average.	Ratio of Triplets, 5 Year Average.
7	109,305	1,042	13	.00953	.01247	—	—
8	110,491	1,065	6	.00963	.00563	—	—
9	112,921	1,142	14	.01011	.01225	.00996	.01057
10	115,609	1,189	13	.01028	.01093	.01011	.01053
11	120,957	1,236	14	.01022	.01132	.01021	.01034
12	131,726	1,350	16	.01025	.01185	.01024	.00946
13	134,343	1,369	8	.01019	.00584	.01031	.00870
14	136,576	1,406	11	.01029	.00782	—	—
15	133,444	1,417	10	.01062	.00706	—	—
Totals	1,105,372	11,216	105	.01015 *	.00936 *	—	—

* 1908-1914 gave 0.010311 and 0.00931. See p. 314 herein.

The 5-year averages shew the regularity of the ratios, and justify the combination of the results of a series of years for the purpose of examining the characteristics of multiple diisogeny.

46. **Twin and triplet frequency according to ages.**—The data for determining the ratios m_2 and m_3 according to formulae (589) and (590) are given immediately by the records of births, and—unlike the maternity ratios m_1, formula (588)—are independent of the Census results. The

numbers of unspecified cases are negligibly small. There is some uncertainty in the numbers for ages 18 to 21 inclusive, owing to misstatements as to age. The following table, based upon 1,035,439 mothers, 10,533 twins, and 104 triplets, gives the available results for 9 years.

TABLE CXXXII.—Shewing the Frequency of Twins and Triplets in Cases of Maternity,* according to the Age of the Mother and of the Father. Australia, 1907-1915.

Age† Groups.	Ages of Mothers. Mothers Under 19	19	20	21–24	25–29	30–34	35–39	40–44	45–54	All Ages of Mothers.	25–39	Triplets ÷ Twins 25–39	Triplets ÷ Twins All Ages.
Fathers Under 19	0 ; 3 527 .0057	0 ; 8 380 .0211	.. .. ..	.. .. ..	.. .. ..	.. .. ..	.. .. ..	.. .. ..	.. .. ..	0 ; 11 907 .0121	.. .. ..	.. .. ..	.. .. ..
19–20	0 ; 11 2,413 .0046	0 ; 9 1,923 .0046	0 ; 11 1,584 .0069	0 ; 4 2,154 .0002	0 ; 2 335 .0059	.. .. ..	.. .. ..	.. .. ..	.. .. ..	0 ; 37 8,409 .00440	0 ; 2 335 .0059	.. .. ..	.. .. ..
21–24	0 ; 33 7,985 .0041	0 ; 55 9,439 .0058	1 ; 83 12,159 .0068	5 ; 366 54,749 .0067	2 ; 137 15,413 .0089	1 ; 21 1,630 .0129	0 ; 6 275 .0218	.. .. ..	.. .. ..	9 ; 701 101,650 .00690	3 ; 164 17,318 .0095	.0183	.0128 .. ..
25–29	0 ; 13 4,664 .0028	0 ; 26 6,446 .0040	0 ; 55 10,271 .0054	2 ; 603 94,452 .00636	7 ; 1,075 117,756 .00913	6 ; 269 21,278 .0126	2 ; 34 2,723 .0124	0 ; 5 287 .0174	.. .. ..	17 ; 2,080 257,877 .00807	15 · 1378 151,757 .00908	.01088	.0082 .. ..
30–34	0 ; 3 1,204 .0025	0 ; 7 1,773 .0039	0 ; 18 3,355 .0054	3 ; 251 39,785 .00631	7 ; 928 103,618 .00895	5 ; 1,097 84,976 .01291	6 ; 219 13,976 .0157	0 ; 19 1,263 .0150	.. .. ..	21 ; 2,542 249,950 .01017	18 ; 2,244 202,570 .01108	.00525	.0083 .. ..
35–39	0 ; 0 407 ..	0 ; 3 594 .0051	0 ; 5 1,039 .0048	0 ; 73 12,564 .00581	3 ; 446 46,867 .00951	7 ; 905 72,990 .01240	10 ; 838 54,557 .01536	3 ; 67 5,707 .0117	0 ; 2 156 .0128	23 ; 2,339 194,881 .01200	20 ; 2,189 174,414 .01255	.00548	.0098 .. ..
20–44	.. .. ..	.. .. ..	0 ; 4 643 .0062	0 ; 26 4,093 .0064	1 ; 143 15,664 .00913	4 ; 429 35,051 .01224	5 ; 752 48,109 .01563	5 ; 282 22,723 .01241	0 ; 3 590 .0051	15 ; 1,639 126,873 .01292	10 ; 1,324 98,824 .01339	.00831	.0092 .. ..
45–49	.. .. ..	.. .. ..	0 ; 3 218 .0138	0 ; 9 1,379 .0065	2 ; 58 5,516 .0105	0 ; 152 11,936 .01274	8 ; 357 23,485 .01520	2 ; 229 19,418 .0118	0 ; 18 2,703 .0067	12 ; 826 64,655 .01278	10 ; 567 40.937 .01385	.0176	.0145 .. ..
50–54	.. .. ..	.. .. ..	.. .. ..	0 ; 2 509 .0039	1 ; 14 1,593 .0087	1 ; 56 3,639 .0154	3 ; 94 6,596 .0143	0 ; 72 7,775 .0093	0 ; 14 1,687 .0083	5 ; 252 21,799 .01156	5 ; 164 11,828 .01386	.0304	.0198 .. ..
55–59	.. .. ..	.. .. ..	.. .. ..	.. .. ..	0 ; 4 619 .0064	0 ; 9 946 .0095	1 ; 22 1,837 .0120	1 ; 29 1,828 .0158	0 ; 5 677 .0074	2 ; 69 5,907 .01168	1 ; 35 3.402 .01028	.0285	.0289 .. ..
60–64	.. .. ..	.. .. ..	.. .. ..	.. .. ..	0 ; 2 216 .0092	0 ; 6 342 .0175	0 ; 5 488 .0102	0 ; 8 540 .0148	0 ; 1 116 .0086	0 ; 22 1,702 .01292	0 ; 13 1,046 .01242	.0000	.0000 .. ..
65–89	.. .. ..	.. .. ..	.. .. ..	.. .. ..	.. .. ..	0 ; 9 292 .0308	0 ; 6 536 .0112	.. .. ..	.. .. ..	0 ; 15 828 .01811	0 ; 15 828 .01811	.0000	.0000 .. ..
All ages of Fathers	0 ; 63 17,260 .00366	0 ; 108 20,555 .00525	1 ; 179 29,269 .00612	10 ; 1,334 209,685 .00636	23 ; 2,809 307,597 .00913	24 ; 2,953 233,080 .01267	35 ; 2,333 152,582 .01529	11 ; 711 59,541 .01194	0 ; 43 5,929 .00725	104; 10533 1,035,439 .01017	All Triplets ÷ All Twins = .00988 ..	.. .. ..	.0099 .. ..
25–39	0 ; 16 6,275 .0025	0 ; 36 8,813 .0041	0 ; 78 14,665 .00532	5 ; 927 146.801 .00631	17 ; 2,443 268,241 .00913	18 ; 2,271 179,224 .01267	18 ; 1,091 71,256 .01531	3 : 91 7,257 .01254	0; 2 156 .0128	.. .. ..	.. .. ..	.. .. ..	.. .. ..
Triplets ÷ Twins 25–39	.0000	.0000	.0000	.0054	.0069	.0079	.0164	.0329	.0000	..	..	..	..
Triplets ÷ Twins All ages	.0000	.0000	.0056	.0075	.0082	.0081	.0150	.0155	.0000	.0099	..	..	..

* The table shews for various age-groups of mothers and fathers the numbers of cases of maternity, and of twins and triplets occurring during a period of 9 years. The first number is the number of triplets; the second—divided from the first by a semi-colon—is the number of twins; the numbers beneath, viz., on the second lines are the "cases of maternity"; the numbers on the third lines are the ratio of the occurrences of twins to the cases of maternity. These ratios are calculated by the formula (589). † The ages are "ages last birthday."

An examination of the individual columns in the table for any given age-group of wives discloses the fact that there are no systematic differences for various ages of the husband. This is confirmed by the combination of the results for considerable groups.[1] *The age of the husband*, though it has an unmistakable influence on the maternity ratio, *has no influence whatever on the twin-ratio.* It is equally clear that the age of the wife is correlated with the frequency of twins.

The graph of the results indicates that the initial part of the curve (*i.e.*, wives' ages up to 20 inclusive) does not conform to the general curve (owing perhaps to misstatements of age). The curve has a maximum at about 37½ years of age (*i.e.*, age 37 last birthday), and is nearly a straight line almost up to the maximum value.

The following table gives the probabilities for the *exact ages*, not "age last birthday."

TABLE CXXXIII.—Shewing, according to Age of the Mother, the Relative Frequency with which at least a Second Child is Born.* Australia, 1907-1915. Twin-ratios.

Age of Mother	Twin Ratio.	Age of Mother.	Twin Ratio.	Age of Mother.	Twin Ratio.	Age of Mother.	Twin Ratio.
11	.00100	22	.00605	33	.01299	44	.01070
12	.00137	23†	.00659	34	.01370	45	.00997
13	.00180	24	.00714	35	.01440	46	.00937
14	.00217	25	.00770	36	.01499	47	.00880
15	.00260	26	.00827	37	.01526	48	.00823
16	.00305	27	.00885	38	.01526	49	.00772
17	.00352	28	.00944	39†	.01502	50	.00725
18	.00400	29	.01007	40	.01470	51	.00680
19	.00449‡	30	.01075	41	.01380	52	.00636
20	.00500	31	.01146	42	.01260	53	.00593
21	.00552	32	.01221	43	.01155	54	.00551
22	.00605	33	.01299	44	.01070	55	.00510

* The table shews the ratio of cases of birth of two or more children to cases of maternity, the age being *exact* (*i.e.*, not age last birthday).

† The ratios have been ascertained with great precision for all the ages from 23 to 39. Later they are less accurate, but the number of cases is relatively small.

‡ The ratios are somewhat uncertain owing to misstatements of age.

[1] For example the following results were obtained :—

Ages of Husbands.	Ages of Wives.								
	Under 19.	19.	20.	21–24.	25–29.	30–34.	35–39.	40–44.	45–54.
Under 30	.0039	.0054	.0062	.0064	.0091	.0127	.0133	.0174	
25 to 39..	.0025	.0041	.0053	.0063	.0091	.0127	.0153	.0124	.0128
All ages of husbands ..	.0037	.0053	.0061	.0064	.0091	.0127	.0153	.0119	.0073

These clearly establish the fact that the age of the husband has no influence whatever.

47. **Apparent increase of frequency of twins with age of husbands.**—If ages greater than 40 be left out of consideration, and the material for ages (wives and husbands respectively) of 25 to 39 years of age alone be embraced, the values of the twin-ratios according to the ages of husbands and wives are given approximately by the following expressions, viz. :—

$$(592)\ldots\ldots m_2 = 0.0034 + 0.000228\,x\ ;\quad m'_2 = -0.0076 + 0.00060\,y.$$

These give the following results for husbands and wives respectively :—

Ages of Husbands	19–20	21-24	25-29	30-34	35-39	40-44	45-49	50-54	55-59	60-64	65-89
Data	.0059	.0095	.0091	.0111	.0125	.0134	.0139	.0139	.0103	.0124	.0181
Formula ..	.0080	.0086	0097	.0108	.0120	.0130	.0142	.0154	.0165	.0176	.0188
Ages of Wives	Under 19	19	20	21-24	25-29	30-34	35-39	40-44	45-54		
Data	.0025	.0041	.0053	.0063	.0091	.0127	.0153	.0125	.0128		
Formula	.0038	.0041	.0047	.0062	.0089	.0119	.0149	.0179*	.0224*		

* The straight line does not hold good for these ages.

This increase with the age of the husband is not, however, due to any influence the husband may be supposed to have upon multiple-births, but wholly to the fact of association in pairs according to age. The smaller coefficient 0.000228 (as compared with 0.000600) arises from the greater "spread" of the ages of the husbands.[1]

Although the attribution of the increased frequency with age to the husband is physiologically meaningless, nevertheless for rough estimates the method is valid, and so long as it is remembered that the effect is not due to increasing age of the husbands, there is no objection to this method of estimation.

48. **Triplet diisogeny.**—The numbers of triplets shewn on Table CXXXII. are quite insufficient to determine with any exactitude the digenous relations of triplets. The age-groups are too small. But if 30 be made a dividing age we get the following result :—

Husbands.	Wives.		Wives.	
	Ages under 30.	Ages 30 & over.	Ages under 30.	Ages 30 & over
Ages under 30	17 ; 2497 342650.	9 ; 335 25236.	.0068 .000050	.0269 .00036
Ages 30 and over	17 ; 2011 241775.	61 ; 5603 424284.	.0084 .000070	.0109 .000144
	Numbers.		Ratios.	

[1] If the ages were identical of husband and wife throughout, the maximum effect would be on a line making an angle of 45 degrees with either axis : consequently the ratio of multiple births if attributed to either sex would yield the same result.

As it has no influence on the occurrence of twins, it may be assumed as extremely unlikely that the age of the husband has any influence on the occurrence of triplets. This is confirmed by the above partitioning of the results, which shews opposite apparent influence. The results given in the final column of Table CXXXII. may therefore be taken as exhibiting the influence of the age of the wife. This influence can be expressed

$$(593)\ldots\ldots m_3 = -0.0044 + 0.00047\ y.$$

y being the exact age of the wife.

The results as ascertained from the data and as given by the formula are :—

Age	20	21–24	25–29	30–34	35–39	40–44	45–49
Data	.0056	.0075	.0082	.0081	.0150	.0155	.0000
Formula ..	.0050	.0064	.0085	.0109	.0132	.0155	?

Having regard to the number of available cases it is certain that the frequency of triplets increases with the age of the wife. The rate of increase 0.00047 agrees well with that of the rate of increase 0.00060 in the case of twins. On plotting the results according to the age of the husband it was found that the points on the graph constituted a curve, not a straight line.

49. **Frequency of twins according to age and according to order of confinement.**—The relation between the frequency of the birth of twins according to age and according to order of confinement can be roughly seen from the results given hereunder. According to the order of confinement the frequency is very closely given by the equation :—

$$(594)\ldots\ldots m'_2 = 0.0082 + 0.00114\ n - 0.0000185\ n^{2\cdot5},$$

the calculated and observed results being respectively :—

Previous confinements	0	1	2	3	4	5	6	7	8	9	10
Formula	.0083	.0094	.0105	.0115	.0123	.0129	.0135	.0138	.0140	.0141	.0139
Data	.0082	.0096	.0107	.0117	.0124	.0130	.0134	.0136	.0138	.0139	.0140
Corresponding age ..	26.78	28.25	29.92	31.32	32.24	33.01	33.58	33.86	34.14	34.28	34.42

And if the age corresponding to these values be inserted from Table CXXXIII., the values on the final line are obtained.

50.—**Unexplored elements of fecundity.**—To distinguish between the *effect of previous births* and age upon the frequency of maternity, of twins, etc., more comprehensive data are required than at present exist for

Australia. The effect is one which, so far as the maternity-ratio is concerned, reflects social tradition in a larger measure than the physiological law: the latter is modified but not obliterated. In the case of twins, triplets, etc., the physiological laws doubtless alone operate.

The records necessary to ascertain the characteristics of *digenous masculinity at birth* exist for only two years, and disclose the fact that the variations according to age are too large to admit of satisfactory analysis, unless say 10 years' material is available.

Assertions from time to time have been made to the effect that the characteristics of first-born children are often sharply differentiated from those of later children. If in the record of cases of mortality the ages of father and mother were also given, and the order of the birth of the deceased, the data for the consideration of this question as regards length of life according to age and to place in order of birth would be available.

This question, however, belongs more properly to the subject of mortality, and will not be further considered in this monograph.

XV.—MORTALITY.

1. **General.**—Human mortality may be considered statistically under two aspects, viz.,

(i.) A *general* one ; that is, the aspect which has regard to the aggregate *mortality from all causes of death* ; and

(ii.) A *particular* one ; that is, the aspect which takes account of *mortality from particular causes* or by particular modes of death.

Both will be referred to.

Deaths from particular causes or by particular modes are, in general, functions both of age and time, i.e., an individual rate of mortality μ', viz., the ratio of the number of deaths D' from a particular cause to that part of the population P' subject to the risk of such death, is :—

$$(595) \ldots\ldots\ldots \mu' = D' / P' = f(x, t)$$

x denoting age, and t time. It also varies with sex. In estimating the general rate of mortality it is convenient, although in many respects unsatisfactory, *virtually* to regard all persons in the population as equally subject to the risk of death from each cause [1] : hence the general rate of mortality may be regarded—subject to some limitations—as made up of the sum of the supposititious rates δ', δ'', etc., from each cause ; that is to say, the general rate of mortality is the ratio of total deaths during a unit of time to the total mean population, or algebraically :—

$$(596) \ldots\ldots\ldots \delta = \delta' + \delta'' + \text{etc.} \equiv \frac{D'}{P} + \frac{D''}{P} + \text{etc.} = \frac{D}{P}.$$

$D = D' + D'' +$ etc., denotes the total deaths, and P the mean population during the unit of time in question. This rate is known as the *crude death-rate*, and is obviously inconsistent with (595). If the age-distributions of all populations were substantially identical, this method of evaluating the rate of mortality could be regarded, for many purposes, as fairly satisfactory.[2] Inasmuch, however, as each particular rate, μ', etc., is a special function of age, the general rate δ is obviously also dependent upon the age and sex distribution. The distinction may thus be drawn between *class mortality* and *general mortality*, "class" denoting any section of the population, defined in any appropriate way.

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[1] For example :—Females only are liable to death from say misadventures in parturition, and that only between certain age-limits ; and children are alone liable to death from diseases associated with dentition, etc., etc.

[2] Throughout statistics conceptual precision has often to be sacrificed in order to express results simply, but the simplicity thus attained is usually more or less misleading.

Comparisons, the purpose of which is to measure, in a crude way, the virulence of the death-forces as between one population and another, *may therefore be made on the basis of a common age and sex distribution*, that is, the rates of mortality for each age or age-group, actually experienced by each population, may be applied to a "standard," or preferably to a "normal," population, the resulting totals giving the comparison required. That is to say, if the fact of variation with time (t) be ignored (though this variation will probably not be even approximately identical for any two populations), the quantity to be ascertained will be that indicated in (599) hereunder.

Let the *proportion* of the normal population between the ages x and $x + dx$ be:—

$$(597) \ldots\ldots\ldots \eta_x \, dx = f_1(x)\, dx, \text{ so that } \int_0^\omega \eta_x \, dx = 1$$

and let the instantaneous rate of mortality (the so-called "force of mortality") at age x be :—

$$(598) \ldots\ldots\ldots {}_1\mu_x = {}_1f_2(x), \text{ and } {}_2\mu_x = {}_2f_2(x)$$

for populations 1 and 2 respectively ; then adopting the same function (597) for both populations and applying (598) to each, we have :—

$$(599) \ldots\ldots\ldots \mu'_0 = \int_0^\omega (\mu_x \cdot \eta_x)\, dx \equiv \int_0^\omega \{ f_2(x) \cdot f_1(x) \}\, dx ;$$

with antecedent suffixes denoting whether the result applies to population 1 or 2.

These quantities may be called the *mortality-coefficients* of the respective populations, and generally they will differ somewhat from the "crude death-rates." If the age-distribution of a population happens to be sensibly identical with the "standard" or "normal" distribution, the mortality-coefficient would of course be sensibly the same as the crude death-rate" ; and it might otherwise also agree with it, but only accidentally. For arithmetical convenience it is usual to compute an *index-of-mortality*, by attributing to "standard" or "normal" groups the death-rates actually experienced in the corresponding groups of the population under review.

The preceding rectification of the crude death-rate for the purpose of comparisons, is but one of the possible methods. Its significance depends virtually upon a common distribution of causes of death, these differing only in frequency of operation. If two countries had the same age-distribution, but one was characterised by violence of the diseases which caused mortality in the *earlier*, and the other by those which caused mortality in the *later* years of life, the results would differ even for the

same differences of rate in each disease, inasmuch as with the same rates the diseases characteristic of the earlier years of life levy a larger toll than those characteristic of later years. In short, the influence or "weight" of a cause of death varies, according to the relation of its incidence with age. Crude death-rates and the indexes of mortality are therefore both of restricted application, and need to be interpreted with full regard to their inherent limitations.

The frequency according to age of the occurrence of disease is very diverse, consequently in the aggregate of mortality from all causes the peculiar incidence of each is to a great extent masked; and as regards the secular trend of mortality the intervention of epidemics may produce great irregularities.

Many diseases have a well-defined annual period, while others have not; these periods, however, are not identical in phase. The aggregate of the deaths from all causes, therefore, gives a less definite indication of an annual period. Inasmuch as diversity of phase and of amplitude do not wholly obliterate the periodicity, the general death-rate, viz., $\delta = D/P$, *i.e.*, the deaths divided by the number of the population, is as follows:—

$$(600)\ldots \delta = D/P = D_0 f_1(t)\{1 + a_0 + \Sigma_{n=1} a_n \sin n\,(\theta + \alpha_n)\} / \{P_0 f_2(t)\}$$

in which θ is a *fraction* of a unit of time (say of a year), $n = 1, 2, 3$, etc., and both D_0 and P_0 are means over a unit of time, as at a particular epoch. Thus the graph of a death-rate, extending over several units of time (years), is made up of a non-periodic curve—representing the *general trend*—upon which is superimposed a periodic curve repeating itself during each unit upon a scale varying with the death-rate itself.[1]

2. **Secular changes in crude death-rates.**—The general lowering of the general crude death-rate in the western world has been remarkable, and is best exhibited by deducing the *general trend* of the rates for each country. The death-rates for Australia are shewn in Table CXXXIV., from 1881 to 1915, for males, females, and persons: see columns (ii.) to (iv.). In order to partially eliminate the irregularities of results for single years, quinquennial means were formed, see columns (viii.) to (x.), and the smoothing of these for "persons" gives the values in column (xiv.), the maximum value 0.01570 being that for the year 1884 and the minimum 0.01066 that for the year 1911. This fall to about two-thirds of its earlier value in 27 years is remarkable, and is accounted for not only by a still greater decrease in infantile mortality, but also in general mortality up to 60 or 65 years of age. It is worthy of note that the year 1895 was characterised by a halt in the decrease exhibited by the general trend of the death-rate.

[1] So long, of course, as the character of the periodicity is maintained.

The rates of infantile mortality are given in columns (v.) to (vii.), the quinquennial means in columns (xi.) to (xiii.), and the smoothed result or *general trend* in column (xv.). Here again the fall has not been continuous, see values for 1894-5. The character of the lowering of the rates does not therefore fall under any law susceptible of simple mathematical expression.

TABLE CXXXIV. Shewing Secular Changes of the Death-rates, and of the Infantile Mortality-rates in Australia, from 1881 to 1915.

Year.	Death Rates × 100,000.			Infantile Mortality Rates × 10,000.			Quinquennial Mean Death Rates × 100,000.			Quinquennial Mean Rates of Infantile Mortality × 10,000.			General Trend of Death Rates (Smoothed) × 100,000.	General Trend of Inf'ntile Mort'lity (Smoothed) × 10,000.
	Males.	Females.	Persons.	Males.	Females.	Persons.	Males.	Females.	Persons.	Males.	Females.	Persons.		
(i.)	(ii.)	(iii.)	(iv.)	(v.)	(vi.)	(vii.)	(viii.)	(ix.)	(x.)	(xi.)	(xii.)	(xiii.)	(xiv.)	(xv.)
1881	1,589	1,328	1,469	1,232	1,095	1,165	1,636	1,348	1,504	1,372	1,203	1,293	1,528	1,293
1882	1,746	1,419	1,596	1,446	1,265	1,357	1,675	1,380	1,540	1,363	1,195	1,284	1,552	1,284
1883	1,654	1,381	1,529	1,302	1,138	1,222	1,708	1,404	1,569	1,353	1,186	1,274	1,569	1,272
1884	1,804	1,460	1,646	1,348	1,168	1,260	1,722	1,417	1,582	1,342	1,176	1,263	1,570	1,261
1885	1,747	1,434	1,604	1,360	1,221	1,292	1,689	1,397	1,555	1,330	1,166	1,251	1,562	1,251
1886	1,659	1,392	1,537	1,348	1,189	1,271	1,676	1,381	1,541	1,316	1,155	1,238	1,546	1,238
1887	1,583	1,317	1,461	1,235	1,091	1,164	1,651	1,386	1,520	1,300	1,140	1,222	1,526	1,221
1888	1,589	1,300	1,456	1,251	1,072	1,164	1,611	1,336	1,485	1,281	1,122	1,203	1,500	1,203
1889	1,672	1,385	1,540	1,400	1,234	1,319	1,603	1,323	1,474	1,260	1,101	1,182	1,474	1,185
1890	1,554	1,287	1,431	1,152	1,009	1,082	1,570	1,289	1,440	1,237	1,078	1,159	1,444	1,163
1891	1,618	1,328	1,484	1,232	1,074	1,155	1,553	1,274	1,424	1,212	1,053	1,135	1,410	1,138
1892	1,419	1,144	1,291	1,142	970	1,058	1,496	1,223	1,369	1,188	1,024	1,108	1,368	1,112
1893	1,502	1,227	1,374	1,240	1,072	1,149	1,459	1,186	1,332	1,167	998	1,086	1,326	1,086
1894	1,386	1,128	1,266	1,107	952	1,031	1,419	1,147	1,292	1,158	993	1,076	1,292	1,074
1895	1,372	1,102	1,245	1,099	921	1,012	1,403	1,131	1,276	1,161	997	1,079	1,280	1,078
1896	1,414	1,135	1,283	1,202	1,045	1,126	1,411	1,139	1,285	1,177	1,012	1,096	1,282	1,096
1897	1,342	1,065	1,212	1,126	967	1,048	1,416	1,145	1,289	1,196	1,031	1,115	1,284	1,114
1898	1,540	1,267	1,412	1,364	1,175	1,272	1,404	1,130	1,275	1,204	1,038	1,125	1,283	1,125
1899	1,411	1,156	1,291	1,246	1,085	1,167	1,395	1,157	1,263	1,198	1,034	1,117	1,273	1,117
1900	1,314	1,026	1,178	1,086	915	1,002	1,403	1,123	1,270	1,181	1,019	1,097	1,255	1,097
1901	1,366	1,064	1,222	1,122	947	1,037	1,362	1,086	1,231	1,145	993	1,062	1,231	1,062
1902	1,383	1,102	1,249	1,142	997	1,071	1,322	1,052	1,194	1,103	946	1,019	1,201	1,020
1903	1,837	1,080	1,215	1,183	1,025	1,105	1,302	1,037	1,176	1,053	892	970	1,172	971
1904	1,212	988	1,105	891	756	825	1,269	1,019	1,150	1 005	852	930	1,144	920
1905	1,214	950	1,088	906	724	818	1,235	994	1,120	952	800	878	1,118	872
1906	1,201	973	1,092	901	760	833	1,212	974	1,098	887	734	813	1,098	827
1907	1,211	977	1,099	884	734	811	1,200	957	1,084	867	711	791	1,085	792
1908	1,224	981	1,107	855	697	778	1,188	952	1,075	849	702	777	1,076	770
1909	1,151	906	1,033	787	642	716	1,184	946	1,070	820	671	748	1,070	751
1910	1,154	924	1,043	817	675	748	1,192	947	1,074	804	650	729	1,067	733
1911	1,182	940	1,066	759	607	685	1,186	941	1,069	790	641	718	1,066	718
1912	1,251	984	1,123	801	630	717	1,189	946	1,072	791	640	717	1,070	708
1913	1,193	953	1,078	788	653	722	1,200	944	1,079	776	626	703	1,079	702
1914	1,167	927	1,051	791	635	715	..	..	..	..	..	..	..	..
1915	1,208	916	1,066	743	605	675	..	..	..	..	..	..	..	..

The results in the above Table are shewn in Curves A and B of Fig. 98, the dots shewing the quinquennial means and the continuous line the general trend. The correlation between the two curves is fairly well indicated, because, although the ratio of the annual number of cases of deaths of children under 1 year of age, to the annual number of deaths of all ages is somewhat variable, there is some degree of general correspondence when a mean is taken over a number of years. See Fig. 98, p. 377.

The following example sufficiently illustrates the variable character of the ratio of infantile to total deaths, shewn in lines (*a*) and (*b*) hereunder :—

TABLE CXXXIV.A.—*Ratio x 10,000, of Infantile to Total Deaths, according to Sex. Australia.

Year	1902.	1903.	1904.	1905.	1906.	1907.	1908.	1909.	1910.	1911.	1912.	1913.	1914.	1915.
Males (*a*) ..	2,155	2,206	1,890	1,930	1,971	1,925	1,834	1,804	1,880	1,720	1,798	1,832	1,871	1,672
Females (*b*) ..	2,477	2,469	2,039	2,030	2,099	2,041	1,915	1,907	1,966	1,787	1,874	1,973	1,955	1,798
Females (*c*) ..	2,419	2,295	2,202	2,123	2,056	2,003	1,960	1,930	1,905	1,886	1,873	1,864	1,858	1,855
Males (*d*) ..	2,138	2,078	2,024	1,976	1,923	1,893	1,863	1,839	1,819	1,807	1,798	1,792	1,788	1,786
$(d) \div (c) = (e)$	.887	.905	.919	.931	.935	.945	.951	.953	.955	.958	.960	.961	.962	.963

* The figures on lines (*a*) and (*b*) are the ratios of the annual numbers of male and of female infantile deaths to the annual number of total male and of total female deaths respectively. The figures on lines (*c*) and (*d*) are the smoothed ratios for females and males respectively. The figures on line (*e*) are the ratios of male to the female ratios as determined from the smoothed ratios (*c*) and (*d*).

Although the ratio oscillates between somewhat wide limits, the female ratio is invariably higher than the male-ratio: the general death-rate of females, however, is lower than that for males.

These results indicate that the *proportion of infantile deaths to total deaths for both sexes is rapidly decreasing* ; *the decrease for females being more rapid than for males.* This is best seen by forming quinquennial means from which the general trend can be readily ascertained. The magnitude and general trend of the ratios of infantile to total mortality in the case of females and also in the case of males, are shewn respectively by curves M and N in Fig. 98, p. 377.

3. **Secular changes in mortality according to age.**—The death-rate for any age-group is the ratio of the number of deaths per unit of time (per annum) therein to the average number of persons in the group during that unit, *i.e.*, to the number at risk.[1] This ratio is markedly different for the two sexes. The following table, viz., CXXXV., based upon the censal results and intercensal experience since 1881[2], shews that for nearly all ages a remarkable diminution in the death rates has taken place. That this must be so is obvious from the results given in Table CXXXIV.

In a later Table, viz., CXXXVI., the average, also according to age, of the ratios between the death-rates of the sexes is given. These average ratios are the ratios of the sum of the four ratios given in each age-group for females to those given for males, and may be referred to the epoch 1900.0 for all comparisons as to any possible change with time.

[1] Actuarially, the ratio of the number of deaths experienced by persons between limits of age to the total number of units of time (years of life) lived within those age-limits by the population considered.

[2] The results for 1911 are really based upon the deaths occurring during the nine years 1907 to 1915 inclusive. The actual populations for these years are assumed to be distributed according to age as at the Census of the middle year, viz., 1911, which must be substantially correct.

It is obvious from the table that *estimations of the frequency of death based upon tables compiled on the experience of past years are erroneous*, if applied at the present time.[3] We shall investigate hereinafter the law of change.

TABLE CXXXV.—Shewing the Mean Death-rates in Age-groups deduced for Various Epochs, and Illustrating their Secular Changes. Australia, 1881 to 1915.

Age or Age-group	Males. Rate × 100,000.				Females. Rate × 100,000.				Average ratio of Female to Male Death Rate.	
	1881–1891, say 1886.0	1891–1901, say 1896.0	1901–1911, say 1906.0	1907–1915, say 1911.0	1881–1891, say 1886.0	1891–1901, say 1896.0	1901–1911, say 1906.0	1907–1915, say 1911.0	Age.	Data. Sm'thed result.
0–0*	25,439	23,473	19,341	16,360	21,340	19,333	15,562	12,867	0.0	‡.1866
0	14,366	12,738	10,112	8,540	12,414	10,786	8,349	6,862	0.5	.8395
1	3,576	2,685	1,804	1,559	3,427	2,519	1,684	1,389	1.5	.9371
2	1,379	982	677	642	1,336	963	631	575	2.5	.9524
3	891	628	441	409	834	617	412	382	3.5	.9477
4	692	497	350	301	648	488	325	300	4.5	.9571
0–4†	4,549	3,777	2,801	2,455	4,035	3,276	2,365	2,023	2.5	.8614
5–9	384	310	222	222	355	293	201	202	7.5	.9236
10–14	253	219	192	173	235	192	171	153	12.5	.8973
15–19	528	366	300	256	406	315	272	221	17.5	.8372
20–24	793	541	410	364	597	447	370	341	22.5	.8326
25–29	870	651	473	431	781	586	468	432	27.5	.9349
30–34	890	737	552	508	813	703	539	475	32.5	.9416
35–39	1,007	902	714	666	976	847	674	586	37.5	.9374
40–44	1,236	1,029	918	841	1,090	836	746	641	42.5	.8233
45–49†	1,591	1,311	1,222	1,120	1,262	1,000	890	794	47.5	.7525
50–54	2,085	1,737	1,522	1,511	1,568	1,273	1,044	1,050	52.5	.7199
55–59	2,803	2,454	2,091	2,153	2,037	1,793	1,497	1,473	57.5	.7157
60–64	3,717	3,624	3,095	3,174	2,694	2,677	2,293	2,177	62.5	.7231
65–69	5,528	5,207	4,708	4,678	4,423	3,753	3,619	3,471	67.5	.7587
70–74	7,488	7,104	7,584	6,972	6,218	5,704	6,074	5,523	72.5	.8069
75–79	11,778	11,686	11,845	10,900	10,076	9,967	9,378	9,162	77.5	.8350
80–84	15,275	16,210	16,450	16,815	14,490	13,984	13,306	14,575	82.5	.8704
85–89	27,169	26,041	27,372	26,783	24,227	21,960	22,836	21,701	87.5	.8427
90–94	24,661	26,917	30,577	30,896	28,455	26,497	29,433	28,960	92.5	1.0026
95–99	45,050	37,500	36,974	39,111	32,207	45,941	41,188	38,319	97.5	.9938
100–4	24,188	39,844	33,724	113,043	18,621	47,312	39,224	107,229	102.5	1.0075

* Nominally at the instant of birth, but not really so. For the first week after birth the curve is quite distinct from the general death-rate curve after that period. The values given are deduced from the results for the five age-groups, 0 to 4 inclusive, by formula (197), p.68 herein. If computed on the basis of $\mu = A + Cx + Bc^x$, see C. H. Wickens' Journ. Austr. Assoc. Adv. Sci. XIV., 1913, p. 535. The values for 0 will be .27640, .26330, .22790, .19460 and .22740, .21470, .17840 and .15090. But true values of μ_0 are really much greater than these.

† Between these limits (inclusive) the ratio is 0.8593.

‡ The ratio of death-rates using .27640 + etc., to .22740 + etc., is 0.8017.

4. **The changes in the ratio of female to male mortality according to time and age.**—The ratio of female to male mortality, according to time, may be deduced from the rates given in Table CXXXIV., and those according to age from the rates given in Table CXXXV.

To avoid the irregularities of individual years the former ratio is obtained by dividing the results in column (ix.) by those in column (viii.), Table CXXXIV. The quotients are given in Table CXXXVI., and are shewn by the dots on curve C, Fig. 98. This is the ratio for *general mortality*. For *infantile mortality* the results in column (xii.) of Table CXXXIV. are divided by those of column (xi.), and these are shewn by

[3] Thus the actuarial tables used by insurance societies err on the side of conservatism; they are based upon death-rates which are now excessively high.

small crosses on curve D, Fig. 98. The firm lines denote the general trend of these results. They give some indication of correlation with the general and infantile death-rates, see Curves A and B, and the difference between the two curves is less marked ; see Fig. 98, p. 377.

TABLE CXXXVI.—Shewing Ratios of Female to Male Death-rates, and Female to Male Rates of Infantile Mortality. Based upon Quinquennial Means. Australia, 1881-1913.

Year of Decade.	Ratios of Female to Male Death-Rates (Curve C).				Ratios of Female to Male Rates of Infantile Mortality (Curve D).			
	1880.	1890.	1900.	1910.	1880.	1890.	1900.	1910.
0	..	.821	.800	.795	..	.872	.863	.809
1	.824	.820	.797	.793	.877	.869	.867	.811
2	.824	.818	.796	.796	.877	.862	.858	.809
3	.822	.813	.796	.788	.876	.855	.847	.807
4	.823	.808	.803	..	.876	.858	.848	..
5	.827	.805	.805	..	.877	.859	.840	..
6	.824	.807	.804	..	.878	.860	.828	..
7	.840	.809	.797	..	.877	.862	.820	..
8	.829	.805	.801	..	.876	.862	.829	..
9	.825	.829	.799	..	.874	.863	.818	..
*2.0	..	..	..	.7925	..	..	..	.8089
*2.5	..	.8160	.7986	..	.8766	.8630	.8565	..
*3.0	.8232	..	..	..	..	..	..	..
*7.5	.8290	.8112	.8013	..	.8762	.8611	.8270	..

* These are means of five quinquennial means, except in two instances where they are means of four quinquennial means.

That the ratio of female to male mortality varies with time, having changed from 0.824 in 1881 to 0.788 in 1913, shews that life-tables for males and females, based on experience dating many years back, canno longer represent the facts with sufficient exactitude.

The curve, shewing the ratio of female to male mortality according to age, may be deduced from Table CXXXV., and in view of the overlap or the partial overlap of the 1907-15 results on those of 1901-11, the epoch to which the ratio may be referred is 1900.[1] These ratios are based upon the sums of the four ratios for each sex, given in the table. The result is shewn in Fig. 98, curve K. There are two maxima and two minima in the curve, at the ages indicated in the table ; see p. 377.

The dotted curve L, from which the curve K departs *during the reproductive period of life*, is symmetrical about an axis passing through the age 47. It is not unlikely that this departure from the curve L is due to the vicissitudes of reproduction ; see the reference hereinafter to the *gestate force of mortality*.

[1] Though not strictly exact, this assumption is sensibly correct.

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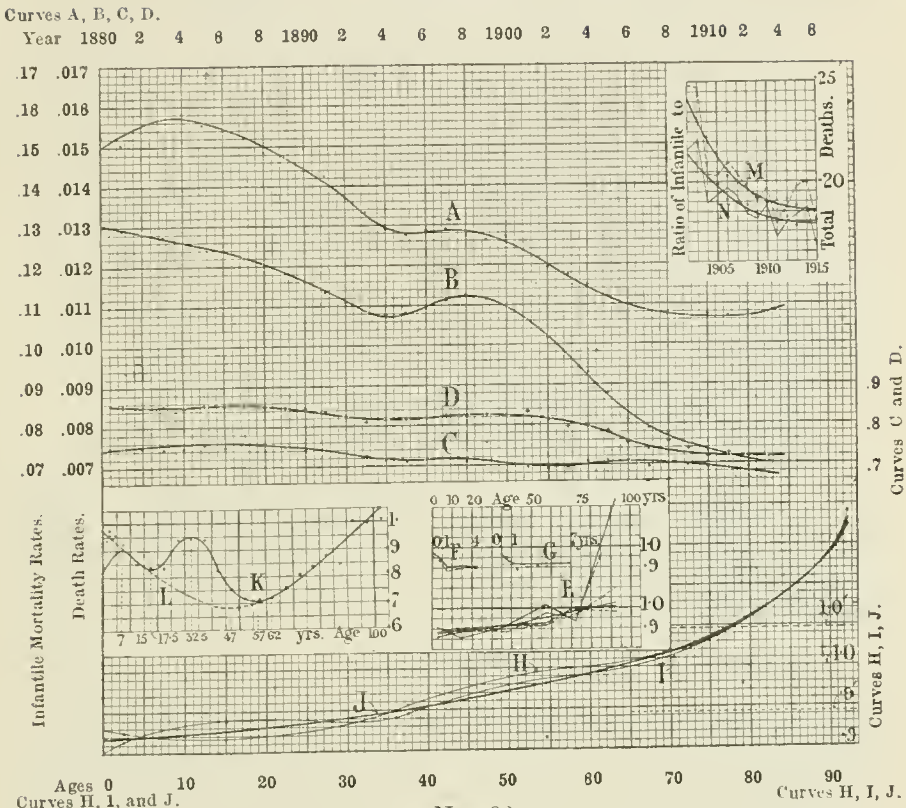


No. 98.

Curve A shews the trend of the quinquennial means of the annual death-rates for "persons" from 1880 to 1913 for the Commonwealth of Australia: the dots shew the quinquennial means themselves; see Table CXXXIV., p. 373.

Curve B similarly shews the trend of the quinquennial means of the infantile mortality rates: the dots shewing, as before, the quinquennial means; see Table CXXXIV., p. 373.

Curve C.—The dots shew the ratios of the quinquennial means of the death-rates for females to the quinquennial means of the death-rates for males, and the continuous line shews the general trend of these results; see Columns viii. and ix. of Table CXXXIV., p. 373.

Curve D.—The minute crosses shew the ratios of the quinquennial means of the rates of female infantile mortality to the quinquennial means of the rates of male infantile mortality, and the continuous line shews the general trend of these results; see columns xi. and xii. of Table CXXXIV., p. 373.

Curves E.—The firm lines are the graphs for males and the broken line the graphs for females, of the results given in the vertical columns of the lower part of Table CXXXVIII., p. 379.

Curves F. shew the changes in the ratios of decrease of mortality for ages 0 to 4, the firm line indicating the results for males and the broken line those for females; see pp. 378-380.

Curve G shews the mean of the results for ages 0 to 4, so reduced that the mean agrees with curve J; see pp. 379-380.

Curves H and I are drawn through the terminals of ordinates representing the means of the factors of decrease and increase. They shew the effect of age; see pp. 379-380.

Curve J may be regarded as the corresponding line for "persons." The scale needs modification. The line denoting unity may be taken at 0.9547: thus 0.9′ and 1′.0 are the correct places for 0.9 and 1.0 in relation to the curve; see p. 380.

Curve K.—The ratios of female to male mortality according to age, are shewn by curve K, the data being indicated by the dots, and the smoothed result by the continuous curve. The smoothed results are given in Table CXXXVII.; see p. 377.

Curve L.—This curve is symmetrical about age 47, and is coincident with curve K from age 62 years onwards; for its significance see p. 376.

Curve M.—The broken lines joining the points shew the ratio of female infantile to total female deaths for successive years. The dots shew the quinquennial means of these, and the firm line shews their general trend; see p. 374.

Curve N.—Similarly the fine zigzag lines are the lines joining the points defining the ratios of male infantile deaths to total male deaths for successive years. The dots shew the quinquennial means of these, and the firm line their general trend. The ratios of the ordinates to curve N to the ordinates to curve M, are given in line (e) in Table CXXXIV. A; see p. 374.

TABLE CXXXVII.—Shewing for the Period of 1881 to 1915 the Average Ratio of Female to Male Mortality, according to Age. Australia.

DATA.						SMOOTHED RESULT. Average Ratios of Females to Male Death-rates.								
Age-group.	Ratio × 1000	Age-group.	Ratio × 1000	Age-group.	Ratio × 1000	Age.	Ratio × 1000	(a)	Age.	Ratio × 1000	(a)	Age.	Ratio × 1000	(a)
0.0	817	15–20	837	65–70	759	0.0	817	971	35.0	944	710	70.0	773	773*
0–1	840	20–25	833	70–75	807	5.0	894	937	40.0	882	693	75.0	810	810*
1–2	937	25–30	935	75–80	835	7.0	903†	909	45.0	787	682	80.0	850	850*
2–3	952	30–35	942	80–85	870	10.0	883*	883*	47.0	753	680‡	85.0	892	892*
3–4	948	35–40	937	85–90	843	15.0	836*	836*	50.0	730	683	90.0	935	935*
4–5	957	40–45	823	90–95	1.003	17.5	834‡	819	55.0	713	695	95.0	980	980*
5-year means	..	45–50	753	95–100	994	20.0	839	802	57.0	710	703	100.0	1026	1028*
0–5	861	50–55	720	100–105	1.007	25.0	894	767	60.0	716	715	102.0	1044	1044*
5–10	924	55–60	716	..	..	30.0	944	736	62.0	724*	724*	..	..	..
10–15	897	60–65	723	..	..	32.5	950†	723	65.0	741*	741*	..	..	..
15–20	837	65–70	759	..	..	35.0	944	710	70.0	773*	610*	..	..	..

* Curve of ratios identical with curve L in Fig. 98, shewn by broken lines.
† Maximum values. ‡ Minimum values.
(a) Columns (a) are the values to curve L shewn by broken lines in Fig. 98. This curve is symmetrically situated about an axis, passing through the axis of abscissæ at age 47.0. For the significance of curve L reference should be made to the text.

5. **Secular changes in mortality vary with age.**—For any age or group of ages, let μ_0 denote the mortality at a particular date, adopted as time origin; and let ρ denote its *rate* of change—the sign being negative if it be decreasing—so that

$$(601)\ldots\ldots\ldots\mu_t = \mu_0 e^{\rho t} = \mu_0 e^{\phi(t)t}.$$

The last form is necessary only if ρ be *not* constant. It will be found probably in all cases that ρ is a function of time, and it is also a function of age. The results for small age-groups are of course irregular, so that it is only in extended age-groups that the laws of the secular changes according to age and time are rendered obvious and unequivocal. This can be seen by an analysis of the results given in Table CXXXV.,[1] and it is important to know whether for any given age ρ is sensibly constant for any sensible period.

The analysis is effected by forming a series of sums of age-group-results from Table CXXXV., and calculating the coefficients which, multiplied into the results of any period will give those of a later period.

[1] For example the sum of the rates 0 to 49 gave the following indication :—

Year (*a*)	1886.0		ρ		1896.0		ρ		1906.0		ρ		1911.0	
Males (*b*)	.12101	×	.8314	=	.09843	×	.7928	=	.07804	×	√(.8219)	=	.07036	Data
Males (*c*)	.12118	×	.8050	=	.09755	×	.8050	=	.07853	×	√(.8050)	=	.07046	Computed
Females (*d*)	.10550	×	.8052	=	.08495	×	.7882	=	.06696	×	√(.7679)	=	.05868	Data
Females (*e*)	.10636	×	.7909	=	.08412	×	.7909	=	.06659	×	√(.7909)	=	.05922	Computed

The constant ratios .8050 and .7909 therefore reproduce the results fairly well, for males and females respectively, though with a decennium as unit for the ratio-value, we find the value of the ratio is

$${}_{10}\rho = 0.8052 - 0.000127t - 0.0001573t^2$$

for females, t being expressed in years reckoned from 1886.0. The results are computed by taking the square root of the quantities .8314 and .7928 : allowing each the weight 2 and .9016 = √(.8219) the weight 1. This gives 0.89723, the square of which is .8050. The factors to divide into .12101, .09843, etc., are respectively 1, .8050, .64802 and .58142 ; the division gives .12101, .12227, .12043 and .12101 the mean of all being .12118 from which by inverting the procedure the above values for males are deduced ; similar results give .10636, etc., for females.

The values, found as shewn, suggest that, for the purpose of obtaining values for successive dates, multiplication by a factor and its powers or say an *annual quinquennial* or *decennial coefficient of variation*, has advantages over the employment of differences.

The quantities in columns ii. to iv., and vii. to ix., of this table, for males and females respectively, are deduced for the corresponding series of age-groups shewn ; the ratios are assumed to be true for the centres of the ranges of ages, an assumption which is sufficiently exact for the purpose in view.[1]

TABLE CXXXVIII. Shewing the Changing Ratios for different age-groups as between different dates. Australia, 1881 to 1915.

AGE GROUPS.	MALES. 1886 to 1896	1896 to 1906.	1906 to 1911.	Means	Ratio to Total.	FEMALES. 1886 to 1896.	1896 to 1906.	1906 to 1911.	Means	Ratio to Total.
(i.)	(ii.)	(iii.)	(iv.)	(v.)	(vi.)	(vii.)	(viii.)	(ix.)	(x.)	(xi.)
00	$.9606^2$	$.9077^2$	.8459	.9165	1.0528	$.9518^2$	$.8972^2$	.8268	.9050	1.0474
0	$.9116^2$	$.8910^2$	.8445	.9019	1.0361	$.9321^2$	$.8798^2$	.8219	.8891	1.0291
1	$.8665^2$	$.8197^2$	.8642	.8473	.9733	$.8573^2$	$.8175^2$	.8248	.8349	.9654
2	$.8439^2$	$.8182^2$	.9480	.8544	.9815	$.8490^2$	$.8094^2$	.9112	.8456	.9777
3	$.8395^2$	$.8380^2$	.9274	.8565	.9838	$.8601^2$	$.8171^2$	.9272	.8563	.9901
4	$.8469^2$	$.8392^2$	.8600	.8464	.9723	$.8678^2$	$.8161^2$	.9231	.8582	.9923
Means ..	.8832	.8523	.8817	.8705	1.0000	.8864	.8395	.8725	.8649	1.0000
0–4 ..	$.9112^2$	$.8612^2$	.8765	.8843	.9251	$.9011^2$	$.8496^2$	.8554	.8714	.9138
[5–14] ..	$.9113^2$	$.8847^2$	.9541	[.9092]	..	$.9067^2$	$.8758^2$	.9543	[.9038]	..
[14–24] ..	$.8286^2$	$.8848^2$	.8732	[.8600]	..	$.8716^2$	$.9179^2$	.8754	[.8908]	..
5–24 ..	$.8564^2$	$.8847^2$	.9030	.8770	.9175	$.8858^2$	$.9018^2$	.9043	.8955	.9392
25–49 ..	$.9098^2$	$.9153^2$	.9193	.9139	.9561	$.8983^2$	$.9138^2$	.8827	.9014	.9453
50–64 ..	$.9530^2$	$.9265^2$	1.0191	.9557	.9998	$.9548^2$	$.9174^2$	.9723	.9433	.9893
65–79 ..	$.9838^2$	1.0029^2	.9343	.9815	1.0268	$.9683^2$	$.9909^2$	.9520	.9741	1.0216
80–104 ..	1.0366^2	1.0054^2	2.5304	1.1229	1.1747	1.1481^2	$.9680^2$	1.4448	1.1354	1.1908
Means 1	.9418	.9327	1.0305	.95588	=1.0000	.9592	.9236	1.0019	.95351	=1.0000
Means 2	$.9436^2$	$.9343^2$	..	..	..	$.9635^2$	$.9247^2$	..	..	..

The quantities shewn in the table for the 10-year intervals are the square roots of the quantities originally given. In the totals these are counted twice. In the means 1 denotes the arithmetical mean, 2 the mean of the squares.

The irregularities of the results are doubtless due in part to actual irregularities in the death-rates themselves, and in part to errors in the data. They shew unmistakeably that the death-rate up to age say 60 decreases with time, and that, at any rate above age 80, the rate for males increases with time. The results exhibiting this are illustrated by curves E, F, G, H and I, Fig. 98, E shewing the six results given in Table CXXXVIII., for males by firm lines, and the six results for females by broken lines. The thick line divides the values under unity, viz., those

[1] Let a series of quantities, a and A, be respectively the numerators and denominators which give the ratio for any range of the variable. Then it is assumed that :—

$$(\alpha)\ldots\ldots\left(\frac{1}{n}\Sigma a\right)\Big/\left(\frac{1}{n}\Sigma A\right) \equiv a_m / A_m = a_0 / A_0$$

where a_0 and A_0 are the values for the middle range, the suffix notation being $-k, \ldots\ldots -1, 0, 1 \ldots\ldots k$. Obviously in general such an assumption is invalid ; the true range is that which gives a value of a' / A' equal to a_m / A_m. Later the assumption will also be made that the mean of a series of ratios may also be ascribed to mid-point of the entire range. The error of such an assumption is best illustrated by setting out the two results thus :—

$$(\beta)\ldots\rho_m = (\rho_1 + \rho_2 + \ldots + \rho_n)/n \equiv \left(\frac{a_1}{A_1} + \frac{a_2}{A_2} + \ldots\ldots + \frac{a_n}{A_n}\right)\Big/ n$$

$$(\gamma)\ldots\ldots\rho'_m = \frac{a_1 + a_2 + \ldots + a_n}{A_1 + A_2 + \ldots + A_n} \equiv \left(\frac{a_1}{A_m} + \frac{a_2}{A_m} + \ldots + \frac{a_n}{A_m}\right)\Big/ n$$

Although in general ρ'_m is not equal to ρ_m, if the successive ratios are in arithmetical progression, they are in agreement, and $\rho_m = a_m / A_m$ above. If these successions of ratios are sensibly linear in their changes, the error will be negligible.

which represent a decrease, from those which represent an increase (on the upper side). It would appear from this figure that the change is somewhere between 70 and 80, and that the rate of *decrease* of mortality unmistakably diminishes as age increases.

Curve G shews the mean of the results multiplied by a factor so as to make the average agree with curve J. Curves F shew the changes in the ratios of decrease for ages 0 to 4, the firm line denoting the results for males and the broken line those for females. Curves H and I are drawn through the ratios, to the total, of the means of the factors of decrease (or increase): they illustrate the general correspondence in the male and female cases of the effect of age, the curve J being the probable general indication, *i.e.*, for persons. The line denoting unity may be taken as at 0.9547: thus the broken line at 1.0474 will be really unity in relation to the curve.[1]

It is obvious that advances in hygiene, therapeutics, and social condition will be marked by diminished mortality. Whether that will extend over all ages or will characterise all but the older ages, depends upon whether the term of life is virtually sharply fixed or not. We shall consider the matter further in a later section.

6. **Fluent life-tables.**—For many purposes (much of insurance business for example) the ordinary tables of rates of mortality (μ_x or m_x), of probabilities of living or dying within a year (p_x or q_x), or of expectations of life ($\mathring{e}_x$), of the population survivors (l_x), at age x, etc., are satisfactory because they represent not only a considerable body of past experience, but also are 'on the safe side' for the major part of the uses to which they are applied (determination of insurance premiums, etc.). For the accurate prediction of life, however, existing tables are not at all satisfactory, because, representing past experience, they take no account of the fact that the rates of mortality for the major part of life are rapidly diminishing, that is the probability of life is increasing for every age, say up to 60 for both sexes in *many* and probably in *all*, civilised countries. Hence for estimations of the true probability of life, for the evaluation of payments for annuities, etc., existing life-tables are seriously defective.

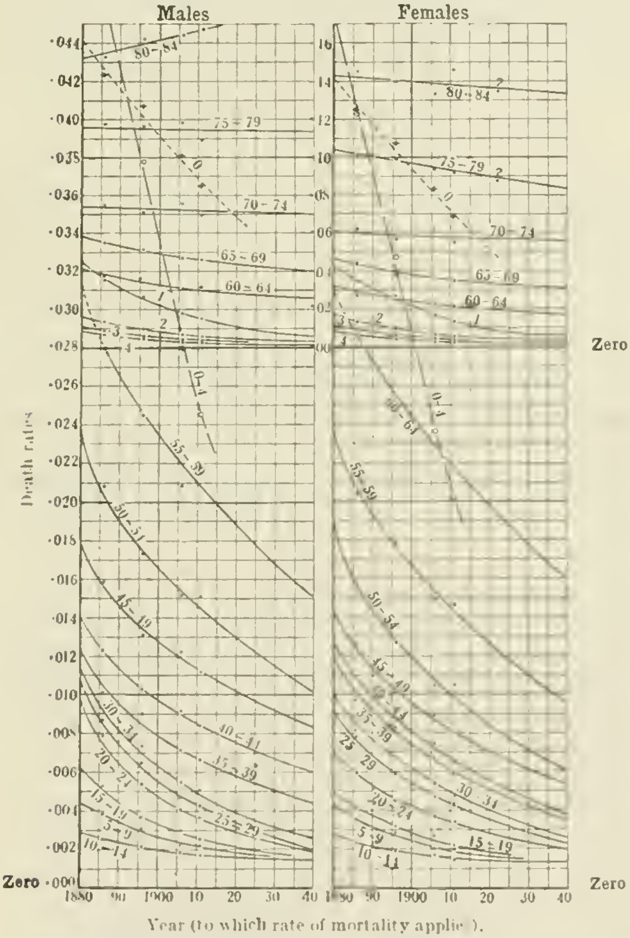
To avoid this difficulty *it is necessary to construct fluent life-tables*, extrapolated for as many years as may seem safe. Such tables are, to the extent they are extrapolated, *prediction tables*. In these, past experience is brought under review in two ways: that is (*a*) as to the values of the various functions as they existed at a given moment, and (*b*) their trend, or variation with time. As the variation with time is not linear probably an *annual coefficient of variation* would best attain the object in view, and could be readily applied.[2]

[1] Any resulting "error of scale" may almost be ignored.

[2] Thus if this were 0.993 for example, the values of the factors for successive years would be —to three decimals—0.993, 0.986, 0.979, 0.972 . . .0.9454, the last being for the 8th year forward. A linear diminution of .007 would have given .9444.

It is *only by means of fluent life-tables* that accurate predictions of survivors for any given age can be ascertained. In Fig. 99, shewing the change of death-rates with time, the dots denote the values according to the data: the system of curved lines shews what may be regarded as the general trend of the mortality-rate for the various age-groups. The results for individual age-groups are irregular, but they unmistakably point to a diminution of the type e^{-mt}, where t denotes the period elapsed, m however having a different value for each age and sex. This index factor (m) has no simple relation to age or to the magnitude of the mortality-rate itself, but is probably related to the two combined; that is, *it is a function of μ and x.* We shall first deal with the method of evaluating it, and it will simplify the matter if m be *not* treated as a function of time as in the final form of (601).

Change of Rates of Mortality according to Age and Time.



NOTE.—The scale for the older ages, viz., ·00 to ·16, is shewn between the graph for the males and that for the females, the zero corresponding to ·028 in the graphs for the lower ages. The curves shew the general trend of the improvement.

Fig. 99.

The dots shew the rates of mortality according to the data; the curved lines denote the general trend. The scale of the upper part of the graphs is shewn in the middle, the divisions representing ten times as great a quantity as in the lower part. The extrapolation of the curves to the year 1940 give an indication of the continuation of the improvement.

7. **Determination of the general trend of the secular changes in mortality.**—The results given in Table CXXXV., shewing a decrease with time—except for very great ages—in the rates of mortality, are best studied in Fig. 99. As this figure, however, gives only the rates of mortality as ordinates, and the epochs to which they refer as abscissæ ; and does not shew the *ratio* of the improvement, it is necessary to evaluate this ratio. To do this the mortality at any epoch must be divided by that at some epoch of reference. Thus we may assume—see Fig. 99—that over greater or lesser stretches of time, the curve of variation of the mortality is of the form (601) with m constant ; that is :—

$$(602)\ldots\ldots\mu_t = \mu_0 e^{rt}\ ; \text{ hence } \log \mu_t = \log \mu_0 + rt \log e$$

The logarithmic homologue of this relation being a straight line, as shewn, the values of μ_0 and r may be found by the " method of least squares." Or, put $R_t = \mu_t / \mu_0$; then, reckoning t from the year for which μ_0 is taken, the general trend of the change in mortality can be computed by the following formulæ, the derivation of which from (602) is obvious.

$$(603)\ldots\ldots r' \log e = \frac{1}{n}\left(\frac{\log R_1}{t_1} + \frac{\log R_2}{t_2} + \text{etc.}\right)$$

In this expression n is one less than the number of dates for which μ is known : r' is, of course, the mean value of r. Having found $r' \log e$ the mean initial value of the rate of mortality is :—

$$(604)..\log \mu'_0 = \{\log \mu_0 + \log \mu_1 + \log \mu_2 + .. - r' \log e\,(t_1 + t_2 + \ldots)\} / (n+1)$$

and μ'_0 is the mean value to be substituted for the original μ_0 to compute later rates; that is, the general trend may then be taken as $\mu'_t = \mu'_0 (e^{r'})^t$ the value of $e^{r'}$ being determined according to the unit of t (*i.e.*, for a year, a quinquennium, a decennium, etc.).[1]

Within what limits an assumption of the relation expressed by (602) may be supposed to exist is of course to be ascertained by graphing the results on a suitable scale.

8. **Modification of the general trend by age.**—In order to discover the relation between age and the present secular improvement in mortality, it will suffice to take the terminal values only into account ;[1] provided we restrict ourselves to the most consistent results. The improvement for 25 years has therefore been computed, and is as follows, the tabulated results being the values of μ_{25} / μ_0 :—

[1] The following instance will suffice to disclose the significance of the method :—

Year	1886	1896	1906	1911	Sum of Squares of residuals.
(*a*) Date	.01379	.00982	.00677	.00642	
(*b*) Adopting terminals	.01379	.01016	.00748	.00642	
(*c*) By (603) and (604)	.01386	.00993	.00711	.00602	
(*b*) − (*a*)	.00000	+ .00034	+ .00071	.00000	.00000062
(*c*) − (*a*)	+ .00007	+ .00011	+ .00034	− .00040	.00000029

The values of $e^{r'}$ for a unit of 5 years, by (*b*), *i.e.*, adopting terminal values 0.8582 : by (*c*), *i.e.*, by above method 0.8463.

TABLE CXXXIX.—Shewing the secular improvement for 25 years in the Rates of Mortality. Australia, 1886-1911.

	Age 1	2	3	4	7.5	12.5	17.5	22.5	42.5	47.5	67.5
Males . .	.436	.466	.459	.435	.544	.684	.485	.459	.680	.704	.846
Females	.405	.430	.458	.463	.533	.651	.544	.571	.588	.629	.785
Ratio*	1.076	1.084	1.002	.940	1.021	1.051	.890	.804	1.157	1.119	1.078

* Ratio of male to female ratio of improvement. The smaller the ratio the greater the diminution of the mortality.

These results shew (i.), that in general tho *diminution of mortality is more marked in young life than in old*; *and* (ii.), *that the diminution is not identical for males and females.*

Changes in rates of mortality, whether due to causes outside human control or otherwise, may be regarded as due to changes in the relation between the human organism and its environment. Factors known to be operative in various organisms, and which are possibly operative in the human case, are :—

(i.) Evolution of the protective reaction between the organism and its environment.

(ii.) Changes of the food supply in amount and quality.

(iii.) Changes due to the reactions of the organism to economic conditions, in respect of its nutritional and neural apparatus, etc.

(iv.) Changes in individual and general hygiene, in therapeutical and surgical knowledge, and in prevailing traditions which affect the vitality of the organism ; etc.

For our present purpose it is *not material whether the change* is what may be called *internal*—as (i.) above— *or external* : either or both may be regarded as *changes in environment, i.e.*, provided they are regarded as either *actual* or *virtual* changes. In short, the effect upon the death and morbidity rates, of any given change in human environment, necessarily varies with the modifiability or "plasticity" of the human organism. The plasticity, however, is not the only element which influences the results. The rate of a general improvement in environment will probably be masked to some extent by evolutionary disturbances, as, for example, by dentitional and puberal changes and, in the case of females, by the demands made on the organism by the exercise of the reproductive function. Hence, *a priori*, it is not to be expected that the secular variation of mortality according to age will reveal any simple progression with age. Moreover, to maintain the same rate of improvement for the ages of least mortality, as for those of greater mortality, is probably, from the nature of the case, very difficult.

Let R_x denote the ratio of change in μ_x in a given unit of time ; R being supposed to vary only with age (x). Excepting at the age of minimum mortality, a given value of μ is characteristic of two ages, viz., one less and the other greater than this minimum age. Since the *plasticity*

of the organism[1] diminishes with age, a given (absolute) change in environment will *tend* to have less effect on the later than on the earlier age, other things being equal. It follows, therefore, that, *in so far as plasticity alone is concerned*, R_x will be greater than R_{x+k}. If the plasticity degrade continuously with age we may suppose that it could be expected to vary probably either as $1 / (x + a)^{b+cx}$ or else as $1 / e^{ax^h}$, the value of a in the former representing the interval between fertilisation and birth, or say 0.75 year, since the plasticity is initially a maximum, and is greatest *in utero*. Consequently if it were necessary to take plasticity alone into account the *reciprocal* of the last quantity should be a factor distinguishing between the equal values of μ for different ages. The former expression, it is found, does not represent the facts ; the latter possibly would do so but for the other elements influencing the result. For the purpose of analysing these complex relations between age, the change in the rate of mortality, and the magnitude of that rate, we shall make use of the Census Life Tables for Australia for 1881-1890, and 1901-1910, see Census Report, Vol. III., pp. 1209-1218. For exact ages 0 and 1, the ratios of μ_x are used, and for the purpose in hand it will be abundantly accurate to take $\mu_x = \frac{1}{2}(m_{x-1} + m_x)$ for ages 2 and above 2,[2] m being the central death-rate for each age in question. In order to fix upon values of the mortality with which to associate the ages and ratios of change, the *geometric means* of the mortalities used in computing the ratios have been adopted, which is consistent with the first form of formula (601). It will also be assumed that the tabular values may be referred to the central point of time of the period from which the data are derived.[3]

As already defined, R_t denotes the ratio of change for the time, that is $R_t = \mu_t / \mu_0$ as before, see formula (603). But there will be some advantage in fixing our attention upon the ratio of improvement rather than upon the ratio of reduction of mortality. Thus if there be no improvement (diminution) in the death-rate with the lapse of time, the quantity considered should be 0, and on the other hand the vanishing of death altogether would be denoted by unity. Let B denote this ratio of betterment (or of improvement), then :—

$$(605) \ldots\ldots {}_0B_t = 1 - {}_0R_t = (\mu_0 - \mu_t) / \mu_0$$

[1] The fixation of plastic elements, by means of which the growth and reconstitution of the cellules of the organism are ensured, or *anabolism*, and the production of heat and energy by the oxidation of dynamic elements, or *katabolism*, constitute together the *metabolism* of the organism. The rate of metabolism or of waste and repair may appropriately be said to measure the plasticity of the organism. The plastic and dynamic elements, for example, the albumins, fats, hydrocarbons, etc., require also the presence of mineral salts and vitamines, in order to properly fulfil their nutritive and dynamic functions. The modifiability of the organism may of course be affected by its environment as well as by age : but its *potential modifiability* may be regarded as the measure of its plasticity.

[2] The error of this assumption is, of course, nearly negligible for most purposes for almost any ages, and for the present purpose is quite negligible. The central death-rate is the number of deaths occurring between any age limits divided by the mean population.

[3] That is, the table for the period 1881-1891 can be regarded as referable to the *point of time* 1886.0, and the table for 1901-1911 to the moment 1906.0.

with suffixes to denote the age to which the formula refers. As already indicated, the magnitude of B will be influenced by various circumstances. For example, the ratio of improvement will probably be low (and as a matter of fact is low) for those ages which are characterised by the lowest rates of mortality[1]; that is *for the ages when vitality is greatest a favourable advance in the environment will produce a relatively small effect.* To analyse the effect of the value of the death-rate upon the improvement we may divide B by the geometric mean of the death-rates measuring the change; that is by:—

$$(606) \ldots\ldots \mu_m = \sqrt{(\mu_0 \, \mu_t)},$$

and call the ratio of the betterment to this quantity, λ, or the *relative betterment*,[2] thus:—

$$(607) \ldots\ldots \lambda_t = B_t \,/\, \mu_m \equiv (1 - \mu_t \,/\, \mu_0) \Big/ \sqrt{(\mu_0 \mu_t)}$$

Since the limits of B are 0 and 1, this quantity can attain to considerable magnitude when t is considerable, and is therefore a sensitive measure of any improvement in the rate of mortality.

The following Table gives for males and females the values of μ_m, B, and λ, the values for μ_0 and μ_t being those given by the analysis of the Census results for thirty years, and the interval being referable therefore to the period between 1886.0 and 1906.0. For values of R, if required, we have simply $1 - B$.

The values of B are shewn in Fig. 100, curves B and B′; in which also the mean death-rates $\sqrt{(\mu_0 \mu_t)}$ are shewn, viz., curve A male, and curve A′ female. These exhibit the following characteristics:—

Curves of Relative Improvement for 20 Years in Death-rates.

	Initial Point.	1st Maximum Age.	1st Maximum Amount.	1st Minimum. Age.	1st Minimum. Amount.	2nd Maximum. Age.	2nd Maximum. Amount.	Remarks.
Males	0.175	2.8 yrs.	0.508	12.8 yrs.	0.209	23.3 yrs.	0.491	Later values.
Females	0.215	2.7 yrs.	0.520	13.2 yrs.	0.224	24.5 yrs.	0.400?	are irregular

Upon plotting the ratio of the betterment, viz., the values of λ for males and females, we obtain the results as shewn upon Fig. 100 by curves C and C′, representing the ratio of improvement in the case of males, and curves C″ and C‴, representing the ratio of improvement in the case of females. These exhibit the following characteristics:—

Ratio of the Relative Improvement to the Death-rate for 20 Years.

	Initial Value.	1st Maximum. Age.	1st Maximum. Amount.	Minimum. Age.	Minimum. Amount.	2nd Maximum. Age.	2nd Maximum. Amount.	Remarks.
Males	0.70	9.2 yrs.	164.6	13.8 yrs.	94.8	16.8 yrs.	109.4	Results after-
Females	1.07	9.5 yrs.	176.4	(13.8 yrs.	108.3)	(16.8 yrs.	100.5)	wards irregular

[1] This corresponds with the age at which the reproductive function commences to unfold, viz., at about age 12. Probably what may be called the age of efflorescence of the organism is generally its period of highest vitality.

[2] This is suggested by the word βελτίωσις, i.e., betterment; β is already appropriated for birth rate, etc.

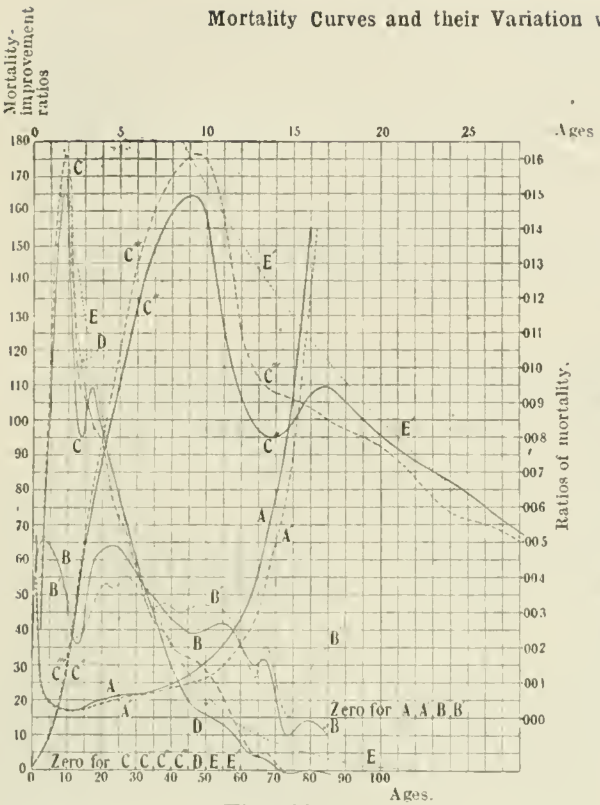
The values for age 0 cannot be deemed to closely represent the facts ; to obtain these a table of deaths occurring on successive days after birth would be needed, and not merely extrapolated results based upon successive years. For all other ages, however, they represent the facts with considerable accuracy.

It will be convenient to call the ratio λ the *mortality improvement ratio.*

TABLE CXL.—Shewing the Mean Mortality, the Relative Improvement in Mortality in 20 Years and the Ratio of this Relative Improvement to the Mean Mortality for Males and Females. Australia, 1886.0 to 1906.0.

Exact Age.	Males.				Females.			Exact Age.	Males.				Females.		
	Mean Death Rate 1896.0	Improvement. Relative.	Improvement. Ratio to Death Rate	Plasticity Curve	Mean Death Rate 1896.0	Improvement. Relative.	Improvement. Ratio to Death Rate.		Mean Death Rate 1896.0	Improvement. Relative.	Improvement. Ratio to Death Rate.	Plasticity Curve	Mean Death Rate 1896.0	Improvement. Relative.	Improvement. Ratio to Death Rate.
0	25100	175	.70	279.0	20150	215	1.07	45	01217	244	20.1	27.7	00963	310	32.2
1	04640	451	9.7	265.0	04270	463	10.8	46	01283	239	18.6	26.4	00988	312	31.6
2	01750	499	28.5	251.8	01660	514	31.0	47	01353	239	17.7	25.0	01018	317	31.1
3	00796	507	63.7	239.2	00752	519	69.0	48	01426	244	17.0	23.8	01055	321	30.4
4	00559	500	89.5	227.2	00522	503	96.4	49	01503	249	16.6	22.6	01095	322	29.4
5	00441	488	110.7	215.9	00409	492	120.3	50	01583	251	15.9	21.5	01140	320	28.1
6	00354	469	132.5	205.5	00324	470	145.0	51	01668	256	15.3	20.4	01190	319	26.8
7	00299	449	150.1	194.8	00269	432	160.6	52	01758	260	14.8	19.4	01249	317	25.4
8	00267	423	158.4	185.1	00235	395	168.1	53	01855	263	14.2	18.4	01319	316	23.9
9	00243	387	159.3	175.8	00215	377	175.3	54	01964	267	13.6	17.5	01398	310	22.2
10	00222	331	163.1	167.0	00201	354	176.1	55	02081	268	12.9	16.6	01488	304	20.4
11	00208	264	126.9	158.7	00193	305	158.0	56	02209	266	12.0	15.8	01586	295	18.8
12	00206	219	106.3	150.7	00195	244	125.1	57	02349	259	11.0	15.0	01694	284	16.8
13	00216	210	97.2	143.2	00204	225	110.7	58	02503	246	10.6	14.2	01814	267	14.7
14	00241	229	95.0	136.0	00221	238	107.7	59	02667	229	8.6	13.5	01948	244	12.5
15	00284	282	99.3	129.2	00243	257	105.8	60	02842	212	7.5	12.9	02091	217	10.4
16	00335	359	106.9	122.8	00273	282	103.3	61	03030	192	6.3	12.2	02249	185	8.2
17	00385	421	109.3	116.6	00309	306	98.9	62	03234	172	5.3	11.6	02418	155	6.4
18	00429	450	104.9	110.8	00343	333	97.1	63	03465	156	4.5	11.0	02605	127	4.9
19	00466	466	100.0	105.3	00375	357	95.2	64	03746	149	4.0	10.5	02831	115	4.1
20	00490	476	97.2	100.0	00404	373	92.3	65	04098	155	3.8	9.9	03137	138	4.4
21	00531	483	91.9	95.0	00431	380	88.2	66	04520	166	3.7	9.5	03514	179	5.1
22	00556	487	87.6	90.3	00457	379	82.9	67	04971	167	3.4	9.0	03898	201	5.2
23	00577	491	85.1	85.7	00479	371	77.5	68	05423	151	2.5	8.5	04253	194	4.6
24	00598	490	81.9	81.5	00504	372	73.8	69	05862	118	2.0	8.1	04596	165	3.6
25	00616	485	78.7	77.4	00535	383	71.6	70	06285	069	1.1	7.7	04945	116	2.3
26	00630	473	74.9	73.5	00565	396	70.1	71	06721	013	.2	7.3	05314	053	1.0
27	00642	458	71.3	69.8	00593	403	67.9	72	07240	—030	—.4	6.9	05740	—005	—0.1
28	00651	440	67.6	66.3	00619	398	64.3	73	07883	—051	—.7	6.6	06260	—042	— .7
29	00660	426	63.5	63.0	00638	392	61.4	74	08647	—052	—.6	6.3	06893	—049	— .7
30	00668	411	61.5	59.9	00652	382	58.6	75	09484	—044	—.5	6.0	0763	—033	— .4
31	00680	392	57.6	56.9	00664	361	54.4	76	1035	—032	—.3	5.7	0843	—006	— .1
32	00696	374	52.6	54.0	00676	338	50.0	77	1126	—021	—.2	5.4	0926	+019	+0.2
33	00714	360	50.4	51.3	00692	324	46.8	78	1222	—012	—.1	5.1	1013	042	0.4
34	00736	352	47.8	48.8	00714	315	44.1	79	1321	—008	—.1	4.9	1108	064	0.6
35	00763	342	44.8	46.3	00737	314	42.6	80	1422	—011	—.1	4.6	1210	085	.7
36	00795	330	41.5	44.1	00762	317	41.6	81	1530	—013	—.1	4.4	1321	104	.8
37	00826	318	38.5	41.8	00790	321	40.6	82	1645	—018	—.1	4.2	1440	121	.8
38	00862	303	35.2	39.7	00816	324	39.7	83	1775	—029	—.2	4.0	1569	134	.9
39	00902	289	32.0	37.7	00838	323	38.5	84	1915	—037	—.2	3.8	1705	141	.8
40	00943	281	29.8	35.9	00859	316	36.8	85	..	..	..	3.6	..	..	..
41	00987	274	27.8	34.1	00876	307	35.0	86	..	..	..	3.4	..	..	..
42	01037	266	25.6	32.4	00896	301	33.6	87	..	..	..	3.2	..	..	..
43	01094	258	23.6	30.7	00919	301	32.7	88	..	..	..	3.1	..	..	..
44	01154	250	21.7	29.2	00941	307	32.6	89	..	..	..	2.9	..	..	..

Mortality Curves and their Variation with Time.



Curves A and A′ represent the geometric means, according to age, of the rates of mortality for 1886.0 and 1906.0, for males and females respectively.

Curves B and B′ are the ratios of the diminution in the rate of mortality in 20 years to the geometric mean of the rates, in the case of males and females respectively.

Curves C and C′ are the ratios of the improvement (last mentioned) divided by the geometric mean rates of mortality.

Curves C″ and C‴ are the curves C and C′ respectively, plotted to an extended horizontal scale but with the same vertical scale.

Curves E and E′—the plasticity curve—shews, in a roughly approximate way, the general trend of the mortality-improvement ratio : see § 10.

Fig. 100.

9. **Significance of the variations in the mortality improvement ratio.**—The following relation between the changes in mortality and in the mortality-improvement-ratio is important.

The variations of the curve of the mortality improvement ratio are reciprocal to those of the mortality itself; that is, x and η being the ordinates to the mortality-improvement-ratio curve, and x and y the ordinates to the mortality curve, we have, practically for all ages,[1]:—

$$(608) \ldots\ldots \eta' / \eta = Ky / y' \text{; or } s = K / r$$

η, η' and y, y' being successive ordinates, and s and r their respective ratios.[2] K, however, is not a constant; nor is it any simply expressed function of x, though generally it is a little less or a little greater than unity.

[1] Certainly for all ages for which the mortality ratio can be very accurately evaluated.

[2] That is η and y are values for x, and η' and y' values for $x + 1$.

This reciprocal relationship reveals the fact that as the mortality at the beginning of life decreases with the successive years, the relative-improvement-ratio increases in very similar proportion. This reciprocal movement of the mortality-ratio, as compared with the mortality-improvement-ratio with increasing age, probably continues throughout life, and certainly continues till at least age 70. The values of the coefficient K in (608) above, are given in Table CXLI., K_m denoting those which apply to males and K_f those which apply to females. The ratios $s = \eta'/\eta$ and $1/r = y/y'$ are also shewn, viz., by the smaller figures between the values of η and y respectively. This coefficient K may be called the *beltiotic coefficient.*[3]

TABLE CXLI.—Shewing the ratios between the mean mortalities and the mortality-improvement-ratios for successive ages. Australia, 1886-0 to 1906-0.

Exact Ages from to (x).	Ratio of Mean Mortalities ($1/r$)	Ratio of Mortality Improvement ratios. (s)	Values of K: Males. K_m	Values of K: Females. K_f	Exact Ages from to (x).	Ratio of Mean Mortalities ($1/r$)	Ratio of Mortality Improvement ratios. (s)	Values of K: Males. K_m	Values of K: Females. K_f	Exact Ages from to (x).	Ratio of Mean Mortalities ($1/r$)	Ratio of Mortality Improvement ratios. (s)	Values of K: Males. K_m	Values of K: Females. K_f
0–1	5.409	13.945	2.578	2.154	28–29	.986	.940	.953	.983	56–57	.941	.918	.976	.952
1–2	2.651	2.934	1.107	1.110	29–30	.988	.968	.980	.973	57–58	.938	.962	1.026	.930
2–3	2.198	2.232	1.016	1.011	30–31	.982	.936	.953	.945	58–59	.939	.809	.862	.914
3–4	1.424	1.401	.984	.970	31–32	.977	.913	.934	.936	59–60	.938	.869	.927	.888
4–5	1.268	1.237	.976	.978	32–33	.975	.958	.983	.959	60–61	.938	.850	.906	.854
5–6	1.246	1.197	.961	.956	33–34	.970	.949	.978	.972	61–62	.937	.837	.893	.837
6–7	1.184	1.133	.957	.919	34–35	.965	.936	.970	.997	62–63	.933	.849	.910	.821
7–8	1.120	1.055	.942	.914	35–36	.960	.927	.965	1.009	63–64	.925	.882	.954	.904
8–9	1.099	1.005	.914	.955	36–37	.962	.927	.964	1.012	64–65	.914	.950	1.039	1.202
9–10	1.095	1.024	.935	.939	37–38	.958	.913	.953	1.009	65–66	.907	.970	1.069	1.295
10–11	1.067	.780	.731	.862	38–39	.956	.911	.953	.997	66–67	.909	.916	1.008	1.126
11–12	1.010	.838	.830	.800	39–40	.956	.930	.973	.978	67–68	.917	.929	.795	.966
12–13	.954	.914	.958	.926	40–41	.955	.932	.976	.970	68–69	.925	.824	.891	.853
13–14	.896	.977	1.090	1.054	41–42	.952	.924	.971	.981	69–70	.933	.545	.584	.670
14–15	.849	1.045	1.231	1.079	42–43	.948	.920	.971	.997	70–71	.935	.931	.427	.459
15–16	.848	1.077	1.270	1.090	43–44	.948	.918	.968	1.023	71–72	.928	.926	..	..
16–17	.870	1.022	1.175	1.085	44–45	.948	.927	.978	1.009	72–73	.918	.917	..	..
17–18	.897	.959	1.069	1.089	45–46	.949	.928	.978	1.006	73–74	.912	.908	..	..
18–19	.921	.953	1.035	1.072	46–47	.948	.948	1.000	1.015	74–75	.912	.903	..	..
19–20	.951	.972	1.022	1.045	47–48	.949	.963	1.014	1.013	75–76	.916	.905	..	..
20–21	.923	.946	1.025	1.019	48–49	.949	.974	1.026	1.003	76–77	.919	.910	..	..
21–22	.955	.953	.997	.998	49–50	.949	.958	1.009	.993	77–78	.921	.914	..	..
22–23	.964	.972	1.009	.979	50–51	.949	.967	1.019	.997	78–79	.925	.914	..	..
23–24	.965	.963	.998	1.003	51–52	.949	.965	1.017	.993	79–80	.929	.916	..	..
24–25	.971	.961	.990	1.030	52–53	.948	.959	1.011	.997	80–81	.929	.916	..	..
25–26	.978	.952	.973	1.034	53–54	.945	.958	1,014	.982	81–82	.930	.917	..	..
26–27	.981	.951	.969	1.017	54–55	.944	.946	1.002	.982	82–83	.927	.918	..	..
27–28	.986	.948	.961	.987	55–56	.942	.935	.993	.980	83–84	.927	.920	..	..

The ratios in the Table ($1/r$) are the values of the mortality at any age divided by the mortality at the age greater by one year; that is, the tabular values are the quantities μ_x/μ_{x+1}.

The tabular ratios of the mortality-improvement-ratios are the values obtained by dividing the mortality-improvement-ratio for any age by that of the age *less* by one year; that is the tabular values are the quantities λ_{x+}/λ_x

The coefficient K is that quantity which, multiplied into the ratio of the mean mortalities, gives reciprocally the ratio of the mortality-improvement-ratios.

[3] From βελτιωτικός, bettering or amending

If the value of the ratio y is required for a single unit of time (1 year), we have, on the assumption of a geometrically progressive decrease in mortality, $\mu_1 = \mu_t^{\frac{1}{t}}$; consequently :—

$$(609) \ldots B_1 = [1 - (\mu_t/\mu_0)^{\frac{1}{t}}]/\mu_0 \text{ and } \lambda_1 = (1 - \mu_1/\mu_0)/\sqrt{(\mu_0\,\mu_1)}$$
$$= B_1/(\mu_0^{2t-1}\mu_t)^{\frac{1}{2t}}$$

$$(610) \ldots\ldots \xi^t = \mu_t / \mu_0 ; \text{ and } \mu_1 = \mu_0 \xi$$

The form of the expression for λ is independent of the unit of time, though of course its numerical value is dependent on that unit.

10. **The plasticity curve.**—If we except the period between exact ages 14 and 17, the beltiotic coefficient continually decreases in value from age 10. If a curve be drawn representing the general result, it is found (from the 20 years' improvement in the mortality conditions) that it is fairly well represented by the equation $y = 278.95\ (0.95)^x$. This curve, viz., E and E′ on Fig. 100, may be called the *plasticity curve*, and its ordinates are given in Table CXL. The amount by which the beltiotic curve (*i.e.*, the curve of the mortality-improvement-ratio) falls short of the plasticity curve, does not, however, and least of all initially, constitute a measure of the great difficulty of attaining to the limit, which plasticity would admit of, were it not for the great difficulty of initial adjustment to a new environment, and to the exhaustion of energy involved by puberal developments. For the analysis of these questions, however, the available data appear to be inadequate,[1] and they will not be further discussed here.

No simple relation expresses the variation of the constants 278.95 and 0.95 with the unit of time over which the improvement is measured.

11. **Rate of mortality at the beginning of life.**—The mortality at the beginning of life is probably considerably affected by local circumstances ; consequently for the first two weeks and perhaps even the first month of life it would be difficult to assign any particular law of change of mortality with age.[2] Statistics for Saxony gave a first minimum rate at 8 days, and a lessor maximum 15 days, and those for Sweden gave

[1] It may be noted that for the relative improvement to be unity we must have $\mu_t = 0$ in (605), that is to say, death must vanish. But no diminution of mortality in a geometrical ratio can reach zero, for though $\mu_0 \xi^t$ may be as small as wo please, it cannot become zero with ξ positive and t finite : moreover, when the death-rate is large the value of λ cannot be great with any practicable change of death-rate.

[2] See " The improvement in infantile mortality : its annual fluctuations and frequency according to age in Australia." by G. H. Knibbs, Journ. Australas. Med. Congress., Sept. 1911, pp. 670-679 ; see also " Die Sterblichkeit im ersten Lebensmonat, Zeit. f. Soz. Mediz., Leipzig Bd. v., p. 175, 15th April, 1910.

a somewhat similar indication[1], while Australian records do not lend any support to this recrudescence of the rate of mortality. Prussian statistics shew a minimum rate for 9 days and a rise to 14 days.[2]

The statistics in Australia are imperfect, and some distributing was necessary owing to the want of precision in stating the *exact interval after birth.* The defective statement of age does not, however, affect the deductions hereinafter. In the following table the results for the fractions of the first day are merely computed; the rates, calculated without regard to migration, the effects of which are nearly negligible, and are not accurately ascertainable, are determined by deducting the deaths from the total births in order to ascertain the numbers of survivors.

The rates so found shew that from the end of the first day the law of mortality is expressed by $\mu_x = \mu_1/x$, for 5 or even 6 days. The generality of this expression can be extended, if it be put in the following form, viz. :—

$$(611)..\mu_x = \mu_1 [1 + f(x)]/x, \text{ consequently } 1 + f(x) = x\,\mu_x/\mu_1$$

and $f(x)$ for the first 5 or 6 days is zero. The shorter expression indicates that after the first 24 hours, and for about the first week of life the *probability of death diminishes as the length of time lived, reckoned from the moment of birth.* The following rates are computed for "persons" (*i.e.*, males and females) from the records of about 500,000 births and the deaths which resulted from them.

TABLE CXLII.—Death-rates per diem at the Beginning of Life. Based upon 499,674 Births, and the Deaths occurring therein. Australia, 1909, 1910, 1912 and 1913.

Age-group or Age Days.	Death-rates per Diem.	Age Days.	Death-rates per Diem.	Age Days.	Death-rates per Diem.	Age Days.	Death-rates per Diem.
0*	.015000	4	.001416	40	.0002237	200	.0001233
0–⅛*	.014061	5	.001137	50	.0002117	225	.0001142
⅛–¼*	.012355	6	.000975	60	.0002035	250	.0001063
¼–½*	.010143	7	.000853	70	.0001948	275	.0000986
½–¾*	.007934	8	.000767	80	.0001875	300	.0000923
¾–1*	.006330	9	.000703	90	.0001804	325	.0000865
0–1*	.009404	10	.000653	100	.0001740	350	.0000821
1	.005743	15	.000497	125	.0001594	365	.0000802
2	.002927	20	.0003961	150	.0001464	..	..
3	.001899	30	.0002678	175	.0001337	1095	.00001084
4	.001416	40	.0002237	200	.0001233	..	..

* Approximate estimates only. There are no available statistics for the accurate estimation of the frequency of death during each of the first 24 hours of life.

[1] Op. cit., p. 676. The results are given on graph No. 7 on the page mentioned.

[2] See Handbuch d. Med. Statistik., Fr. Prinzing, 1906, pp. 281-2; also G. Lommatzsch, Zeit. f. sächs. stat. Bureau, 1897, Bd. xliii., p. 1.

We may take the mean of μ_x for the first 5 days as the value of the mortality *at* the end of the first day; this gives the rate 0.005729 per diem. Using this to determine $1 + f(x)$, we find that its values are as follow:—

TABLE CXLIII.—Shewing the Values of $x\mu_x/\mu_1$, that is $1 + f(x)$ in (611).

Exact Age, Days.	$1 + f(x)$ Crude.	$1 + f(x)$ Smoothed.	Exact Age, Days.	$1 + f(x)$. Crude.	$1 + f(x)$. Smoothed.	Exact Age, Days.	$1 + f(x)$. Crude.	$1 + f(x)$. Smoothed.
1–5	1.0000	1.0000	30	1.4023	1.4023	125	3.4779	3.4373
6	1.0211	1.0200	35	1.4442	1.4442	150	3.8331	3.8067
7	1.0422	1.0438	40	1.5619	1.5620	175	4.0841	4.1022
8	1.0710	1.0714	45	1.7038	1.7043	200	4.3044	4.3342
9	1.1014	1.1028	50	1.8476	1.8466	225	4.4851	4.5131
10	1.1398	1.1380	55	1.9894	1.9889	250	4.6387	4.6493
12.5	1.2280	1.2280	60	2.1313	2.1312	275	4.7329	4.7532
15	1.3013	1.3013	70	2.3802	2.3930	300	4.8333	4.8352
17.5	1.3526	1.3526	80	2.6182	2.6330	325	4.9071	4.9057
20	1.3828	1.3828	90	2.8340	2.8500	350	5.0157	4.9749
25	1.3990	1.3990	100	3.0372	3.0450	365	5.1096	..
30	1.4023	1.4023	125	3.4779	3.4373	..	..	..

From 5 to 10 is a second degree curve, the 1st diff. for a unit being = +·0200, 2nd diff. = +.0038. From 40 to 60 is a straight line, the common difference for a unit being +.02846. The curve from 60 to 120 is a second degree curve, the 1st diff. being +0.2622 and the 2nd diff. −0.02238. From 125 to 350 is a third degree curve, the first rank of difference being + 0.3694, − 0.0739, and + 0.0104, the last being the common differences.

The results in the above table shew that although for the first few (five) days the death-rate diminishes as the duration of life, *this rapid rate of diminution is not continued*, but the rate falls off more slowly—and on the whole continually—till the minimum death-rate occurs.

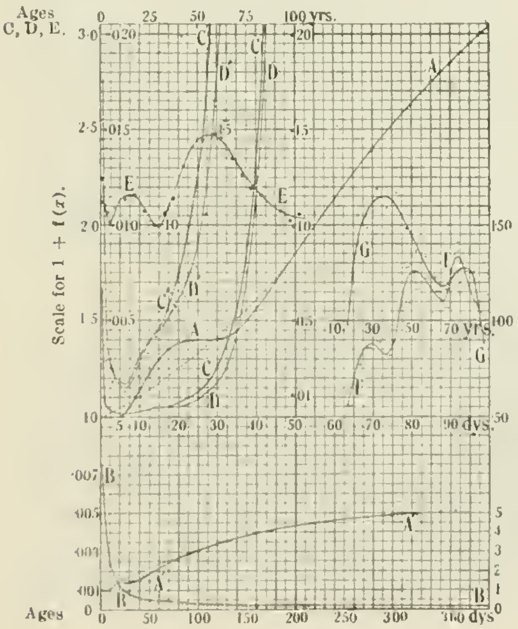


Fig. 101.

Mortality Curves.

Curve A A shews the values of 1 + f (x) for 90 days, see (611) p. 390.

Curve A′ A′, shews on a smaller scale the values of 1 + f (x) for 360 days.

Curve B B is the curve of rates of mortality for 360 days; the dotted line shews what *would* be the curve if the hyperbolic law held throughout.

Curve C C is the curve of rates of mortality for males, and curve D D is that for females.

Curves C′ and D′ are the same as C and D, except that the vertical scale is increased tenfold.

Curve E E is the curve of masculinity of the rates of mortality according to age; see Table CXLVI.

Curve F F is the curve of the ratios of the rates of mortality for males "not married" to those for married males, according to age.

Curve G G is the curve of the ratios of the rates of mortality for females "not married" to those for married females, according to age.

The characteristics of the diminution of the initial death-rate may be summed up as follows :—

(*a*) For the first 24 hours of life satisfactory data do not exist to determine the characteristics of the death-rate (see below).

(*b*) From the end of the first to the end of the fifth day the rate varies inversely as the duration of life.

(*c*) From the end of the fifth day the rate of diminution rapidly falls off till about the 20th day, then less rapidly till the 30th day, then the rapidity of the falling off of the rate of diminution approximates to what it was from the 5th to the 20th days, but after that decreases slowly and fairly regularly.

(*d*) No simple function expresses these changes in the variation of the death-rate, and they probably differ somewhat in different countries.

If the expression (611) is put into the form :—

$$(612)\ldots\ldots\mu_x = \frac{\mu_1}{x} e^{a(x-1)^b}$$

this can be fitted to a considerable range of the curve, *provided that minor fluctuations are ignored.* It cannot, however, represent with sufficient accuracy a year's results. To fit any two points on the curve besides the origin we have :—

$$(613)\ \ldots\ldots\log\frac{\log y'}{\log y} = b\log\frac{x'-1}{x-1}$$

in which $y = x\mu/\mu_1 = 1 + f(x)$. When b is found a can be readily obtained from (612).

For the values of μ for fractions of the first day it may be assumed that the curve is $\mu_0 e^{-x}$. For this to give .995729 at the end of the first day we must make $\mu_0 = 0.015573$ (per diem), and this would be the mortality for $x=0$, viz., at the moment of birth, and is equivalent to a death-rate of 5.684 per annum. This may be put in another way, viz., it is equivalent to a rate of unity per 64.21 days (*i.e.*, 365 ÷ 5.684), and implies that such a rate, if operating uniformly for that period on a group of children for 64.21 days, the group being kept constant, would in that time account for the death of all born.

12. **Composite character of aggregate mortality according to age.**— Before dealing further with the variation with age of the rate of mortality, it is desirable to review the nature of the aggregate rate of mortality.

The general rate of mortality for any age, $D_x/P_x=\mu_x$, viz., the aggregate number of deaths of persons between given infinitesimal limits of age occurring in a unit of time, divided by the average number of persons of the same age[1] (the average being taken over the unit period in which the

[1] In practice D and P are taken between limits x and x', say, in which case μ is not given but instead the average over the range. The difference is dealt with later.

deaths occurred) is made up of the rates from each cause, and if regarded from the summation point of view —see (596), p. 370—is compounded of a series of rates, the graphs of which are by no means similar. For example, in "causes of deaths," Nos. 31 and 32, the real number at risk are those shewn in line 2 below, the variation with age is quite unlike the variation with age of the total mortality, and is by no means identical in the two cases, as will be at once seen from a Table given hereinafter. The results are as follows :—

1. Age-group ..	10–14	15–19	20–24	25–29	30–34	35–39	40–44	45–49	50–54	Ttl.
2. Cases of Maternity 1907-1915*	211	54,527	262,866	317,815	238,746	155,813	60,970	6,075	79	..
3. Cases of Puerperal Septicæmia	1	96	370	515	459	307	112	12	1 ?	1872
4. Ratio	.00474	.00176	.00141	.00162	.00192	.00197	.00184	.00197	.01266 ?	..
5 Cases of other Accidents of Pregnancy & Labour	4	161	587	816	799	829	397	55	0 ?	3648
6. Ratio	.01896	.00295	.00223	.00257	.00335	.00532	.00651	.00905	.00000 ?	..

* Actually births. These are, however, only slightly too great. The correction may be neglected for the present purpose.

The results shew that out of a total of 100 deaths at all ages from puerperal septicæmia and other accidents of pregnancy and labour, 34 will arise from the former cause, and 66 from the latter ; and also that the distribution according to age differs considerably for septicæmia ; the proportion dying at different ages remaining more nearly constant than in the case of deaths from other accidents of pregnancy and labour. The fall to a minimum occurs at about age 23.4, when the ratio is about 0.00219. The minimum in the case of septicæmia is at about age 23.1, and the ratio is about 0.00139, the proportion of the deaths from other accidents of pregnancy, etc., being here 0.61 of the two combined.[1]

Causes of death may be classified, as regards their relative frequency according to age, as follow, viz.[2] :—

(i.) *Normal*, viz., those in which the relative frequency is similar to the relative frequency of death from all causes combined ;

[1] See formulæ (292) and (294), p. 92 herein.

[2] The causes of death given in a Table hereinafter may be classified according to the scheme indicated, and are as follows, viz. :—

Class (i.) **Normal.**—9. Influenza ; 12. Epidemic Diseases ; 16*a*. General Diseases ; 18. Cerebral Hæmorrhage, etc. ; 18*a*. Other Diseases of the Nervous System ; 20. Acute Bronchitis ; 22. Pneumonia ; 23. Other Diseases of the Respiratory System ; 24. Diseases of the Stomach ; 25. Diarrhœa and Enteritis ; 27. Hernia and Intestinal Obstruction ; 28*a*. Diseases of the Digestive System ; 29. Acute Nephritis, etc. ; 30*a*. Other Diseases of the Genito-Urinary System ; 32*a*. Diseases of the Skin and Cellular Tissue ; 32*b*. Diseases of the Organs of Locomotion ; 35. Violent Death ; 38. Ill-defined Diseases.

Class (ii.).—**Infantile, Sub-classes** (*a*), (*b*) **and** (*c*).—5. Measles (*b*) ; 7. Whooping Cough (*a*) ; 8. Diphtheria and Croup (*b*) ; 14. Tubercular Meningitis (*b*) ; 15. Other Forms of Tuberculosis (*c*) ; 17. Simple Meningitis (*a*) ; 33. Congenital Debility and Malformations (*a*) ; 33*a*. Other Diseases of Infancy (*a*).

Class (iii.).—**Senile.**—16. Cancer and other Malignant Tumours ; 19. Organic Diseases of the Heart ; 19*a*. Other Diseases of the Circulatory System ; 21. Chronic Bronchitis ; 28. Cirrhosis of the Liver ; 34. Senile Debility.

Class (iv.).—**Median.**—1. Typhoid Fever ; 13. Tuberculosis ; 26. Appendicitis, etc. ; 31. Puerperal Septicæmia ; 32. Accidents of Pregnancy and Labour ; 36. Median ; 30. Non-cancerous tumours of the female genital organs.

Organic diseases of the heart and other diseases of the circulatory system are hardly to be included in the "normal" series, because the death-rate in the first year of life is not very great.

(ii.) *Infantile*, viz., those which characterise infancy only ;
(iii.) *Senile*, viz., those which characterise old age only ;
(iv.) *Median*, viz., those which characterise middle age only.

The infantile causes of death may be subdivided into three sub-classes, viz. (*a*) those in which the mortality is greatest in the first year of life ; (*b*) those in which it is later than the first year ; and (*c*) those in which the mortality is greatest in the first year, but is followed by an irregular mortality for all ages.

It is obvious that, apart from variations in the distribution according to age, and general differences in local salubrity, epidemics will cause differences in mortality rates according to age, hence to be representative of a country, the deduced mortality rates must be taken over a sufficient period of time. The results in the Table CXLIV. hereinafter are based upon 9 years' experience, viz., from 1907-1915 in Australia, and the distribution of the population, according to age and sex is assumed to be as at the Census of 3rd April, 1911. Before analysing these results it will be necessary to consider the character of curves of organic increase or decrease.

13. **The curve of organic increase or decrease.**—The curve e^x (or e^{-x}) and its variants, may, for obvious reasons, appropriately be called the curve of organic increase or (organic decrease). In considering its application to the increase of population by birth or the reduction of population by death, etc., certain characters of the curve deserve notice, and will now be considered. If to adapt it to a given instance, the expression be put in the more general form hereunder, we may note that :—

$$(614)\ldots\ldots y = Ae^{nx+a} = (Ae^a)e^{nx} = A'e^{nx} = A'\mathrm{m}^x$$

in which $\mathrm{m} = e^n$. Hence the addition of a constant to the index of e *affects only the vertical scale of the graph* of the curve, *while n affects its horizontal scale.* If n be constant the final form in the above expression is satisfactory, but if it change with x, then the appropriate expression is—

$$(615)\ldots\ldots y = Ae^{nx\phi(x)} = Ae^{x\cdot n\phi(x)} = Ae^{n'x} = A\phi(x)^x$$

and the form of $\phi(x)$ will be determined by the law of change in n'. Geometrically this is equivalent to changing the x-scale as x increases.

In order to ascertain the form of $\phi(x)$ the quantities, group or other, may be set out as shewn hereunder, and the quotients B/A = b, C/B=c, etc., computed. If b, c, d, etc., are not equal, then the curve Be^{bx} will not satisfy the data. If on computing also the values of b (c—1) / (b—1); c (d—1)/ (c—1) , etc., it is found that they are not equal, the curve $A+Be^{bx}$ will also not satisfy the values. The last step may be shortened by putting the above in the form (c—1)/(1—1/b). Similarly, if the quotients of the differences of the ordinates are not equal the equation will not apply.

Scheme of Examination of Data.

Value, of ..	x_1	x_2	x_3	x_4	x_5	, etc.
or Range between	x_1 to x_2 ,	x_2 to x_3 ,	x_3 to x_4 ,	x_4 to x_5 ,	x_5 to x_6 ,	etc.
Group value ..	A ,	B ,	C ,	D ,	E ,	etc.
Ratio B/A, etc.	b	c	d	e	f	
Reciprocals A/B, etc.	1/b	1/c	1/d	1/e	1/f	
or Ratio of differences	$\frac{y_3-y_2}{y_2-y_1}$;	$\frac{y_4-y_3}{y_3-y_2}$; etc.				

If the values are increasing in the order A . . . E, n is positive, if diminishing n is negative, x being regarded as positive throughout.

There is, of course, no universal guide for deciding what form of function to adopt, but if b, c, d ascend by a common difference, the function will be of the form $Be^{bx(1+ax)}$; if by a common multiple, it will be of the form Be^{bx^2}; if by a common power of x, of the form Be^{bx^p}, and so on. Successive values of $m = e^n$ may therefore be analysed on the same principles as the original data.

14. **Exact value of abscissa corresponding to the quotient of two groups.**—It is obvious that if there be two distributions (*e.g.*, the number of deaths occurring in a population of a given magnitude in a given unit of time, both set forth according to age) and the average over a range of the variable be ascertained (*e.g.*, the average death-rate of all persons between ages x_1 and x_2) the quotient found, by dividing one group by the other group, with the same range, will, in general, be the *exact* value for some given value of the abscissa; and ordinarily this value must not be referred to the middle of the range in question, when high precision is desired. Let the two distributions be denoted by G and H, and let G be the numerator group and H the denominator group; and let the five quantities x_1, x_g, x_h, x_m. and x_2 denote respectively :—(i.) the value of the abscissa at the beginning of the group; (ii.) the abscissa of the mean ordinate of the group G, and (iii.) that of H; (iv.) the abscissa where the ratio becomes exact; and (v.) the value of the abscissa at the end of the range. That is, if ${}_gy_m$ and ${}_hy_m$ denote the ordinates at x_m for the two distributions, then we must have :—

$$(616) \ldots\ldots G/H = {}_gy_m/\ y_m$$

The following laws hold as to the position of x_m in relation to x_g and x_h the latter being the abscissæ of the ordinates equal to the respective means of the two distributions ;—

(i.) The two distributions increase linearly,[1] then $x_m = \frac{1}{2}(x_1+x_2)$.

(ii.) The relative increase of the two distributions, though not linear, is identical throughout (*i.e.*, they increase in the same proportion); then x_m is the common abscissa of the means of the group-ordinates.

[1] Provided, however, that the prolongation of the bounding lines does not meet on the axis of the abscissæ, since, in this case, the required ratio holds for any value of x; that is to say, in this case x_m is indeterminate. This, however, does not vitiate the adoption of the middle of the range, or indicate that it should not be adopted.

(iii.) The ordinates of one distribution are constant throughout; the values of the other are variable; x_m is the abscissa of the ordinate equal to the mean of the ordinates of the variable distribution.

(iv.) The relative increase of the ordinates of distribution G is more rapid than that of the ordinates of distribution H; then $x_g > x_m > x_h$.

(v.) The relative increase of the ordinates of distribution G is less rapid than those of distribution H; then $x_g < x_m < x_h$.

(vii.) Where the distributions G or H include maxima or minima (either one or both), no general law applies as regards the value of the abscissa x_m, and it may have more than one value. In general also the position of x_m in such cases is not accurately determinable from the group-data.

The most general supposition that can be made regarding the curve of instantaneous values which, between given limits, will satisfy a particular group-value (not near a maximum or minimum, at a point of inflexion, or very near the terminals) is that it is approximately represented by Be^{bx} or Bn^x in which e^b or n is the ratio of any group to the adjoining group. A curve of this type will satisfy three groups G_{-1}, G and G_1, in ascending or descending order of magnitude, see Part XIII., §25 (508), p. 266. Let the value of G/G_{-1} be n_1, and of G_1/G be n_2, then the value of $\mathfrak{m} = e^b$, which will give a curve satisfying the three group totals, is[1] :—

$$(617) \ldots\ldots \mathfrak{m} = n_1 (n_2 - 1)/(n_1 - 1)$$

The common quantity χ to subtract so as to get three groups with the common ratio $\mathfrak{m}$ between the second and first, and third and second, is :—

$$(618) \ldots\ldots \chi = G (n_2 - n_1)/(1 + n_1 n_2 - 2n_1)$$

where G is the central group, and the position of the ordinate corresponding to the group-height, $G / (x' - x)$ say, is wholly dependent upon $\mathfrak{m}$,[2] which should be substituted for either n_1 or n_2.

The abscissa of intercept of the group-rectangle with the curve of distribution is obviously independent of the scale, or of the zero of the

[1] It is important to bear in mind when high accuracy is desired, that the mean of n_1 and n_2 is not necessarily at all near the value $\mathfrak{m}$. For example suppose the groups are 1000, 1200, and 2040, the value of n_1 is 1.2, and of n_2, 1.7. The arithmetic mean is 1.45, and geometric 1.42829. While the value of $\mathfrak{m}$ is 4.2. For the ratios (1000—937.5) : (1200—937.5) : (2040—937.5) are identical, and are 4.2. Similarly the groups 2040, 1200, and 1000 give for n_1 the value $1/1.7 = 0.0588235$, and for that of n_2 the value $1/1.2 = 0.0833333$ give $1/4.2 = 0.0238095$, as may be seen by applying the formula for the value of $\mathfrak{m}$. The position of the abscissa of the ordinate to the curve corresponding to the mean height is the same in either case, as also is the position of the centroid vertical.

[2] The value of x_m is that of the abscissa of the point (or points), where a line parallel to the axis of abscissæ and distant therefrom the average of the group-ordinates cuts the bounding curve (the curve of distribution); or—in a graph—the abscissa of the intersection of the group rectangle with the curve of distribution.

figure. We suppose the range to be 0 to 1, and the curve to be $y = Ae^{ax}$, in which e^a is $\mathfrak{m}$. Hence since $\log. \mathfrak{m} = a \log. e$, we have by integrating between the limits 0 and 1 :—

$$(619)\ldots\ldots y_m = e^{ax_m} = \frac{1}{a}(e^a - 1) = \mathfrak{m}^{x_m} = \frac{\log e}{\log m}(m - 1)$$

Consequently by taking logarithms of this last form of the equation, we have, since $\log._{10}\ 2.3025851 = 0.3622157$:—

$$(620)\ldots\ldots x_m = \frac{\log_{10}(\mathfrak{m} - 1) - \log_{10}(\log_{10} \mathfrak{m}) - 0.3622157}{\log_{10} \mathfrak{m}}$$

To find the value of x_m when there are two curves we have, writing ξ for x_m for convenience, and in order to distinguish this case from the previous one :—

$$(621)\ldots\ldots \frac{A(e^a-1)/a}{B(e^b-1)/b} = \frac{Ae^a\xi}{Be^b\xi};\ \text{wherefore}\ \frac{(\mathfrak{m}_a - 1)/a}{(\mathfrak{m}_b - 1)/b} = \frac{\mathfrak{m}_a^\xi}{\mathfrak{m}_b^\xi}$$

But $1/a$ divided by $1/b = \log. \mathfrak{m}_b / \log. \mathfrak{m}_a$ consequently :—

$$(622)\ldots\ldots \left(\frac{\mathfrak{m}_a}{\mathfrak{m}_b}\right)^\xi = \frac{\log \mathfrak{m}_b}{\log \mathfrak{m}_a} \cdot \frac{\mathfrak{m}_a - 1}{\mathfrak{m}_b - 1},\ \text{and therefore}$$

$$(623)\ldots \xi = \frac{(\log \mathfrak{m}_a - 1) - \log(\mathfrak{m}_b - 1) + \log(\log_e \mathfrak{m}_b) - \log(\log_e \mathfrak{m}_a)}{\log \mathfrak{m}_a - \log \mathfrak{m}_b}$$

which put in suitable form for computation with Briggsian logarithms is

$$(623a)\ldots\ldots \xi = \frac{\log_{10}(\mathfrak{m}_a - 1) - \log_{10}(\mathfrak{m}_b - 1) + \log_{10}\log_{10}\mathfrak{m}_b - \log_{10}\log_{10}\mathfrak{m}_a - 0.3622157}{\log_{10}\mathfrak{m}_a - \log_{10}\mathfrak{m}_b}$$

The fraction ξ can thus be readily tabulated in a table of double entry, with the arguments $\mathfrak{m}_a$ and $\mathfrak{m}_b$. If in these last formulæ (623) or (623a) we put $\mathfrak{m}_b = 1$, we get (620). If (623) is used it is important to note that the Napierian logarithms are to be used where indicated. Formula (622) may be regarded as the fundamental equation for the determination of x_m or ξ.

The preceding formulæ are unsuitable when $\mathfrak{m}_a = \mathfrak{m}_b$. But by putting $\mathfrak{m}_a = \mathfrak{m}_t + h$ where h is a very small quantity, we obtain, after expanding both sides of the following equation, viz. :—

$$(624)\ldots\ldots \left(\frac{\mathfrak{m} + h}{\mathfrak{m}}\right)^\xi = \frac{\log \mathfrak{m}}{\log(\mathfrak{m} + h)} \cdot \frac{\mathfrak{m} - 1 + h}{\mathfrak{m} - 1}$$

which is (621) recast, and remembering that the powers of h are negligible :—

$$(625)\ldots\ldots \xi' = \frac{\mathfrak{m}}{\mathfrak{m} - 1} - \frac{1}{\log_e \mathfrak{m}}$$

ξ', therefore, is not really indeterminate, when $\mathfrak{m}_a = \mathfrak{m}_b$, as might have easily been wrongly inferred from (623). When both $\mathfrak{m}_a$ and $\mathfrak{m}_b$ are 1 it may readily be shewn that $\xi' = 0.5$ by expansions applied to (625), which is but a special case of the following, viz. :—

If $\mathfrak{m}_a = 1/\mathfrak{m}_b$, then we have at once from (622) that $\xi = \frac{1}{2}$. And finally if $\mathfrak{m}_a = k\mathfrak{m}_b$ we have

$$(626)\ldots\ldots k^{\xi} = \{(k\mathfrak{m}_b - 1) \log \mathfrak{m}_b\} / (k-1) \log k\mathfrak{m}_b,$$

by means of which also tables may be constructed.

The following table will enable the value to which any group-ratio should be referred to be readily found, after the values of $\mathfrak{m}_a$ and $\mathfrak{m}_b$ for the two distributions have been ascertained. Where the original distributions are increasing with x the value ξ given in the table is read from the left-hand toward the right; where diminishing, with increase of x, from the right-hand toward the left. Thus if $\mathfrak{M}_a = 1/\mathfrak{m}_a$ and $\mathfrak{M}_b = 1/\mathfrak{m}_b$, then :—

$$(627)\ldots\ldots \xi_{\mathfrak{M}} = 1 - \xi_m$$

consequently the table may be entered for the reciprocals of *both*[1] the ratios without altering the result provided the point to be determined is taken either ξ_m from one end of the range or $\xi_{\mathfrak{M}}$ from the other.

TABLE CXLIV.—**Shewing the values of ξ, viz., the relative Distance from the Initial Value of any Range on the Axis of abscissæ, to the Ordinate, to which the Ratio of any Two Groups should be ascribed, the whole Range being regarded as Unity.**

VALUES OF ξ.

m_a or m_b	.05	.10	.15	.20	.25	.5	1.0	1.25	1.50	1.75	2.0	3	4	5	7.5	10	15	20	25	40	m_a or m_b
.05	.281	.302	.315	.324	.332	.357	.383	.392	.399	.406	.411	.427	.439	.447	.462	.474	.488	.500	.508	.525	.05
.10	..	.323	.337	.347	.355	.382	.408	.417	.424	.430	.436	.452	.464	.473	.488	.500	.514	.526	.534	.550	.10
.15	..	..	.351	.361	.369	.396	.423	.432	.440	.446	.451	.467	.479	.489	.504	.516	.530	.542	.550	.566	.15
.20	..	..	..	.371	.380	.406	.434	.443	.451	.457	.463	.479	.491	.500	.515	.527	.541	.553	.561	.577	.20
.25	..	..	..	..	.388	.415	.443	.452	.460	.466	.472	.488	.500	.509	.524	.536	.550	.561	.569	.586	.25
.50	..	..	..	..	..	.443	.471	.480	.488	.495	.500	.516	.528	.537	.552	.564	.578	.589	.597	.613	.50
1.0	..	..	..	..	..	..	.500	.509	.517	.523	.529	.545	.557	.566	.581	.592	.607	.617	.624	.639	1.0
1.25	..	..	..	..	..	..	..	.519	.526	.532	.538	.554	.566	.575	.590	.601	.614	.625	.633	.648	1.25
1.50	..	..	..	..	..	..	..	..	.534	.540	.546	.562	.574	.582	.597	.608	.622	.634	.641	.656	1.50
1.75	..	..	..	..	..	..	..	..	..	.547	.552	.568	.580	.588	.602	.614	.628	.639	.646	.661	1.75
2.0	..	..	..	..	..	..	..	..	..	..	.557	.573	.585	.594	.607	.618	.632	.643	.650	.665	2.0
3	..	..	..	..	..	..	..	..	..	..	..	.590	.602	.611	.624	.635	.649	.659	.666	.680	3
4	..	..	..	..	..	..	..	..	..	..	..	..	.612	.621	.634	.645	.658	.668	.675	.689	4
5	..	..	..	..	..	..	..	..	..	..	..	..	..	.629	.642	.653	.666	.676	.683	.696	5
7.5	..	..	..	..	..	..	..	..	..	..	..	..	..	..	.658	.669	.681	.691	.698	.711	7.5
10	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	.677	.689	.698	.705	.718	10
15	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	.701	.710	.716	.729	15
20	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	.719	.724	.737	20
25	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	.727	.742	25
40	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	.755	40

If x, x_m and x' are respectively the beginning of the range, the point at which the ordinates to the two distributions are in the exact ratio of the corresponding groups, and the end of the range, ξ is the ratio of the distance x to x_m to the distance x to x'.

Most of the quantities in the table have been directly calculated, and are less than .0005 in error. The greatest error in the interpolated part of the table will be about .002.

Let h and k be any two small quantities such that $m'_a = m_a + h$, and $m'_b = m_b - k$, then by expanding and neglecting powers of h and k higher than the first we obtain :—

$$\frac{h}{k} = -\frac{m_a}{m_b} \cdot \frac{\xi_{m_a m_b} - \xi_{m_b m_b}}{\xi_{m_a m_b} - \xi_{m_a m_a}}$$

which shews that if we add (or subtract) any small quantity to (or from) m_a to get a tabular value we must subtract (or add) m_a/m_b times the quantity from (or to) m_b in order to obtain the true value of ξ

This follows from the fact that the ξ quantity alone is approximately unity in all practical cases, and gives the required quantity by a single interpolation.

If reciprocals be taken of *both* m_a and m_b, the value of ξ is the arithmetical complement of its value for m_a and m_b, *i.e.* :—

$$\xi_{m_a m_b} + \xi_{\frac{1}{m_a}\frac{1}{m_b}} = 1$$

[1] It hardly needs to be added that the table must not be entered for *one* reciprocal only.

15. **Absence of climacterics in mortality.**—A general impression exists that death is more than normally frequent at some age between the ages of 50 and 60 in the case of males, and an age between 40 and 50 in the case of females. The crude figures for individual years are very irregular, and no precise deduction can be based upon them until they are smoothed. The sums for 9 years (1907-1915) from Australian records are as follows :—

Ages	48	49	50	51	52	53	54	55	56	57
Deaths of Males	2841	2723	3667*	2323	3071	2827	2987	2903*	3010	2781
Smoothed Means	..	2912	2925	2922	2926	2921	2960	2902	2938	2931
Ages	..	37	38	39	40*	41	42	43	44	45*
Deaths of Females ..	..	1402	1619	1540	1608	1206	1695	1396	1449	1673
Smoothed Means	..	..	1529	1475	1534	1489	1478	1470	1527	1493
Ages	58	59	60	61	62	63	64	65	66	67
Deaths of Males	3011	2705	3636*	2232	2896	2995	3068	3905*	3044	3286
Smoothed Means	2930	2873	2896	2893	2959	3032	3182	3260	3307	..
Ages	46	47	48	49	50*	51	52	53	..	..
Deaths of Females ..	1423	1524	1601	1741	1782	1319	1733	1601	..	..
Smoothed Means	1534	1590	1619	1593	1635	1635	1584	..	..	..

* If the errors be supposed to accrue mainly through ages 1 and 2 years above and below the true age, the true value can be found approximately by taking the mean of 5 years, since the results are usually linear for small ranges of age. The excess over this can then be distributed among the ages plus and minus 1 and 2, according to the probabilities of the case. The general trend can then be found. One-third of the excess over one-fifth was added to the age below, one two-thirds to the age above. The quinquennial means so corrected were formed, and are as given opposite "smoothed means."

The instances where the age ends in the integers 0 and 5 shew the usual defect, viz., a tendency to give approximate ages ending in those numbers. When the necessary distribution of the excess, however, is effected no indication exists of climacterics (marked in the death-rates of males or females) in the range shown. This is evident from the following rates deduced from the adjusted population results of the Census, after making allowance for the difference from the mean of the 9 years included,[1] viz. :

Ages (Males)	49	50	51	52	53	54	55	56	57	58	59	60	61	62	63	64	65	66
Death-rates'..	0124	0130	0138	0148	0159	0173	0183	0202	0220	0241	0255	0276	0295	0319	0344	0377	0404	0430
Smoothed ..	0124	0130	0138	0148	0160	0174	0180	0198	0218	0240	0253	0267	0292	0318	0345	0373	0402	0432
Ages (Females	38	39	40	41	42	43	44	45	46	47	48	49	50	51	52	53		
Death-rates'..	00607	00598	00635	00632	00645	00658	00701	00702	00740	00789	00830	00852	00924	00991	01031			
Smoothed ..	00598	00613	00628	00643	00658	00673	00690	00711	00738	00773	00818	00873	00928	00983	01038			

It is worthy of note that the actual number of deaths of males oscillates very little on either side of the average, 2921, between the ages 49 and 62 ; and also that the actual number of deaths of females between the ages 38 and 45, and between 45 and 52, oscillate but small amounts on either side of the averages, viz., 1499 and 1598 respectively. The death-rates of course all increase appreciably.

16. **Fluctuations of the ratio of female to male death-rates according to age.**—The average ratio of the female rate of mortality to the male rate of mortality for 1886 to 1915, according to age, is given in the final column of Table CXXXV., p. 375, and this ratio for the death-rate for all ages is

[1] The ratio of males and females as at the Census of 1911, and for the sum of the mean populations, gave the following results :

Sum Mean Male Populations 1907–15 ÷ Males at Census 1911 = 21,150,358 ÷ 2,313,035 = 9.143985
,, Female ,, ,, ÷ Females ,, ,, = 19,620,889 ÷ 2,141,970 = 9.160207

The divisors used to obtain the mean numbers were 9.1440 and 9.1602 for males and females respectively.

given in Table CXXXVI., p. 376. It is analysed in Table CXXXVII. p. 377. The fluctuations with time of the rates according to age are shewn in the following table :—

TABLE CXLV.—Shewing the Variation in the Ratio of Female to Male Mortality-rates according to Age ; 1886 to 1911. Australia.

Age-group.	Epoch. 1886.0	1896.0	1906.0	1911.0	Age-group.	Epoch. 1886.0	1896.0	1906.0	1911.0
0 ..	.864	.975	.826	.803	40–44	.882	.812	.813	.763
1 ..	.958	.938	.933	.818	45–49	.793	.763	.728	.709
2 ..	.969	.981	.932	.896	50–54	.752	.733	.686	.695
3 ..	.936	.982	.934	.933	55–59	.727	.731	.716	.684
4 ..	.936	.982	.929	.996	60–64	.725	.739	.741	.686
0–4 ..	.887	.867	.844	.824	65–69	.800	.721	.769	.742
5–9 ..	.924	.945	.905	.913	70–74	.830	.803	.801	.792
10–14 ..	.929	.877	.891	.884	75–79	.855	.853	.792	.840
15–19 ..	.769	'861	.907	.862	80–84	.949	.863	.809	.867
20–24 ..	.753	.826	.902	.935	85–89	.892	.843	.834	.810
25–29 ..	.898	.900	.989	1.002	90–94	1.154	.984	.963	.937
30–34 ..	.913	.954	.976	.936	95–99	.715	1.225	1.139	.979
35–39 ..	.969	.939	.944	.880	100–104	.770	1.187	1.163	.949
0–4 ..	.887	.867	.844	.824	15–49*	.854	.865	.894	.870
5 14* ..	.926	.911	.898	.897	50–89*	.816	.786	.769	.765

* Average of rates merely.

The results in the table shew that, on the whole, there is a decrease in the ratio of the death-rates for females, *i.e.*, the environment or its effect has become more favourable in the case of females than in the case of males in Australia. This result is well defined for ages 0 to 4 ; fairly well-defined for ages 5 to 14, and is not well defined for later ages. Moreover for all ages the results are rather irregular. The reciprocal of this ratio, viz., the male divided by the female rate for the years 1907-1915, is given in Table CXLVI. hereinafter, and is shewn on Fig. 101 by curve E.

17. **Rates of mortality as related to conjugal condition.**—The effect of conjugal condition upon death-rates is well marked, and is shewn in the following table, based upon three years' results, viz., 1910-1912. For convenience of comparison the mortality results for nine years for all males and all females is repeated[1] in the table, the masculinity ratio between the death-rates being also given ; see curve E, Fig. 101. The ratio of the death-rate of the "not married" to that of the married fluctuates considerably from age-group to age-group, both for males and females. For males between 20 and 85 years of age the death-rate is considerably higher for the "not-married." For females the advantage lies with the "not married" until the child-bearing period has been passed, after which it lies with the "married." The exact age at which the death-rates become equal in the case of females is probably about 43 years.

[1] These results were given in Table CXXXV., p. 375, and the average "femininity" of the death-rate was also given for each age-group.

TABLE CXLVI.—Shewing in Age-groups the Average Death-rates of all Males and all Females, 1907-1915, and of Married and Unmarried Males and Married and Unmarried Females, 1910-1912. Australia.

Age-Group.	Age.	1907-1915. Mortality Rate.		Ratio Male to Female Rate.	Mortality Males, 1910-1912.			Mortality, Females. 1910-1912.		
		Males.	All Females.		Married. (m)	Not Married. (u)	Ratio. u/m	Married. m'	Not Married. u'	Ratio. u'/m'
0 ..	0.43	.08540	.06862	1.2445						
1 ..	1.46	.01559	.01389	1.2226						
2 ..	2.48	.00642	.00575	1.1153						
3 ..	3.49	.00409	.00382	1.0712	..	.01034		..	.00858	—
4 ..	4.49	.00301	.00300	1.0037						
0-4 ..	1.98	..	..	..						
5-9 ..	7.42	.00222	.00202	1.0949						
10-14 ..	12.60	.00173	.00153	1.1315						
15-19 ..	17.58	.00256	.00221	1.1597	.00257	.00251	0.977	.00388	.00215	0.554
20-24 ..	22.49	.00364	.00341	1.0699	.00244	.00378	1.549	.00418	.00321	0.768
25-29 ..	27.46	.00431	.00432	.9984	.00329	.00504	1.532	.00454	.00390	0.859
30-34 ..	32.50	.00508	.00475	1.0684	.00405	.00682	1.684	.00482	.00429	0.890
35-39 ..	37.50	.00666	.00586	1.1363	.00564	.00896	1.589	.00616	.00506	0.821
40-44 ..	42.51	.00841	.00641	1.3108	.00752	.01099	1.665	.00643	.00609	0.947
45-49 ..	47.47	.01120	.00794	1.4101	.01039	.01535	1.477	.00765	.00974	1.273
50-54 ..	52.39	.01511	.01050	1.4394	.01406	.01917	1.363	.01065	.01237	1.162
55-59 ..	57.45	.02153	.01473	1.4615	.02039	.02709	1.329	.01459	.01778	1.219
60-64 ..	62.48	.03174	.02177	1.4578	.02947	.03614	1.226	.02097	.02416	1.152
65-69 ..	67.46	.04678	.03471	1.3479	.04523	.05266	1.164	.03529	.03766	1.067
70-74 ..	72.41	.06972	.05523	1.2624	.06730	.08364	1.243	.05504	.07433	1.351
75-79 ..	77.27	.10900	.09162	1.1898	.10721	.13670	1.275	.09348	.11829	1.265
80-84 ..	82.15	.16815	.14575	1.1537	.16415	.20613	1.256	.14615	.14664	1.003
85-89 ..	86.96	.2678	.2170	1.2342	.2640	.2199	0.883	.2106	.2385	1.133
90-94 ..	91.88	.309	.2895	1.0669						
95-99 ..	96.70	.391	.3832	1.0207	.354	.327	0.924	.328	.505	1.541
100 & over	101.4?	1.13	1.07	1.0542						
All ages'	..	.001194	.000945	1.2636	..	..	..	..	..	..

The "married" include "widowed" and "divorced."

The graphs of the ratios of the death-rates of the "not-married" to the death-rates of the married are curve G for males, and curve F for females, Fig. 101.

18. **Exact ages of least mortality.**—The ages of least mortality appear to vary slightly, but cannot be ascertained with a very high degree of precision, owing to the limitations of the data. They may be taken, however, to be as follows :—

Males (Year)	1886.0	1896.0	1906.0	Females	1886.0	1896.0	1906.0
Exact Age	12.0	11.8	11.0	..	12.0	12.2	10.6
Death-rate	.00232	.00213	.00178	..	.00223	.00178	.00159
*General trend (.889)	.00232	.00206	.00183	(.830)	.00223	.00185	.00154
† or (.896)	.00231	.00207	.00185	(.823)	.00225	.00185	.00152

* Allowing twice the weight to the ratio for the 20-years period to that for the 10-years period.

† Allowing equal weight to the 20-years and 10-years periods. This result is less probable than the former.

These results, deduced from the values given in the Life Tables of the Australian Census, Vol. III., pp. 1209 to 1218, are probably nearly correct, and indicate a minimum mortality at "exact age" 11.6 for both males and females, and not only a less mortality, but also a greater improvement in the case of females : the factors for 10 years being respectively 0.889 and 0.830. These for 25 years would be respectively 0.745

and 0.628. The ratios of female to male deaths were—from the data—0.961, 0.836, 0.893; from the general trend—0.961, 0.898, 0.843. Thus the improvement increases at the rate indicated by multiplying by 0.99362 per annum, or by 0.938 for a 10-year period, that is as 0.959, 0.899, 0.843, at 1886, 1896 and 1906.

19. **General theory of the variation of mortality with age.**—From time to time attempts have been made to present a rational theory of the variation of the death-rate with age.[1] On the other hand it has been held that such attempts are merely efforts to clothe what is really an empirical "fitting of the curve," with a rational guise.[2] It is certain, however, that, in a general way, the aggregate of the menaces to life may be subsumed under certain elementary conceptions, which we now proceed to indicate. Actuaries have adopted the term "*force of mortality,*" to denote the death-rate *at* a given age (*i.e.*, between the ages x and $x + dx$). This may be deemed to be *composite,* and to consist of several forces of mortality which, operating over a considerable range of ages, have their maximum effect, however, at different ages. Thus the deaths D_x of persons whose ages are (sensibly) x, in a population P_x of the same age (within the same limited age limits) is :—

$$(628)\ldots\ldots D = P\,(\mu' + \mu'' + \mu''' + \text{etc.}) = P\mu$$

(with the same suffix—denoting age—throughout).

The following conception of the nature of the life-and-death struggle of a living organism represents the phenomena in a general way.[3]

(i.) The plasticity of an organism, *i.e.*, its modifiability in reacting to its environment, is a maximum at its origin,[4] and continually diminishes during life;

[1] Prof. Karl Pearson, adopting the "Vision of Mirza" conception, suggests an analysis by means of systems of progressive elimination, viz., first of deaths due to "old-age mortality," then those due to other elements, and finally those due to infantile mortality. He divided the deaths into five elements, viz. (i.) those from old-age mortality (mode, at age $72\frac{1}{2}$, mean, age $67\frac{1}{2}$); (ii.) those from middle-age mortality (mode, age 42); (iii.) those from mortality of youth (mode, age 23); (iv.) mortality of childhood (mode, age 23); and (v.) infantile mortality. This last should start—so it is averred—at -0.75, *i.e.*, nine months before birth. The "recorded deaths" are the post-natal, which to the ante-natal are as 246 : 605. See K. Pearson, "The Chances of Death," etc., Vol. I., 1897, pp. 1–41.

[2] Prof. Harald Westergaard says :—(See his paper on the "Scope and Method of Statistics," Journ. Americ. Stat. Assoc., Vol. XV., Sept., 1916, p. 254):—"Several mathematicians have erred in thinking that it would be possible to find a mathematical law of mortality, a physiological law, as it were. We have several formulas of this kind, by Lambert, Moser, Gompertz, Makeham. For a certain period of life Makeham's formula is exceedingly practical, *but after all it is only a beautiful formula of interpolation.*"

[3] It may be added that the similar problem *in utero,* though important to physiologists, can be solved only when a sufficient number of women, having become competent and interested observers of their own careers, supply the necessary data.

[4] This is probably a measure of the rapidity of metabolism in the organism.

(ii.) In virtue of its plasticity an organism is both *vulnerable* and *recuperable*.

(iii.) On "birth," *i.e.*, on the introduction into a new environment, the *inimical force*, *i.e.*, the difficulty of adjustment to the new environment, is very great, but this difficulty diminishes continually and with great rapidity. The initial difficulty of adaptation to the new environment may be called the *natal force of mortality*.

(iv.) This falls off so rapidly that it may be regarded as operating for the very limited number of days[1] that constitute what may be called the initial or *natal adaptative effort*.

(v.) The adaptation having been established, a new condition supervenes during which the mortality is markedly less, and characterises what may be called the *infantile adaptative effort*, covering roughly the first twelve months of life.

(vi.) The inimical force, now greatly diminished as a consequence of successful initial adaptation, may be called the *infantile force of mortality*. This, like the natal force of mortality, also degrades, but nothing like so rapidly.

(vii.) Since organic life is maintained in virtue of its plastic endowment (adaptability to its environment) the inimical forces (or measure of the difficulty of adaptation) increase continually from the moment of birth (or more strictly from the moment of origin *in utero*). This growing incapacity for adaptation may be called the *senile force of mortality*.

(viii.) The pressure put on the organism of social life (education, etc.), and by the arising of puberty, and the assertiveness of the reproductive forces, constitute an inimical force, characteristic of the period of life commencing in childhood and vanishing at the end of the disturbing (reproductive) period. This may be called the *genesic force of mortality*.

So far the consideration has embraced both sexes, but in the case of the female another force must be assumed, when a differentiation between reproductive and non-reproductive females is taken into account, viz., the following :—

(ix.) The exhaustion and general dangers of reproduction, initiating on reaching puberty, and continuing till the end of the reproductive period, constitute an inimical force which may be called the *gestate force of mortality*.

[1] From what has preceded, see § 11 of this part, it would appear that this is about 5 days.

(x.) There exist also dangers to life which are of a purely casual nature. The aggregate of these may be called the *adventitious force of mortality.* This, however, probably need not be separated from the other forces of mortality.[1]

These several forces of mortality can be so evalued as to be *additive* in character, as in formula (628), so that the (average) aggregate force of mortality is their sum. This aggregate of inimical forces thus gives the measure at any age of the force determining the rate of death for persons of the age in question. We shall later refer again to this element. We may also suppose that there are in addition what may be called *special forces of mortality.* The indication given may be regarded as the condition of things when general hygiene is fairly satisfactory throughout life. Not only, however, do individual instances differ from this, but so also do the characteristics of particular communities.[2]

---

[1] The conception of life as a play between conservative and inimical forces has been presumed by some to be inadequate. Thus although the rapid diminution of the "natal" and "infantile" forces of mortality may be supposed to measure the quick and slower elements of the adaptation attained, the "secular" force of mortality specially characteristic of old age, to measure the decrease of adaptability, and the "genesic" and "gestate" forces of mortality—analogous to one another—to measure the stress put on the organism by the play of the reproductive function, and its consequential effects, entering as it were, as a disturbing factor the effect of which ultimately vanishes, yet there is another factor, acting throughout life, which, as Gompertz considered, is apparently independent of the progressive deterioration with age of the organism, and of course independent also of its adaptativeness. This chance element, viz., the "adventitious force of mortality," would, of course, include death by accident or misadventure, is certainly not a constant ; it is a function of age, and differs strikingly as between the sexes. The real vicissitudes of life of a chance nature are, however, not on the average uniform, and probably are not very dissimilar in relative frequency to the relative frequency of death from such causes as have already been indicated. To the extent this is so they may, of course, be regarded as embraced in the other inimical forces.

In considering the whole question, it is to be remembered that we are not really dealing with individual lives, but with a *multiplex-organism,* viz., an aggregate of lives or population ; and we are measuring the progressive reduction of that organism by the elimination of theoretically infinitesimal elements (removal by death). And from this point of view it is obviously *very doubtful whether the conception of an adventitious force of mortality is necessary at all,* and it is certain that to the extent the relative frequency conforms to the other types of inimical forces it may be regarded with advantage not as merely masked by, but included in them. Prof. Westergaard says, however, op. cit., p. 254 : "If we seek a formula for the combined effects of all the causes in action, we run the risk of overlooking some, which it would really be exceedingly important to take into consideration."

[2] In Saxony, for example, the mortality apparently falls till the eighth day after birth after which there appears to be a recrudescence of mortality till the fifteenth or sixteenth day before the final continuous fall of the infantile mortality. In Sweden the mortality shews a less marked and irregular recrudescence till the twelfth or thirteenth day. In Australia the rapid fall continues till the end of the fourth or fifth day, then continues at somewhat the same rate for twelve months. Thus for the *period of high infantile mortality* Australia seems normal for good infant hygiene, Sweden more normal than Saxony, and Saxony abnormal, *i.e.*, the infant hygiene is probably not at all good. From this it is obvious that each large population will probably have to be treated independently in regard even to the *form* of the curve representing the earlier stages of the force of *infantile mortality.* This has already been shewn, see pp. 389-392.

20. **The Gompertz-Makeham-Lazarus theory of mortality.**—In 1825, B. Gompertz,[1] suggested that death was possibly "the consequence of two generally coexisting causes," viz.: (i.) *Chance*, without previous disposition to death or deterioration: and (ii.) *Deterioration*, or lessened ability to withstand destructive agencies. Assuming that exhaustion of the resisting power to disease proceeds in constant ratio for equal increments of age, that is, that the *force of mortality* increases in geometrical progression, he deduced his well-known formula, viz.:—

$$(629)\ldots\ldots\mu_x = -\frac{1}{l_x}\cdot\frac{dl_x}{dx} = Bc^x$$

B and c being constant, independent of the age (x), and determinable from the data of a mortality table, l_x being the number living at the age x, and dl_x the change in l_x in the time (*i.e.*, change of age) dx.

In January 1860, Makeham, having examined a number of mortality tables for the ages 20 to 80, found it was necessary to modify the Gompertz formula. He shewed that, for the age-period mentioned, the so-called "force of mortality," μ_x, as given in several *mortality tables*, could be closely represented without changing the constants of formula (629) by adding a constant A, viz., by an expression of the form:—

$$(629a)\ldots\ldots\mu_x = A + Bc^x.$$

in other words, the force of mortality, assumed by Gompertz to be a geometrical progression, should, according to Makeham, be represented by a geometrical progression plus a constant.

[1] Gompertz showed that if the chance of disease were equal at all ages, and if its effect were independent of age, then it would follow that the number of living and dying, as the age increased in arithmetical progression, would decrease in geometrical progression. But, if liability to death increased with age, the number living would diminish faster than in geometrical progression. He observed that, although the hypothesis was not an "unlikely supposition with respect to a great part of life, the contrary appears to take place at certain periods"; see his paper "On the Nature of the Function expressing the Law of Human Mortality," read before the Roy. Soc., Lond., 16th June, 1825, and appearing in the Phil. Trans., 1825, pp. 513-585. He had given an earlier paper (June 29th, 1820) at the same society, entitled "A Sketch of an Analysis and Notation applicable to the Estimates of the Value of Life Contingencies." Phil. Trans., Pt. I., 1820, pp. 214-294. This obviously led to the later ones; see also a supplement to both read 20th June, 1861, and published in the Phil. Trans., 1862, Vol. 152, pp. 511-559. Prof. De Morgan discussed Gompertz's view; see "On a Property of Mr. Gompertz's Law of Mortality," Journ. Inst. Actuaries, Vol. VIII., July 1859, pp. 181-184; and also Phil. Mag., Nov. 1839. To represent number living at age x, Prof. De Morgan used:—

$$l_x = e^{k+bc^x} = dg^{q^x} \text{ where } d = e^k,\ g = e^b,\ q = e^c$$

Later, viz., 1839, Ludwig Moser published in Berlin his "Die Gesetze der Lebensdauer.

Later he discovered that a further modification, viz., the introduction of a term Cx, that is, an arithmetical progression, gave the formula a wider extension. Thus his second modification was the expression :—

$$(629b)\ldots\ldots\mu_x = A + Cx + Bc^x.$$

The significance of expressions of this type is seen at once from (630) hereinafter, that is :—

$$(629c)\ldots \log_e y = -\int Bc^x dx = K - \frac{B}{\log c} c^x \text{ ; or } y = kg^{c^x}$$

according to Gompertz ; or

$$(629d)\ldots \log_e y = -\int (A + Bc^x)dx = K - Ax - \frac{B}{\log c} c^x \text{; or } y = ks^x g^{c^x}$$

according to Makeham's first modification of Gompertz's formula ; and

$$(629e)\ldots \log_e y = -\int (A + Cx + Bc^x)dx = K - Ax - \tfrac{1}{2}Cx^2 - \frac{B}{\log c} c^x;$$
$$\text{or } y = ks^x h^{x^2} g^{c^x}$$

according to Makeham's second modification.

In these K is merely an integration constant, and is equal to $\log_e k$, and $\log_e g = -B/\log_e c$; $\log_e s = -$ A ; $\log_e h = -\frac{1}{2}C$.

More recently Wilhelm Lazarus, of Hamburg,[1] and later Vitale Laudi,[2] of Trieste, in order to embrace results for earlier ages, abandoned the arithmetical progression represented by the term Cx, and introduced in its place a second geometrical progression making the form of the instantaneous rate of mortality.

$$(629f)\ldots\ldots\mu_x = A + Gb^x + Bc^x.$$

C. H. Wickens has shewn that, for Australia, infantile mortality from birth to age 5 is well expressed by a formula of the type of Makeham's second modification of the Gompertz formula[3] ; see also § 20 hereinafter.

An expression is general, however, if it cover the whole range of life with the one series of constants for any particular epoch : this none of the formulæ will do with the number of terms adopted. Before further developing the matter we shall consider the nature of a constant population, the death-rates of which are also constant.

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[1] See Uber Mortalitätsverhältnisse und ihre Ursachen, Hamburg, 1867. Lazarus' paper was translated by T. B. Sprague, M.A. ; see Journ. Inst. Act., Vol. XVIII., pp. 54-61 ; 212-223. T. S. Lambert published an article on Longevity, in 1869, New York.

[2] In a publication, " Die Rechnungsgrundlagen der k.k. priv. Assicurazioni Generali in Triest," a very concise exposition is given of the biologic basis of the fundamental formula under the title " Die biologische Begründung der Ausgleichungsformel nach Lazarus." See § 2, pp. xxiv.-xxix.

[3] See " Investigations concerning a Law of Mortality," C. H. Wickens, A.I.A., Journ. Aust. Assoc. Adv. Sc. XIV., pp. 526-536.

21. **Theory of an "actuarial population."**—Consider an indefinitely large group of persons, who born at a given moment, are then subject to death, the rate of which (governed solely by age) is characteristic of their environment (and period). If this group be neither increased nor diminished by emigration, and as age increases be lessened only by death, the proportion of survivors at each age may be regarded as furnishing the relative numbers of what may be called an *actuarial population.* A population so constituted plays an important part in actuarial investigations as to the probability of death according to age, and has also been called a "*constant population.*"

Let the *ratio* of such a population after the period x has elapsed, P_x say (the members now being all of age x) to the initial population, viz., that at age 0, P_0 say, be denoted by y_x; that is, let $y_x = P_x/P_0$; then initially y, or $(y_0) = 1$ and $1 - y_x$ will denote the ratio of the aggregate of deaths up to the age x.

Let $\mu = \phi(x)$ denote the *rate* per unit of time[1] at which death occurs at the "exact age" x; then the number dying in a unit of time, whose ages are between x and $x + dx$, is the number living between those age-limits, multiplied by the rate of dying, that is, $y\mu\ dx$. Thus if μ be regarded as positive

$$(630)\ldots\ldots -dy = y\mu dx\,;\ \text{or}\ \frac{dy}{y} = -\phi\,(x)\,dx$$

By integration we obtain :—

$$(631)\ldots\ldots \log y = -\int\phi\,(x)\,dx\ :\ \text{or}\ y = e^{-\int\phi(x)dx}$$

Equations (630) and (631) are the bases of the theory of an "actuarial population." The number of survivors at each age obviously depends on the form of $\phi(x)$, and is completely determined when that function is known. Various forms that have been adopted for $\phi(x)$, and their integrals have already been given, formulæ (629) to (629*f*).

The probability at birth, of living to age x is y_x, as given by (631) above. The probability of dying before age x (ν_x, say), is the arithmetical complement of the probability of living, viz., $1 - y_x$; that is :—

$$(632)\ldots\ldots \nu_x = 1 - y_x = 1 - e^{-\int\phi(x)dx}$$

Similarly the probability (p_x) of persons of age x living to age $x+1$ and (q_x) that of dying before that age, are respectively :—

$$(633)\ldots\ldots p_x = (y_{x+1})/y_x\,;\ \text{and}\ q_x = (y_x - y_{x+1})/y_x = 1 - p_x.$$

The *average of the death-rates* (M) of persons dying between ages x_1 and x_2 is :—

$$(634)\ldots\ldots \mathrm{M} = \frac{1}{x_2 - x_1}\int_{x_1}^{x_2}\mu\,dx = \frac{1}{x_2 - x_1}\int_{x_1}^{x_2}\phi\,(x)\,dx$$

[1] Which may be a day, month, year, etc., but is usually a year

When the range of ages is a unit (or 1 year) we shall denote this quantity by the letter (m). The *group-rate of mortality* (M) for persons dying between the ages x_1 and x_2, is the ratio of the total deaths between the ages in question occurring in a unit of time, to the average population from which the deaths are drawn ; that is :—

$$(635)..M = \frac{y_2 - y_1}{\frac{1}{x_2 - x_1} \cdot \int_{x_1}^{x^2} y\, dx} = \frac{e^{-\int\phi(x_2)dx} - e^{-\int\phi(x_1)dx}}{\frac{1}{x_2 - x_1} \int_{x_1}^{x^2} e^{-\int\phi(x)dx}\, dx}$$

When the group-rate of mortality is taken through a range of age, of one year only, it is known actuarially as the *central death-rate* (m) of the year in question. In this case $x_2 - x_1$ is unity, and disappears in the above expression, and $y_2 = y_1 + 1$. Since through the greater part of life μ does not change very rapidly, $\mu_{x+\frac{1}{2}}$ is approximately equal to m_x, the group-range being really from *exact* age x to exact age $x+1$, and M for a range of k years will—to a very rough approximation—be km.

If, for so small a range of age as one year, it be assumed that the mean population is the mean of the populations at the beginning of, and end of the year of age (which is sensibly correct for a considerable range of the table[1]) the following relations hold between m, m, and μ :—

$$(636)....\mathrm{m} = \operatorname{colog}_e p = -\log(1-q) = q + \tfrac{1}{2}q^2 + \tfrac{1}{3}q^3 + \tfrac{1}{4}q^4 + \ldots$$

$$(637)....m = 2q/(2-q) = q_x(1 - \tfrac{1}{2}q) = q + \tfrac{1}{2}q^2 + \tfrac{1}{4}q^3 + \tfrac{1}{8}q^4 + \ldots$$

$$(638)....\mu_{+\frac{1}{2}} = \qquad\qquad = q \pm \epsilon$$

in which last expression ϵ is usually very small, but is not readily susceptible of any general expression, and must be specially determined for the very early and very late stages of life.

The *instantaneous rate of mortality* μ, at any exact age, is, of course, not immediately furnished by statistical data, but has to be deduced therefrom. It is the value of M in equation (634) or of M in equation (635) when x_1 and x_2 become identical.

22. **The relation between the mortality curve and the probability of death.**—The relation between μ_x and q_x may be established as follows : For any limited range of ages, excepting during the first year of life, the instantaneous mortality may be put in the form $\mathrm{A} + \mathrm{B}\,\mathrm{m}^{\chi}$, the integral of which is $\mathrm{C} + \mathrm{A}\chi + \mathrm{B}\mathrm{m}^{\chi}/\log_e \mathrm{m}$. If three successive values of μ for equal changes of age are μ_0, μ_1 and μ_2, then we shall have[2] :—

$$(639)......\mathrm{A} = \frac{\mu_2\mu_0 - \mu_1^2}{\mu_2 - 2\mu_1 + \mu_0}; \quad \mathrm{B} = \frac{(\mu_1 - \mu_0)^2}{\mu_2 - 2\mu_1 + \mu_0};$$

$$(640)......\mathrm{m} = \frac{\mu_2 - \mu_1}{\mu_1 - \mu_0}; \quad \mathrm{m} - 1 = \frac{\mu_2 - 2\mu_1 + \mu_0}{\mu_1 - \mu_0} = \mathrm{n}$$

[1] See Census Report, Vol. III., pp. 1215-6, for example.

[2] For greater precision, three values one place earlier, μ_{-1}, μ_0 and μ_1 can also be taken, and the corresponding values of A', B', m' and n' computed, entering the corresponding values of μ (the suffixes of which are unity less than those given). Thus for the stretch μ_0 to μ_1 the mean of the two results will—in general—be more accurate than either. Geometrically this is very simple ; m is the ratio of the *differences* of the rates of mortality : so long as the differences are in constant ratio the one value m applies.

If the successive values of μ are for ages k years apart, then the values of A and B remain unchanged for values one year apart, but the value of $\mathfrak{m}$ for 1 year, $\mathfrak{m}_1$ say, is $\mathfrak{m}_1 = \mathfrak{m}_k^{\frac{1}{k}}$ or $\mathfrak{m}_1{}^k = \mathfrak{m}_0$. This quantity is, of course, always positive, being greater than unity for an ascending curve, and less than unity for a descending curve. The use of the quantity $\mathfrak{n}$, enables $\log \mathfrak{m} = \log(1 + \mathfrak{n})$ to be conveniently expanded in the series $\mathfrak{n} - \frac{1}{2}\mathfrak{n}^2 + \frac{1}{3}\mathfrak{n}^3 -$ etc., which is convenient when $\mathfrak{n}$ is small.

Although the above expression for the curve is simple, yet when the value of the integral is applied to (635) in order to find the *average population*, it leads to an arithmetically intractable expression.[1] Consequently a direct general expression for p_x and q_x is not readily obtained in terms of μ. It is usual in actuarial computation to compute the Gompertz-Makeham constants from the values of μ and to find p and q from the values of y (that is, from l_x and l_{x+1}). The relation can, however, be obtained in quite another way. Put

$$(641) \ldots\ldots q = \gamma q' = \gamma \cdot \tfrac{1}{2}(\mu_x + \mu_{x+1});$$

then γ is a correction factor to what is ordinarily an approximate value. It can be computed and entered in a double-entry table for a suitable range of values of $\kappa = \mu_{x+1}/\mu_x$, and a suitable range of values of μ. Such a table would admit of q being readily and accurately found by interpolations, and would simplify the computations of life-tables. Similarly tables could be constructed in which the arguments were m_x, and the ratios $\kappa' = m_{x+1}/m_x$. This, however, will not be dealt with further in this article.

In *actuarial notation*,[2] l_x, the relative number of persons living at age x is so expressed that l_0 is 100,000, 10,000, or some such large unit;

---

[1] Put $K = e^{-c}$; $a = e^{-A}$; $\beta = e^{-B/\log_e \mathfrak{m}}$, then

$$\int e^{-\left\{C + A\chi + B\mathfrak{m}^\chi/\log_e \mathfrak{m}\right\}} dx = K \int a^\chi \beta^{\mathfrak{m}^\chi} d\chi = K \int e^{-z} d\chi$$

Let $a = \mathfrak{m}^t$ and $\beta\mathfrak{m}^\chi = e_\chi{}^{-r}$, consequently taking logarithms $\mathfrak{m}^\chi \log_e \beta = \iota$. Differentiating $\mathfrak{m}^\chi \log_e \mathfrak{m} \log_e \beta . d\chi = dv = v \log_e \mathfrak{m}\ d\chi$ (by substitution).

$$a^\chi = \mathfrak{m}^{t\chi} = (\mathfrak{m}^\chi)^t = (v/\log_e \beta)^t \text{; consequently}$$

$$\int a^\chi \beta^{\mathfrak{m}\chi} d\chi = \int (v \log_e \beta)^t \cdot e^v \cdot dv/v \log_e \mathfrak{m} = \frac{1}{(\log_e \beta)^t \log_e \mathfrak{m}} \int e^v v^{t-1} dv;$$

the value of the integral being $e^v [v^{t-1} - (t-1) v^{t-2} + (t-1)(t-2) v^{t-3} -$ etc.$]$ which gives the required values, if it be remembered that when $x = 0, 1$, etc. $v = \log_e \beta$, $m \log_e \beta$, etc., respectively.

Neglecting the C term we have, however, by expansion,

$$\int e^{-z} d\chi = \int [1 - A\chi + \tfrac{1}{2}A^2\chi^2 - \tfrac{1}{6}A^3\chi^3 + \ldots\ldots - (\beta + A\beta\chi - \tfrac{1}{2}A^2\beta\chi^2)\mathfrak{m}^\chi + \tfrac{1}{2}(\beta^2 - \tfrac{1}{2}A\beta^2\chi)\mathfrak{m}^{2\chi} + \tfrac{1}{6}\beta^3\mathfrak{m}^{3\chi} - \text{etc.}] d\chi.$$

which can be integrated term by term.

[2] Ordinarily actuarial methods are based upon the algorithms of "finite-differences." The connection between these and infinitesimal methods has been dealt with in a paper entitled "On the relation between the theories of compound interest and life contingencies," by J. M. Allen, F.I.A.; see Journ. Inst. Act., Vol. xli., pp. 305-337; see also discussion pp. 337-348, and particularly that by D. C. Fraser.

d_x denotes the number of persons dying between ages x and $x + 1$: and similarly :—

$$(642)\ldots\ldots p_x = \frac{l_{x+1}}{l_x}\ ;\ \text{and}\ q_x = \frac{l_x - l_{x+1}}{l_x} = \frac{d_x}{l_x}$$

$$(643)\ldots\ldots m_x = \frac{d_x}{l_{x+\frac{1}{2}}}\ ;\ \text{and}\ p_x = \frac{2 - m_x}{2 + m_x}$$

Also, since p and q are arithmetical complements.

$$(644)\ldots\ldots q_x = \frac{2m_x}{2 + m_x}\ ;\ \mu_x = -\frac{1}{l_x}\cdot\frac{dl_x}{dx} = -\frac{d}{dx}(\log l_x)^1$$

23. **Limitations of the Gompertz theory and of its developments.**— The conception put forward by Gompertz, and the modifications of that conception by Makeham, have, as already stated, been again modified by Laudi and Lazarus, who, to embrace results for earlier ages, replaced Makeham's result by putting for the value of the number living :—

$$(645)\ldots\ldots y_x = l_x = kH^{3^x} s^x b^{c^x}.\quad ^2$$

More recently C. H. Wickens has shewn that a similar type of expression fits ages up to age five.[3]

These formulæ, however, cannot be made to conform to the whole range of facts, viz., from the earliest to the latest ages, as has already been shewn. Although Mr. Wickens has shewn that, in a general way, a curve of the Gompertz-Makeham type represents the facts for the first few years of life, the formulae given do not conform to the details of the first twelve months of life : in short, it is not consistent with the "natal"

[1] It is generally assumed that $m_x = \mu_{x+\frac{1}{2}}$ approximately, the approximation being quite as accurate as the data. Approximations of greater precision are given in the Text Book of the Institute of Actuaries II., 1887 Edit., p. 25, and by later writers.

[2] Laudi and Lazarus gave the value 1.291219 to H and 0.4 to b, which give factors that become sensibly unity when x is 15. Thus the term for age 15 is 1.000000274, and for age 20 is 1.00000000281. For age 1, however, the term is as high as 1.1076433.

[3] Mr. Wickens' formula gives for the h term (h^{x^2}) and for ages up to 5, the following values for h, viz., 1.00056 for males, and 1.00037 for females, which are obviously not in good agreement with the value 1.1076433 mentioned above for age 1.

Mr. Wickens, putting—

$$A = -\log_e s = \frac{1}{M}(a - \tfrac{1}{2}\gamma)\ ;\ B = -\log_e c \log_e g = \frac{\beta \log c}{M^2(c-1)}\ ;\ \text{and}$$
$$C = -2\log_e h = \gamma/M\ ;\ \text{in which}$$
$$a = -(\log s + \log h)\ ;\ \beta = -(c-1)\log g\ ;\ \text{and}\ \gamma = -2\log h\ ;$$

M denoting $\log_e$, *i.e.*. 0.4342945, gives the result in the form :—

$$\mu_x = 0.00816 - 0.00113x + 0.21971^{0 \cdot 12453x};\ \text{for males.}$$
$$\mu'_x = 0.00645 - 0.00074x + 0.17199^{0 \cdot 14872x};\ \text{for females.}$$

the fit being excellent. See also "Assurance Magazine" (Journ. Inst. Act.), Vol. X., pp. 283-5, 1862. (Letter dated 15th August, 1861).

or "infantile" forces of mortality, as is evident from Fig. 101, p. 319. Formulae of that type can, of course, be made to represent the earlier features of the curve of mortality, viz., the natal and infantile and earlier forces of mortality, that is, the Lazarus-Laudi scheme of addition can be extended, the effect of the earlier terms disappearing when we please. This would give a complete general formula of the type

$$(646)\ldots\ldots\mu_x = a + bc^x + \beta\gamma^x + BC^x + \ldots + Sx^s + Tx^t + \ldots$$

It will be found hereinafter that a somewhat different conception already outlined in § 19 really accords with the facts, and it may be added that Westergaard's opinion that the formulae are really empirical, is substantially supported by the analysis.

24. **Senile element in the force of mortality.**—The senile element in the force of mortality may be assumed to operate from birth, with, of course, increasing potency as age advances. Initially it clearly ought to be zero, and thus it can be simply expressed by[1] :—

$$(647)\ldots\ldots\mu_s = Sx^s, \text{ or } \log\mu_s = \log S + s\log x.$$

Applying this gives for male and female senile death-rates, respectively :—

$$(648)\begin{cases} \text{Males } \mu = 0.0^{13}06100x^{5\cdot9671}\ ; \quad \text{females} = 0.0^{13}2826x^{6\cdot6189}\ ; \text{ or} \\ \text{,, } \log\mu = \bar{1}3.78533 + 5.9671\log x\ ; \quad \text{,,} \quad = 14.45117 + 6.6189\log x, \end{cases}$$

and they give a common value 0.99844 for the annual rate of mortality at age 111.40 years. These formulæ give the results in columns (ii.) and (vii.) of Table CXLVIII., p. 413. To find log. S we have :—

$$(649)..\log S = \log\mu_1 - \log x_1[(\log\mu_2 - \log\mu_1)/(\log x_2 - \log x_1)],$$

---

[1] This was decided upon empirically after examining the applicability of other forms. For example, if senile impairment *accumulated* at a constant rate, it would be expressed by $Se^{\sigma x}$. If the accumulated effect accelerated with age, a linear acceleration would require $Se^{\sigma x^2}$. Thus the index 2 would be too great if the acceleration were greater than linear, and too small if it were more rapid than linear. $Se^{\sigma x^s}$ was examined: this gave ${}_2\mu_s / {}_1\mu_s = e^{\sigma(x_2^s - x_1^s)}$ and $d\mu / dx = S\sigma s e^{\sigma x^s} x^{s-1}$. The value of σ may be readily found by means of a table of values of the x term (in brackets) for various values of s. Taking $S = .0021852$, $\sigma = 0.00064$, and $s = 2$ gave, for "persons"

	Age	0.	7.5	12.5	17.5	. . .	52.5	57.5	62.5	67.5	72.5	77.5	82.5
Calc.	μ	.0022	.0023	.0024	.0027		.0128	.0181	.0266	.0404	.0632	.102	.170
Data	μ	—	—	—	—		.0129	.0182	.0268	.0345	.0631	.103	.162

The result is obviously too high for early ages, when s was made unity. Formulæ of the type $Sx^{\sigma x}$ are inappropriate between 0 and 1, because x^x has a minimum at $x = .0367879$4, i.e., $1/e$, when its value is 0.6922007. $S(e^{\sigma x^s} - 1)$ would, however, probably be satisfactory from age 1 onwards.

s being the quantity in the square brackets. It would probably be preferable to adopt a mortality curve for the older ages, passing *among* the points given by the relatively meagre and uncertain data, than to follow them closely, since the general indication is probably the more reliable.

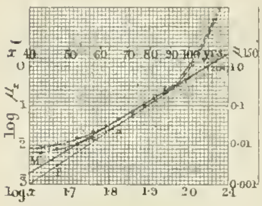

Fig. 102.

The curves shew the senile element in mortality.

The fit of the formula to the data is shewn on Fig. 102, on which the lines represent the logarithmic homologues of the senile curves, the small circles denoting the data for males and the small squares those for females.

25. **The force of mortality in earlier childhood.**—The rates of mortality from age 1.5 to 7.5 were as follow from 1907-1915 in Australia:—

Ages (years)	0.5		1.5		2.5		3.5		4.5		5.5		6.5	7.5
Males ..	.07608		.01550		.00641		.00408		.00301		—		—	.00220
Coefficient		.204		.414		.637		.738				.731		
Females	.06156		.01380		.00574		.00381		.00300		—		—	.00200
Coefficient		.224		.376		.652		.791				.667		

No elementary function will satisfactorily represent these results with precision.[1] The following results, however—empirically found—reproduce the data almost exactly, and shew the fluctuating character of the value of e^{-b} in the expression e^{-bx}, which would represent the curve:—

TABLE CXLVII.—Rates of Mortality in Childhood. Australia, 1907-15.

Exact Age.	Males.	Females	Exact Age.	Males.	Females	Exact Age.	Males.	Females
0.5	.07608	.06156	4.5	.00304	.00300	8.5	.00208	.00185
	.408	.428		.938	.935		.970	.970
1.0	.03104	.02635	5.0	.00285	.00281	9.0	.00202	.00179
	.500	.524		.940	.935		.970	.970
1.5	.01552	.01381	5.5	.00268	.00262	9.5	.00196	.00174
	.596	.610		.946	.935		.970	.970
2.0	.00925	.00842	6.0	.00253	.00245	10.0	.00190	.00169
	.692	.681		.952	.935		.970	.970
2.5	.00640	.00574	6.5	.00240	.00229	10.5	.00184	.00164
	.762	.772		.962	.935		.970	.970
3.0	.00488	.00443	7.0	.00229	.00214	11.0	.00179	.00159
	.832	.862		.970	.935		.970	.970
3.5	.00406	.00382	7.5	.00221	.00201	11.5	.00173	.00154
	.852	.878		.970	.955		1.000	1.000
4.0	.00346	.00335	8.0	.00214	.00191	12.0	.00173	.00154
	.879	.895		.970	.965			

NOTE.—The small figures are the ratios which multiplied into the values immediately above them give those immediately below.

[1] We have seen in § 11 that, for the greater part of first year of life, $\mu = \mu_0 [1 + f(x)] / x$, and that toward the end of the first year $f(x)$ is large—about 4—compared with unity. Also it is evident from curve A′, Fig. 101, that it is approximately a constant at about 320 days to perhaps 400 days, thus $\mu = 5\mu_0 / x$, and would appear to have become constant at least for some range of x. Such, however, is not the case. If it were we should have $x\mu_x = k [1 + f(x)]$ a constant. We obtain, however, the following results :—

.03808	.02325	.01602	.01428	.01354	—	—	.01650
.03078	.02070	.01435	.01333	.01350	—	—	.01500

which shew that $1 + f(x)$ is not expressible by any simple relation. The results for males for $2\frac{1}{2}$, $3\frac{1}{2}$, and $4\frac{1}{2}$ years can be expressed by $\mu_x = \mu_0 e^{-ax^b}$, and for females this expression is also fairly approximate.

26. **Genesic and Gestate elements in mortality.**—If the infantile and juvenile, and the senile elements of the mortality be subtracted from the totals, the residuals will constitute the genesic element in the case of males, and the gestate elements in the case of females. The rate of diminution seemed to be constantly 0.97 per half-year (see Table CXLVII., p. 412) from age 8.5 to 11.5 for both sexes. This is equivalent to 0.73752 for 5 years, and the adoption of this gives the results in columns (iii.) and (viii.) of Table CXLVIII. This may be regarded as the measure of degradation of the power of adjustment to environment. The residuals smoothed as shewn on Fig. 103, are given in columns (iv.) and (ix.). On this figure the heavy curve, M, denotes results for males, and the light one, F, results for females. The computed mortality curves and those given by the crude data, are shewn in columns (v.) and (vi.) for males, and columns (x.) and (xi.) for females. The agreement in general is fair up to 62.5 years. Afterwards the results diverge somewhat. It has, however, to be remembered that these divergencies are not really large, and do not make large differences as between the computed and actual numbers of deaths.

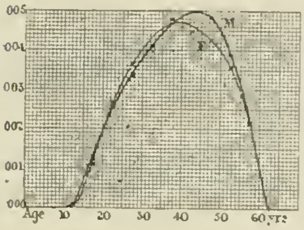

Fig. 103.

The curves shew the genesic (M) and gestate (F) elements in mortality.

TABLE CXLVIII.—Illustrating the component-elements of the Force of Mortality. Australia, 1911.

	Male Rates of Mortality, × 100,000					Female Rates of Mortality, × 100,000.				
	Senile Element.	Juvenile Element.	Genesic Sm'thd.	Total. (Computed.)	Total. (Observed.)	Senile Element.	Juvenile Element.	Gestate Sm'thd.	Total. (Computed.)	Total. (Observed.)
(i.)	(ii.)	(iii.)	(iv.)	(v.)	(vi.)	(vii.)	(viii.)	(ix.)	(x.)	(xi.)
2.5 ..	0	640	0	640	641	0	571	0	574	574
7.5 ..	0	221	0	221	220	0	201	0	201	200
12.5 ..	0	163	10	173	173	0	149	4	153	153
17.5 ..	2	120	133	255	255	0	110	120	230	220
22.5 ..	7	89	252	348	364	3	81	268	352	341
27.5 ..	24	65	343	432	432	10	60	366	436	433
32.5 ..	64	48	413	525	508	29	44	435	508	475
37.5 ..	151	36	467	654	666	74	33	467	574	586
42.5 ..	318	26	497	841	841	170	24	462	656	641
47.5 ..	617	19	484	1,120	1,122	354	18	426	798	796
52.5 ..	1,121	14	387	1,522	1,522	686	13	356	1,055	1,057
57.5 ..	1,930	11	220	2,161	2,161	1,254	10	215	1,479	1,479
62.5 ..	3,173	8	3	3,184	3,179	2,177	7	3	2,187	2,181
67.5 ..	5,022	6	0	5,028	4,693	3,623	5	0	3,628	2,201
72.5 ..	7,693	4	0	7,697	7,034	5,814	4	0	5,818	5,580
77.5 ..	11,455	3	0	11,458	11,136	9,041	3	0	9,044	9,379
82.5 ..	16,635	2	0	16,637	17,387	13,674	2	0	13,676	15,026
87.5 ..	23,632	2	0	23,634	27,557	20,188	2	0	20,190	22 492
92.5 ..	33,926	1	0	33,927	31,673	29,161	1	0	29,162	30,007
97.5 ..	45,071	1	0	45,072	40,475	41,314	1	0	41,315	39,873
02.5 ..	60,744	1	0	60,745	1.23393	57,531	1	0	57,532	1.16876

27. **Norm of mortality-rates.**—A study of mortality rates for the same country at different times, and for various countries, shews that the real nature of the mortality curve will probably be revealed only by

obtaining a norm of mortality rates on a wide basis. Such a norm would necessitate a compilation for a large series of populations, of the following data, viz. :—

(*a*) *Infantile deaths* according to *hours* for the *first week* of life ; then according to *days* for the *first month* of life : and according to *weeks* for the balance of the year.

(*b*) *Deaths in childhood* according to *months* for the second year ; and according to *quarters* for the third year and afterwards ;

(*c*) annually—or better *semi-annually*—till 15.

Afterwards the annual number of deaths.

The "number living" would preferably be deduced for the first 12 months (making corrections, however, for migration), by subtracting the deaths from the recorded births. Afterwards, or at any rate after the second year, the census data would in most cases be preferable to use.

The combination of a large number of results, viz., all deaths in any age-group, and the sum of the populations in the same age-group from which such deaths were drawn, would probably disclose the true laws of the incidence of death. Only in large bodies of figures can it be hoped that the minor chance influences will counteract one another.

28. **Number of deaths from particular causes.**—The actual numbers of deaths according to sex and age, which occurred in Australia during the 9 years 1907-1915 from various causes, were as shewn in the following table, viz., No. CLXIX., their relative frequency from all causes together, but retaining the age-groups, that is their ratios to the totals for the same sex, being shewn on the last two lines, see pp. 416-417 :—

29. **Relative frequency of deaths from particular diseases according to age and sex.**—If for each sex and for each age-group in that sex, the number of deaths from each cause be divided by the total deaths from all causes, the quotients are the relative positions of the disease as regards their contribution to the totality of deaths. Thus they measure the gravity of the incidence of any disease in question. This has been done and the results are shewn in Table CL., on pp. 418-419.

30. **Death-rates from particular diseases according to age and sex.**—It has already been pointed out that the incidence of death according to sex, has diverse characters as regards its relation to age ; see § 12, p. 393 hereinbefore. If the ratio of the number of deaths which occur in one year from any disease, in any age-group, and for either sex, to the average number of persons of the same sex in the age-group be found, this ratio will be the annual death-rate for the particular disease in question.[1] Thus the ratios are exactly analogous to the values with accents in (628) of § 19, p. 402 ; that is, they are the individual components of the death-rate for the same sex and age-group. They represent the ratio of the number of persons of a particular age-group who will (probably) die of the particular disease in question during the one year. These ratios, multiplied by 1,000,000, are shewn in Table CLI. and are thus the (partial) death-rates for each disease and for the two sexes, see pp. 420-421.

The forms of the rate-of-mortality curves for each disease are shewn on Fig. 104, the heavy lines denoting the curves for males and the lighter line those for females. They illustrate the marked differences in the incidence of death as between the sexes for the same disease, and according to age as between different diseases.

31. **Rates of mortality during the first twelve months of life.**—The incidence of death during the first twelve months of life is so varied that the means for the successive years 0, 1, 2, 3, etc., cannot be regarded as giving a satisfactory indication in regard thereto. Even in the first month of life, the frequency of deaths greatly varies for the successive weeks therein, so that a month is clearly too large a unit to adopt for rigorous results. Consequently, a tabulation for the first four weeks is necessary as well as for each of the succeeding eleven months. The population on which the ratios were based was 399,823 male births, and 38,027 females, which was reduced by the deaths themselves and increased by the net immigration of the same sex.[2]

[1] The sum of the mean populations for each sex and for the 9 years under review were distributed according to the Census of 1911, the middle year. This gave the divisors by means of which the rates were computed.

[2] The immigration is by no means wholly negligible for accurate results ; thus it was estimated to be—for each sex—267 for the eleventh to the twelfth month, while the deaths were : males, 933 ; females, 768. Its neglect does not, however, obviously make a large error, since the deaths are drawn mainly from those born in the country under consideration.

TABLE CXLIX.—Actual Number of Deaths in Australia according to Cause, Age, and Sex, during 9 Years, 1907-1915.

Cause	Sex	Ages. 0	1	2	3	4	5 to 9	10 to 14	15 to 19	20 to 24	25 to 29	30 to 34	35 to 39	40 to 44	45 to 49	50 to 54	55 to 59	60 to 64	65 to 69	70 to 74	75 to 79	80 to 84	85 to 89	90 to 94	95 to 99	100 and over	Total. †
1 Typhoid Fever ..	M	8	10	8	16	14	88	134	390	732	653	418	325	238	196	127	71	50	28	13	6	3	1	0	0	0	3,529
	F	5	6	14	18	17	84	203	346	348	263	168	134	99	82	58	34	15	16	13	1	1	3	0	0	0	1,928
5 Measles	M	209	315	146	56	34	100	19	25	60	19	15	7	4	8	11	6	1	3	3	1	2	0	0	0	0	1,044
	F	171	240	108	62	35	103	24	30	18	20	16	17	11	12	6	5	3	2	3	1	0	1	0	0	0	888
7 Whooping Cough* ..	M	1,127	359	101	49	27	44	1	0	0	1	0	0	0	0	0	1	1	1	0	3	0	0	0	0	0	1,715
	F	1,263	406	157	65	31	61	3	1	1	2	0	0	0	0	0	2	1	1	0	0	0	0	0	0	0	1,994
8 Diphtheria and Croup..	M	183	380	370	405	301	787	186	52	20	17	10	8	7	3	4	5	2	1	0	2	0	1	0	0	0	2,744
	F	134	309	338	324	309	961	209	70	26	20	11	12	8	3	5	2	3	1	0	1	0	0	0	1	0	2,747
9 Influenza*	M	183	41	17	16	11	34	17	44	52	62	42	79	94	124	108	112	116	171	228	243	178	80	36	6	0	2,094
	F	129	41	18	12	5	43	28	37	63	49	45	69	53	68	80	66	119	174	246	290	161	98	35	9	2	1,940
12 Other Epidemic Diseases*	M	135	29	11	5	7	16	15	35	80	91	57	61	72	76	83	67	75	77	92	72	52	28	11	2	1	1,250
	F	114	19	16	9	11	12	13	22	28	31	32	13	21	43	46	19	49	58	56	66	38	19	4	3	0	742
13 Tuberculosis of the Lungs	M	77	71	27	16	6	62	105	595	1,391	1,781	1,812	1,868	1,799	1,843	1,574	1,109	771	576	321	135	37	15	2	0	0	15,993
	F	67	47	22	19	14	76	225	1,112	1,978	2,118	1,626	1,371	1,078	866	540	392	311	271	158	68	26	3	1	0	0	12,389
14 Tuberculous Meningitis	M	190	174	113	59	54	139	72	51	35	43	35	25	19	20	5	4	2	1	0	1	0	0	0	0	0	1,042
	F	147	155	99	75	46	140	81	81	63	36	25	13	10	7	5	5	1	1	1	0	0	0	0	0	0	991
15 Other forms of Tuberculosis	M	175	85	42	25	25	80	91	106	120	129	93	106	99	93	86	81	47	31	32	16	1	2	0	0	0	1,565
	F	120	78	37	21	13	58	61	111	143	125	109	106	70	64	47	24	35	19	13	5	4	2	1	0	0	1,266
16 Cancer and other Malignant Tumours ..	M	20	20	22	14	16	53	27	61	87	138	178	355	694	1,278	1,729	1,957	2,117	2,433	2,038	1,471	708	276	32	16	5	15,745
	F	12	23	17	19	10	20	33	46	73	160	310	623	1,077	1,526	1,643	1,683	1,658	1,729	1,549	1,198	561	234	56	11	0	14,271
16a Other General Diseases*	M	975	153	102	104	79	381	331	375	449	372	448	541	669	753	801	728	688	637	478	359	170	48	5	1	0	9,647
	F	702	142	96	77	58	350	305	289	372	425	425	475	431	504	569	579	559	651	500	413	143	37	5	4	0	8,111
17 Simple Meningitis* ..	M	1,096	600	251	122	87	308	175	205	203	142	93	104	87	105	86	61	40	35	27	14	7	1	1	0	0	3,850
	F	898	465	178	122	82	229	134	136	100	71	59	71	57	70	49	26	16	20	18	13	0	1	1	0	0	2,816
18 Cerebral Hæmorrhage and Softening* ..	M	175	43	13	7	4	21	16	37	64	77	136	191	320	492	734	908	1,067	1,225	1,399	1,341	809	315	71	14	2	9,481
	F	137	45	10	9	8	20	16	32	60	66	80	138	254	410	613	649	831	1,152	1,442	1,350	807	363	103	17	1	8,613
18a Other Diseases of the Nervous System* ..	M	2,042	349	201	99	86	207	175	251	290	299	402	494	596	790	739	661	619	699	579	524	252	93	10	1	1	10,459
	F	1,450	353	173	91	83	176	153	195	237	239	232	289	343	351	363	318	385	420	350	341	191	89	24	4	1	6,851
19 Organic Diseases of the Heart	M	38	33	17	9	19	156	231	344	367	395	437	622	959	1,328	1,787	2,071	2,432	3,027	3,266	2,955	1,711	657	146	31	5	23,043
	F	37	23	20	21	21	174	289	345	348	404	433	579	754	984	1,169	1,278	1,585	2,266	2,490	2,350	1,259	532	119	19	1	17,500
19a Other Diseases of the Circulatory System* ..	M	106	26	17	15	16	106	81	135	137	141	173	280	400	551	678	721	623	800	780	711	416	178	37	10	1	7,139
	F	61	28	14	15	20	79	117	133	186	196	212	216	271	274	270	317	268	400	515	440	282	140	34	4	0	4,492
20 Acute Bronchitis* ..	M	1,273	197	74	36	19	31	7	7	11	9	6	12	9	20	29	26	40	56	91	125	87	41	7	0	1	2,214
	F	954	169	60	33	16	33	6	9	4	12	9	8	15	13	16	26	32	48	79	117	86	39	19	2	0	1,805
21 Chronic Bronchitis ..	M	5	3	1	0	3	20	9	7	14	15	18	27	44	101	139	198	363	556	814	1,041	840	406	110	26	1	4,761
	F	8	4	2	2	0	8	10	11	12	24	13	27	24	48	88	124	183	404	664	814	652	340	115	36	5	3,618

22	Pneumonia*	M	1,163	544	230	109	60	234	135	257	429	391	499	658	690	896	841	637	626	610	609	518	324	143	40	1	2	10,691
		F	851	447	168	90	57	178	140	174	241	258	281	357	346	339	285	208	285	418	446	425	267	98	37	2	0	6,491
23	Other Diseases of the Respiratory System*	M	1,899	721	275	133	104	203	76	109	159	180	176	299	433	532	598	588	592	698	707	701	461	206	35	8	1	9,894
		F	1,461	607	241	137	85	178	84	93	133	118	115	165	148	194	227	205	249	364	467	486	345	138	43	9	4	6,296
24	Diseases of the Stomach*	M	259	44	19	11	8	17	12	15	27	42	49	77	74	112	109	112	111	132	142	127	82	34	7	0	0	1,622
		F	224	41	22	10	9	25	16	41	68	63	70	82	84	86	74	88	70	109	127	122	99	39	11	2	0	1,582
25	Diarrhœa and Enteritis (all ages)*	M	12,462	2,551	439	197	89	171	48	43	44	62	66	92	89	121	149	181	220	294	425	436	332	133	39	7	1	18,694
		F	9,889	2,218	422	162	90	191	44	41	65	68	77	104	93	127	110	138	194	305	390	421	329	142	45	13	5	15,683
26	Appendicitis and Typhlitis	M	1	2	9	12	16	137	208	234	254	161	146	122	108	119	87	67	47	33	28	10	5	0	0	0	0	1,809
		F	2	5	7	10	17	128	182	170	120	106	90	81	49	72	56	30	23	27	21	7	4	2	0	0	0	1,209
27	Hernia, Intestinal Obstruction* ..	M	456	67	27	16	11	32	26	48	43	43	58	58	78	111	116	123	155	163	188	180	95	33	6	1	1	2,135
		F	266	24	18	9	11	37	18	15	28	49	62	88	79	115	101	111	120	169	171	166	83	47	8	0	0	1,795
28	Cirrhosis of Liver ..	M	8	7	0	1	1	8	7	7	12	34	64	107	196	285	362	298	239	210	151	115	40	8	0	0	0	2,155
		F	8	5	4	1	0	8	5	5	11	21	35	79	119	171	147	134	129	109	79	66	24	3	0	0	0	1,163
28a	Other Diseases of the Digestive System* ..	M	206	38	35	38	27	127	75	112	117	150	139	180	217	237	272	235	218	263	228	176	85	47	5	1	0	3,228
		F	172	29	30	43	40	159	91	92	162	208	199	207	223	204	246	225	248	258	240	190	97	42	9	1	0	3,415
29	Acute Nephritis and Bright's Disease* ..	M	127	57	28	27	17	92	77	196	297	324	310	437	658	918	1,055	1,120	1,136	1,219	1,144	1,013	501	171	31	9	0	10,964
		F	80	59	27	11	12	85	97	183	354	349	323	420	482	553	639	602	617	668	618	455	240	78	22	7	0	6,981
30	Non-cancerous Tumours of the Female Genital Organs	M	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
		F	4	0	0	0	0	1	1	19	124	168	184	158	157	152	82	59	41	32	37	30	9	5	3	0	0	1,266
30a	Other Diseases of the Genito-urinary System*	M	45	10	1	1	3	10	4	10	38	33	40	73	109	156	179	225	347	524	685	811	538	242	40	7	0	4,131
		F	22	6	3	1	3	8	4	9	25	32	34	46	51	52	58	57	56	51	68	55	30	15	2	0	0	688
31	Puerperal Septicæmia	M	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
		F	0	0	0	0	0	0	1	96	370	514	459	307	112	12	1	0	0	0	0	0	0	0	0	0	0	1,872
32	Other Accidents of Pregnancy and Labour ..	M	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
		F	0	0	0	0	0	0	4	161	587	816	799	829	397	55	0	0	0	0	0	0	0	0	0	0	0	3,648
32a	Diseases of the Skin and Cellular Tissue* ..	M	195	15	10	7	5	13	15	17	19	29	27	34	35	70	56	69	74	107	140	163	132	79	24	7	1	1,343
		F	149	18	9	5	5	13	8	16	15	25	24	25	27	32	35	43	60	79	110	149	108	46	13	7	1	1,022
32b	Diseases of the Organs of Locomotion	M	34	18	13	13	13	56	58	26	17	20	11	18	10	11	19	13	22	15	19	12	4	2	1	1	0	426
		F	17	17	9	7	6	27	20	5	10	10	7	8	6	8	9	8	5	9	14	5	4	1	0	0	0	212
33	Congenital Debility and Malformations* ..	M	17,204	109	43	15	12	21	11	8	2	1	1	0	0	1	0	0	0	0	0	0	0	0	0	0	0	17,425
		F	12,976	85	23	14	13	18	7	2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	13,138
33a	Other Diseases of Infancy	M	2,143	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	2,143
		F	1,638	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1,638
34	Senile Debility	M	0	0	0	0	0	0	0	0	0	0	0	0	2	4	14	81	393	1,142	3,104	4,802	5,057	3,121	1,116	278	78	19,192
		F	0	0	0	0	0	0	0	0	0	0	0	1	0	2	10	21	213	783	2,096	3,470	3,917	2,827	1,148	309	64	14,861
35	Violent Death—Suicide Excepted*	M	903	493	369	248	177	863	902	1,472	1,763	1,661	1,495	1,446	1,510	1,597	1,418	1,110	869	705	622	429	288	120	24	10	3	20 497
		F	672	377	291	225	189	471	294	294	289	251	224	235	203	204	210	188	151	225	243	250	214	128	59	12	4	5,903
36	Suicide	M	0	0	0	0	0	0	13	76	264	362	425	473	523	507	475	367	265	203	112	64	32	13	0	0	0	4,174
		F	0	0	0	0	0	0	8	79	116	113	122	101	90	105	72	48	29	17	11	4	1	0	0	0	0	916
38	Unknown or ill-defined Diseases* .. .	M	672	241	52	33	16	61	42	49	69	84	106	170	242	299	458	439	502	421	282	217	108	50	6	3	0	4,622
		F	612	251	42	23	18	37	26	28	59	70	72	99	115	163	185	154	209	177	113	94	47	28	1	2	1	2,626
	Total Deaths* ..	M	45,797	7,808	3,089	1,914	1,376	4,678	3,401	5,399	7,666	7,961	7,985	9,349	11,084	13,757	14,928	14,447	14,870	17,096	18,747	18,824	13,357	6,544	1,842	440	104	252,463
		F	35,452	6,742	2,695	1,742	1,334	4,191	2,960	4,529	6,840	7,500	6,982	7,553	7,357	7,971	8,114	7,958	8,753	11,433	13,348	13,863	10,029	5,540	1,918	474	89	185,367
	Frequency relative to total according to age-group ..	M	18,140	3,093	1,223	758	545	1,853	1,347	2,138	3 036	3,153	3,163	3,703	4,394	5,449	5,913	5,722	5,890	6,771	7,425	7,456	5,291	2,592	730	174	41	100,000
		F	19,125	3,637	1,454	940	720	2,261	1,597	2,443	3,690	4,046	3,766	4,074	3,969	4,300	4,377	4,293	4,722	6,168	7,201	7,479	5,410	2,989	1,035	256	48	100,000

* Causes of death for which the mortality is relatively great during the first year of life.

† For the relative frequency of each cause of death, see Table CLI. preferably, or CL.

TABLE CL.—Ratio according to Age-groups of Deaths from Particular Causes to the Total Deaths from all Causes in the same Age-group according to Sex. Average for 9 years. Australia, 1907-1915.

Ref No.	Cause.	Sex	0	1	2	3	4	5 to 9	10 to 14	15 to 19	20 to 24	25 to 29	30 to 34	35 to 39	40 to 44	45 to 49	50 to 54	55 to 59	60 to 64	65 to 69	70 to 74	75 to 79	80 to 84	85 to 89	90 to 94	95 to 99	100 and over.	All Ages.
1	Typhoid Fever	M	0002	0013	0026	0084	0102	0188	0394	0722	0955	0820	0524	0348	0215	0142	0085	0049	0034	0016	0007	0003	0002	0002	..	..	..	0140
		F	0001	0009	0052	0103	0127	0200	0686	0764	0509	0351	0241	0177	0135	0103	0071	0043	0017	0014	0010	0001	0001	0005	..	..	..	0104
5	Measles	M	0046	0403	0473	0293	0247	0214	0056	0046	0078	0024	0019	0008	0004	0006	0007	0004	0001	0002	0002	0001	0001	..	..	..	..	0041
		F	0048	0356	0401	0356	0262	0246	0081	0066	0026	0027	0023	0023	0015	0015	0007	0006	0003	0002	0002	0001	..	0002	..	..	..	0048
7	Whooping Cough ..	M	0246	0460	0327	0256	0196	0094	0003	..	..	0001	..	..	..	..	..	0001	0001	0001	..	0002	..	..	..	..	..	0068
		F	0356	0602	0583	0373	0232	0146	0010	0002	0001	0003	..	..	..	..	..	0003	0001	0001	..	..	..	..	..	..	..	0108
8	Diphtheria and Croup	M	0040	0487	1198	2116	2188	1682	0547	0096	0026	0021	0013	0009	0006	0002	0003	0004	0001	0001	..	0001	..	0001	..	..	..	0109
		F	0038	0458	1254	1860	2316	2293	0706	0155	0038	0027	0016	0016	0011	0004	0006	0003	0003	0001	..	0001	..	..	..	0021	..	0148
9	Influenza	M	0040	0052	0055	0084	0080	0073	0050	0082	0068	0078	0053	0084	0085	0090	0072	0078	0078	0100	0122	0129	0133	0122	0195	0136	..	0083
		F	0036	0061	0067	0069	0038	0103	0095	0082	0092	0065	0064	0091	0072	0085	0099	0083	0136	0152	0184	0209	0161	0177	0182	0190	0225	0105
12	Other Epidemic Diseases	M	0029	0037	0036	0026	0051	0034	0044	0065	0104	0114	0071	0065	0065	0055	0056	0046	0050	0045	0049	0038	0039	0043	0060	0046	0096	0050
		F	0032	0028	0059	0052	0083	0029	0044	0049	0041	0041	0046	0017	0028	0054	0057	0024	0056	0051	0042	0048	0038	0034	0021	0063	..	0040
13	Tuberculosis of the Lungs	M	0017	0091	0087	0084	0044	0133	0309	1102	1814	2237	2269	1998	1623	1340	1054	0768	0519	0337	0171	0072	0028	0023	0011	..	..	0634
		F	0019	0070	0082	0109	0105	0181	0760	2455	2892	2824	2329	1815	1465	1086	0666	0493	0355	0237	0118	0049	0026	0005	0005	..	..	0668
14	Tuberculous Meningitis	M	0042	0223	0366	0308	0392	0297	0212	0094	0046	0054	0044	0027	0017	0015	0003	0003	0001	0001	..	0001	..	..	..	..	..	0041
		F	0042	0230	0367	0430	0345	0334	0274	0179	0092	0048	0036	0017	0014	0009	0006	0006	0001	0001	0001	..	..	..	..	..	..	0054
15	Other Forms of Tuberculosis	M	0038	0109	0136	0131	0182	0171	0268	0196	0157	0162	0117	0113	0089	0068	0058	0056	0032	0018	0017	0009	0001	0003	..	..	..	0062
		F	0034	0116	0137	0121	0098	0138	0206	0245	0209	0167	0156	0140	0095	0080	0058	0030	0040	0017	0010	0004	0004	0004	0005	..	..	0068
16	Cancer and other Malignant Tumours ..	M	0004	0026	0071	0073	0116	0113	0079	0113	0114	0173	0223	0380	0626	0929	1158	1355	1424	1423	1087	0781	0530	0422	0174	0364	0481	0624
		F	0003	0034	0063	0109	0075	0048	0112	0102	0107	0213	0444	0825	1464	1914	2025	2115	1894	1512	1160	0864	0559	0422	0292	0232	..	0770
16a	Other General Diseases	M	0213	0196	0330	0543	0574	0815	0973	0695	0586	0467	0561	0579	0604	0547	0537	0504	0463	0373	0255	0191	0127	0073	0027	0023	..	0382
		F	0198	0211	0356	0442	0435	0835	1030	0638	0544	0567	0609	0629	0586	0632	0701	0728	0639	0569	0375	0298	0143	0067	0026	0084	..	0438
17	Simple Meningitis ..	M	0239	0768	0813	0637	0632	0658	0515	0380	0265	0178	0117	0111	0078	0076	0058	0042	0027	0021	0014	0007	0005	0001	0005	..	..	0153
		F	0253	0690	0661	0700	0615	0546	0453	0300	0146	0095	0084	0094	0078	0088	0060	0033	0018	0017	0014	0009	..	0002	0005	..	..	0152
18	Cerebral Hæmorrhage and Softening ..	M	0038	0055	0042	0037	0029	0045	0047	0069	0083	0097	0170	0204	0289	0358	0492	0629	0718	0716	0746	0712	0606	0481	0386	0318	0192	0376
		F	0039	0067	0037	0052	0060	0048	0054	0071	0088	0088	0115	0183	0345	0514	0756	0815	0949	1008	1080	0974	0805	0655	0537	0359	0112	0465
18a	Other Diseases of the Nervous System ..	M	0446	0447	0651	0517	0625	0442	0515	0465	0378	0376	0503	0528	0538	0574	0495	0458	0416	0409	0309	0278	0189	0142	0054	0023	0096	0414
		F	0409	0524	0642	0522	0622	0420	0517	0431	0347	0319	0332	0383	0466	0440	0447	0400	0440	0367	0262	0246	0190	0161	0125	0084	0112	0370
19	Organic Diseases of the Heart	M	0008	0042	0055	0047	0138	0334	0679	0637	0479	0496	0547	0665	0865	0965	1197	1433	1636	1771	1742	1570	1281	1004	0793	0705	0481	0913
		F	0010	0034	0074	0121	0157	0415	0976	0762	0509	0539	0620	0707	1025	1235	1441	1606	1811	1982	1865	1695	1255	0960	0620	0401	0112	0944
19a	Other Diseases of the Circulatory System	M	0023	0033	0055	0078	0116	0227	0238	0250	0179	0177	0217	0299	0361	0401	0454	0499	0419	0468	0416	0378	0311	0272	0201	0227	0096	0283
		F	0017	0042	0052	0086	0150	0188	0395	0294	0272	0261	0304	0286	0368	0344	0333	0398	0306	0350	0386	0317	0281	0253	0177	0084	..	0242
20	Acute Bronchitis ..	M	0278	0252	0240	0188	0138	0066	0021	0013	0014	0011	0008	0013	0008	0014	0019	0018	0027	0033	0049	0066	0065	0063	0038	..	0096	0088
		F	0269	0251	0223	0189	0120	0079	0020	0020	0006	0016	0013	0011	0020	0016	0020	0033	0037	0042	0059	0084	0086	0070	0099	0042	..	0097
21	Chronic Bronchitis ..	M	0001	0004	0003	..	0022	0043	0027	0013	0018	0019	0022	0029	0040	0073	0093	0137	0244	0325	0434	0553	0629	0620	0597	0591	0096	0189
		F	0002	0006	0007	0012	..	0019	0034	0024	0017	0032	0019	0036	0033	0060	0109	0156	0209	0353	0498	0587	0650	0614	0600	0759	0562	0195
22	Pneumonia	M	0254	0697	0764	0570	0502	0500	0397	0476	0560	0491	0625	0704	0623	0651	0563	0441	0421	0357	0325	0291	0243	0218	0217	0023	0192	0424
		F	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	0326	0366	0334	0307	0266	0177	0193	0042	..	0350

23	Other Diseases of the Respiratory System	M	0415	0923	0890	0695	0756	0434	0223	0202	0207	0226	0220	0320	0391	0387	0401	0407	0398	0405	0377	0372	0345	0315	0190	0182	0048	0392
		F	0412	0900	0894	0786	0637	0425	0284	0205	0191	0157	0165	0219	0201	0243	0280	0258	0285	0318	0350	0351	0344	0249	0224	0190	0449	0340
24	Diseases of the Stomach	M	0057	0056	0062	0057	0058	0036	0035	0028	0035	0053	0061	0082	0067	0081	0073	0078	0075	0077	0076	0067	0061	0052	0038	..	..	0064
		F	0063	0061	0082	0057	0068	0060	0054	0090	0099	0084	0100	0109	0114	0108	0091	0111	0080	0095	0095	0088	0099	0070	0057	0042	..	0085
25	Diarrhoea and Enteritis (all ages)	M	2721	3271	1421	1029	0647	0366	0141	0080	0057	0078	0083	0098	0080	0088	0100	0125	0148	0172	0227	0232	0249	0203	0212	0159	0096	0741
		F	2789	3290	1566	0930	0675	0456	0149	0091	0095	0091	0110	0138	0126	0159	0136	0173	0232	0267	0292	0304	0328	0256	0235	0274	0562	0846
26	Appendicitis and Typhlitis	M	0001	0003	0029	0063	0116	0293	0612	0433	0331	0202	0183	0131	0097	0086	0058	0046	0032	0019	0015	0005	0004	..	..	..	..	0072
		F	0001	0007	0026	0057	0127	0305	0615	0375	0175	0141	0129	0107	0067	0090	0069	0038	0026	0024	0016	0005	0004	0004	..	..	..	0065
27	Hernia, Intestinal Obstruction	M	0100	0086	0087	0084	0080	0068	0076	0089	0056	0054	0073	0062	0070	0081	0078	0085	0104	0095	0100	0096	0071	0050	0033	0023	0096	0085
		F	0075	0036	0067	0052	0083	0088	0061	0033	0041	0065	0089	0117	0107	0144	0124	0140	0137	0148	0128	0120	0083	0085	0042	..	..	0097
28	Cirrhosis of Liver ..	M	0002	0009	..	0005	0007	0017	0021	0013	0016	0043	0080	0115	0177	0207	0242	0203	0161	0123	0081	0061	0030	0012	..	..	..	0085
		F	0002	0007	0015	0006	..	0019	0017	0011	0016	0028	0050	0105	0162	0215	0181	0168	0147	0095	0059	0048	0024	0005	..	..	..	0063
28a	Other Diseases of the Digestive System ..	M	0045	0049	0113	0199	0196	0271	0221	0207	0153	0188	0174	0192	0196	0172	0182	0163	0147	0154	0122	0094	0064	0072	0027	0023	..	0128
		F	0049	0043	0111	0247	0300	0379	0307	0203	0237	0277	0285	0274	0303	0256	0303	0283	0283	0226	0180	0137	0097	0076	0047	0021	..	0184
29	Acute Nephritis and Bright's Disease ..	M	0028	0073	0091	0141	0124	0197	0226	0363	0387	0407	0388	0467	0594	0667	0707	0775	0764	0713	0610	0538	0375	0261	0168	0205	..	0434
		F	0023	0088	0100	0063	0090	0203	0328	0404	0518	0465	0463	0556	0655	0694	0787	0757	0705	0584	0463	0328	0239	0141	0115	0148	..	0377
30	Non-cancerous Tumours of the Female Genital Organs	F	0001	..	..	..	..	0002	0003	0042	0181	0224	0264	0209	0213	0191	0101	0074	0047	0028	0028	0022	0009	0009	0016	..	..	0068
30a	Other Diseases of the Genito-urinary System ..	M	0010	0013	0003	0005	0022	0021	0012	0018	0050	0042	0050	0078	0098	0113	0120	0156	0234	0306	0365	0431	0403	0370	0217	0159	..	0164
		F	0006	0009	0011	0006	0023	0019	0013	0020	0037	0043	0049	0061	0069	0065	0071	0072	0064	0045	0051	0040	0030	0027	0010	..	..	0037
31	Puerperal Septicæmia ..	F	..	..	..	..	..	..	0003	0212	0541	0685	0657	0407	0152	0015	0001	..	..	..	..	..	..	..	..	..	..	0101
32	Other Accidents of Pregnancy and Labour ..	F	..	..	..	..	..	..	0014	0355	0858	1088	1144	1098	0540	0069	..	..	..	..	..	..	..	..	..	..	..	0197
32a	Diseases of the Skin and Cellular Tissue ..	M	0043	0019	0032	0037	0036	0028	0044	0032	0025	0036	0034	0036	0032	0051	0037	0048	0050	0063	0075	0087	0099	0121	0130	0159	0096	0053
		F	0042	0027	0033	0029	0038	0031	0027	0035	0022	0033	0034	0033	0037	0040	0043	0054	0069	0069	0082	0108	0108	0083	0068	0148	0112	0055
32b	Diseases of the Organs of Locomotion	M	0007	0023	0042	0068	0095	0120	0170	0048	0022	0025	0014	0019	0009	0008	0013	0009	0015	0009	0010	0006	0003	0003	0005	0023	..	0017
		F	0005	0025	0033	0040	0045	0065	0068	0011	0015	0013	0010	0011	0008	0010	0011	0010	0006	0008	0010	0004	0004	0002	..	..	..	0011
33	Congenital Debility and Malformations ..	M	3757	0140	0139	0078	0087	0045	0032	0015	0003	0001	0001	..	..	0001	..	..	..	..	..	..	..	..	..	..	..	0690
		F	3660	0013	0085	0080	0098	0043	0024	0004	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	0709
33a	Other Diseases of Infancy	M	0468	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	0085
		F	0462	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	0088
34	Senile Debility	M	..	..	..	..	..	..	..	..	..	..	..	..	0002	0003	0009	0056	0264	0668	1656	2551	3786	4769	6059	6318	7500	0760
		F	..	..	..	..	..	..	..	..	..	..	..	0001	..	0003	0012	0026	0243	0685	1570	2503	3906	5103	5985	6519	7191	0802
35	Violent Death (Suicide excepted)	M	0197	0631	1195	1296	1286	1845	2652	2726	2300	2086	1872	1547	1362	1161	0950	0768	0584	0412	0332	0228	0216	0183	0130	0227	0288	0812
		F	0190	0559	1080	1292	1417	1124	0993	0649	0423	0335	0321	0311	0276	0256	0259	0236	0173	0197	0182	0180	0213	0231	0308	0253	0449	0319
36	Suicides	M	..	..	..	..	..	..	0038	0141	0344	0455	0532	0506	0472	0369	0318	0254	0178	0119	0060	0034	0024	0020	..	..	..	0165
		F	..	..	..	..	..	..	0027	0174	0170	0151	0175	0134	0122	0132	0089	0060	0033	0015	0008	0003	0001	..	..	..	..	0049
38	Unknown or Ill-defined Diseases	M	0147	0309	0168	0172	0116	0130	0124	0091	0090	0106	0133	0182	0218	0217	0307	0304	0338	0246	0150	0115	0081	0076	0033	0068	..	0183
		F	0173	0372	0156	0132	0135	0088	0088	0062	0086	0093	0103	0131	0156	0205	0228	0194	0239	0155	0085	0068	0047	0051	0005	0042	0112	0142
	Total	M	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	Total Deaths ..	F	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000

TABLE CLI.—Death-rates in Age-groups or the Number of Deaths in Australia per Million Persons in each Age-group, according to Cause, Age and Sex. 9-year Average. (1907-1915).

	Cause.	Sex	0	1	2	3	4	5–9	10–14	15–19	20–24	25–29	30–34	35–39	40–44	45–49	50–54	55–59	60–64	65–69	70–74	75–79	80–84	85–89	90–94	95–99	100 and over	All Ages	Relative Frequency of Death from each cause.*
1	Typhoid Fever	M	15	20	17	34	31	41	68	185	348	353	266	231	180	160	129	106	107	77	48	35	38	41	0	0	0	167	.01398
		F	10	12	30	39	38	40	105	169	174	151	114	104	86	82	75	63	37	49	54	7	15	118	0	0	0	98	.01040
5	Measles	M	390	629	303	120	74	47	10	12	29	10	10	5	3	7	11	9	2	8	11	6	25	0	0	0	0	49	.00414
		F	331	495	231	136	78	50	12	15	9	11	11	13	10	12	8	9	7	6	12	7	39	0	0	0	0	45	.00479
7	Whooping Cough	M	2,102	717	210	108	59	21	1	0	0	1	0	0	0	0	0	1	2	3	0	17	0	0	0	0	0	81	.00679
		F	2,445	837	335	143	69	29	2	1	1	1	0	0	0	0	0	4	2	3	0	0	0	0	0	0	0	101	.01076
8	Diphtheria and Croup ..	M	341	758	768	866	658	373	94	25	10	9	6	6	5	2	4	7	4	3	0	12	0	41	0	0	0	130	.01087
		F	259	637	722	711	694	464	108	34	13	11	7	9	7	3	7	4	7	3	0	7	0	0	0	808	0	140	.01482
9	Influenza	M	341	82	35	34	24	16	9	21	25	33	27	56	71	101	109	170	248	468	848	1,407	2,240	3,274	6,038	5,333	0	99	.00829
		F	250	85	38	26	11	21	15	18	31	28	31	54	46	68	103	122	296	528	1,018	1,917	2,340	3,839	5,285	7,276	24,096	99	.01047
12	Other Epidemic Diseases	M	252	58	23	11	15	7	8	17	38	49	36	43	55	62	84	100	160	210	342	417	655	1,146	1,845	1,778	10,870	58	.00495
		F	221	39	34	20	25	6	7	11	14	19	22	10	18	43	60	35	122	176	232	436	552	744	604	2,425	0	38	.00400
13	Tuberculosis of the Lungs	M	144	142	56	34	13	29	53	282	661	964	1,152	1,331	1,364	1,501	1,593	1,653	1,646	1,576	1,194	782	466	614	335	0	0	757	.06334
		F	130	97	47	42	31	37	116	543	986	1,219	1,106	1,064	939	863	698	726	774	823	654	449	378	117	151	0	0	631	.06683
14	Tuberculous Meningitis ..	M	354	348	235	126	118	66	36	24	17	23	22	18	14	16	5	6	4	3	0	6	0	0	0	0	0	49	.00413
		F	285	319	211	164	103	68	42	40	31	21	17	10	9	7	6	9	2	3	4	0	0	0	0	0	0	51	.00535
15	Other Forms of Tuberculosis	M	326	170	87	53	55	38	46	50	57	70	59	76	75	76	87	121	100	85	119	93	13	82	0	0	0	74	.00620
		F	232	161	79	46	29	28	32	54	71	72	74	82	61	64	61	44	87	58	54	33	58	78	151	0	0	65	.00683
16	Cancer and other Malignant Tumours	M	37	40	46	30	35	25	14	29	41	75	113	253	526	1,041	1,750	2,917	4,519	6,658	7,580	8,518	8,913	11,296	5,367	14,222	54,347	744	.06236
		F	23	47	36	42	22	10	17	22	36	92	211	483	938	1,521	2,125	3,116	4,124	5,249	6,409	7,917	8,153	9,166	8,455	8,892	0	727	.07699
16a	Other General Diseases ..	M	1,818	306	210	222	173	180	168	178	213	201	285	385	507	613	811	1,085	1,469	1,743	1,779	2,079	2,140	1,965	839	889	0	456	.03821
		F	1,359	293	205	169	130	168	157	141	185	245	289	369	375	502	736	1,071	1,391	1,976	2,069	2,729	2,078	1,449	755	3,234	0	413	.04376
17	Simple Meningitis ..	M	2,044	1,198	521	261	190	146	89	98	96	77	59	74	66	86	87	90	85	96	100	81	88	41	168	0	0	182	.01525
		F	1,738	958	380	268	184	111	69	66	50	41	40	55	50	70	63	48	40	61	74	86	0	39	151	0	0	144	.01519
18	Cerebral Hæmorrhage and Softening	M	326	86	27	15	9	10	8	18	30	42	86	136	243	401	743	1,353	2,278	3,352	5,203	7,765	10,184	12,892	11,909	12,444	21,739	448	.03755
		F	265	93	21	20	18	10	8	16	30	38	55	107	221	409	793	1,201	2,067	3,497	5,967	8,922	11,728	14,219	15,551	13,743	12,048	439	.04646
18a	Other Diseases of the Nervous System ..	M	3,808	697	417	212	188	98	89	119	138	162	256	352	452	643	748	985	1,321	1,913	2,153	3,034	3,172	3,806	1,677	889	10,870	495	.04143
		F	2,807	727	369	200	186	85	79	95	118	138	158	224	299	350	469	589	958	1,275	1,448	2,254	2,776	3,486	3,624	3,234	12,048	349	.03696
19	Organic Diseases of the Heart	M	71	66	35	19	40	74	117	163	174	214	278	443	727	1,081	1,808	3,087	5,191	8,283	12,147	17,111	21,540	26,890	24,488	27,556	54,348	1,089	.09126
		F	71	47	43	46	47	84	149	169	173	232	295	449	657	981	1,512	2,366	3,943	6,879	10,303	15,530	18,296	20,839	17,967	15,360	12,048	892	.09441
19a	Other Diseases of the Circulatory System ..	M	198	52	35	32	35	50	41	64	65	76	110	199	303	449	686	1,075	1,330	2,189	2,900	4,117	5,237	7,285	6,206	8,889	10,870	338	.02828
		F	118	58	30	33	44	38	60	65	93	113	144	168	236	273	349	587	667	1,214	2,131	2,908	4,098	5,484	5,134	3,234	0	229	.02423
20	Acute Bronchitis	M	2,374	393	154	77	40	15	4	3	5	5	4	9	7	16	29	39	85	153	338	724	1,095	1,678	1,174	0	10,870	105	.00877
		F	1,846	348	128	72	35	16	3	4	2	7	6	6	13	13	21	48	80	146	327	773	1,250	1,528	2,869	1,617	0	92	.00974
21	Chronic Bronchitis ..	M	9	6	2	0	7	9	5	3	7	8	11	19	33	82	141	295	775	1,521	3,027	6,028	10,575	16,617	18,450	23,111	10,870	225	.01886
		F	15	8	4	4	0	4	5	5	6	14	9	21	20	48	114	230	455	1,226	2,747	5,379	9,475	13,318	17,364	29,102	60,241	184	.01952
22	Pneumonia	M	2,169	1,086	490	233	151	111	69	122	204	212	317	469	523	730	851	949	1,336	1,669	2,265	3,173	4,079	5,853	6,709	889	21,739	505	.04235
		F	1,647	921	359	197	128	86	72	85	122	148	191	277	311	338	369	552	709	1,269	1,845	2,809	3,880	3,839	5,585	1,617	0	331	.03502

23	Other Diseases of the Respiratory System	M	3,541	1,440	571	284	228	96	39	52	76	97	112	213	328	433	605	876	1'263	1,910	2,629	4,059	5,804	8,431	5,871	7,111	10,870	468	.03919
		F	2,828	1,251	514	300	191	86	43	45	68	68	78	128	129	193	294	380	619	1,105	1,932	3,212	5,014	5,406	6,493	7,276	48,193	321	.03397
24	Diseases of the Stomach	M	483	88	39	24	17	8	6	7	13	23	31	55	56	91	110	167	237	361	528	735	1,032	1,391	1,174	0	0	77	.00642
		F	434	84	47	22	20	12	8	20	34	36	48	63	73	86	96	163	174	331	526	806	1,439	1,528	1,661	1,617	0	81	.00853
25	Diarrhœa and Enteritis (all ages)	M	23,238	5,101	912	421	195	81	24	20	21	34	42	66	68	99	151	270	470	805	1,581	2,525	4,180	5,443	6,541	6,222	10,870	884	.07405
		F	19,140	4,570	901	355	202	92	23	20	32	39	53	81	81	126	142	255	483	926	1,614	2,782	4,781	5,562	6,795	10,509	60,241	799	.08461
26	Appendicitis and Typhlitis	M	7	4	19	26	35	65	106	111	120	87	93	87	82	97	88	100	100	90	104	58	63	0	0	0	0	86	.00717
		F	4	10	15	22	38	62	94	83	60	61	61	63	13	72	72	56	57	82	87	46	58	78	0	0	0	62	.00652
27	Hernia, Intestinal Obstruction	M	850	134	56	34	24	15	13	23	20	23	37	41	59	90	117	183	331	446	699	1,042	1,196	1,351	1,006	889	10,870	101	.00846
		F	515	49	38	20	24	17	9	7	14	28	42	68	68	114	131	205	299	513	708	1,097	1,206	1,841	1,208	0	0	91	.00968
28	Cirrhosis of Liver ..	M	15	14	0	2	2	4	4	3	6	18	41	76	149	232	366	437	510	575	562	666	504	327	0	0	0	102	.00854
		F	15	10	9	2	0	4	3	2	5	12	24	61	104	170	190	248	321	330	327	436	349	118	0	0	0	59	.00627
28a	Other Diseases of the Digestive System ..	M	384	76	73	81	59	60	38	53	56	81	88	128	169	193	275	350	465	720	848	1,019	1,070	1,924	839	889	0	153	.01279
		F	333	60	64	94	90	77	47	45	80	120	135	161	194	203	318	417	617	783	993	1,256	1,410	1,645	1,359	808	0	164	.01842
29	Acute Nephritis and Bright's Disease ..	M	237	114	58	58	37	44	39	93	141	175	197	311	499	747	1,008	1,669	2,425	3,336	4,255	5,866	6,307	6,999	5,200	8,000	0	518	.04343
		F	155	122	58	24	26	41	50	89	176	201	220	326	420	551	827	1,114	1,535	2,028	2,557	3,007	3,488	3,055	3,322	5,659	0	356	.03766
30	Non-cancerous Tumors of the Female Genital Organs	F	8	0	0	0	0	1	1	9	62	97	125	123	136	151	106	109	102	97	153	198	131	196	453	0	0	65	.00683
30a	Other Diseases of the Genito-urinary System	M	84	20	2	2	7	5	2	5	18	18	25	52	83	127	181	335	741	1,434	2,548	4,696	6,773	9,905	6,709	6,222	0	195	.01636
		F	43	12	6	2	7	4	2	4	12	18	23	36	44	52	75	106	139	155	281	363	436	588	302	0	0	35	.00371
31	Puerperal Septicæmia ..	F	0	0	0	0	0	0	1	47	184	206	312	238	98	12	1	0	0	0	0	0	0	0	0	0	0	95	.01010
32	Other Accidents of Pregnancy and Labour	F	0	0	0	0	0	0	2	79	292	469	544	643	346	55	0	0	0	0	0	0	0	0	0	0	0	186	.01968
32a	Diseases of the Skin and Cellular Tissue ..	M	364	30	21	15	11	6	8	8	9	16	17	24	27	57	57	103	158	293	521	944	1,661	3,233	4,026	6,222	10,870	64	.00532
		F	288	37	19	11	11	6	4	8	7	14	16	19	24	32	45	80	149	240	455	985	1,569	1,802	1,963	5,659	12,048	52	.00551
32b	Diseases of the Organs of Locomotion	M	63	36	27	28	28	27	29	12	8	11	7	13	8	9	19	19	47	41	71	69	50	82	168	889	0	20	.00169
		F	33	35	19	15	13	13	10	2	5	6	5	6	5	8	12	15	12	27	58	33	58	39	0	0	0	11	.00114
33	Congenital Debility and Malformations ..	M	32,080	218	89	32	26	10	6	4	1	1	1	0	0	1	0	0	0	0	0	0	0	0	0	0	0	824	.06903
		F	25,116	175	49	31	29	9	4	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	670	.07088
33a	Other Diseases of Infancy	M	3,996	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	101	.00849
		F	3,170	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	83	.00884
34	Senile Debility	M	0	0	0	0	0	0	0	0	0	0	0	0	2	3	14	121	839	3,125	11,544	27,807	63,663	127,737	187,186	247,111	847,825	907	.07602
		F	0	0	0	0	0	0	0	0	0	0	0	1	0	2	13	39	530	2,377	8,673	22,932	56,924	110,737	173,335	249,798	771,085	757	.08017
35	Violent Death—Suicide excepted	M	1,684	984	767	531	387	409	458	699	838	899	951	1,030	1,145	1,300	1,435	1,654	1,855	1,929	2,313	2,484	3,626	4,912	4,026	8,889	32,608	969	.08119
		F	1,300	777	621	493	425	227	152	144	144	144	153	183	174	203	272	348	376	683	1,005	1,652	3,110	5,014	8,908	9,700	48,193	301	.03184
36	Suicide	M	0	0	0	0	0	0	7	36	125	196	270	337	397	413	481	547	566	555	417	371	403	532	0	0	0	197	.01653
		F	0	0	0	0	0	0	4	39	58	65	83	78	78	105	93	89	72	52	46	26	14	0	0	0	0	47	.00494
38	Unknown or Ill-defined Diseases	M	1,253	481	108	71	35	29	21	23	33	45	67	121	184	243	464	654	1,072	1,152	1,049	1,257	1,360	2,046	1,006	2,667	0	219	.01831
		F	1,185	517	90	51	41	17	13	14	29	40	49	77	100	162	239	285	520	537	468	621	683	1,097	151	1,617	12,048	134	.01417
Deaths all Causes.	As at 30th June, 1911 .. Per 1,000,000 same sex and age-group.	M	85,398	15,594	6,415	4,092	3,010	2,215	1,729	2,564	3,643	4,308	5,076	6,659	8,406	11,202	15,107	21,533	31,741	46,782	69,723	109,003	168,152	207,834	308,957	391,111	1,130,435	11,937	1.00
		F	68,619	13,891	5,752	3,820	2,999	2,023	1,528	2,211	3,405	4,315	4,751	5,860	6,413	7,914	10,495	14,733	21,773	34,707	55,231	91,615	145,747	217,008	289,596	383,185	1,072,289	9,447	1.00
Census Results	As at 31st Dec., 1905† ..	M	101,120	18,040	6,770	4,410	3,500	2,212	1,932	3,043	4,042	4,785	5,610	7,038	9,180	12,093	15,558	21,057	30,674	47,596	76,464	116,621	169,497	247,539	360,968	518,718	847,297		
		F	83,490	16,840	6,310	4,120	3,250	2,004	1,763	2,664	3,697	4,685	5,602	6,597	7,569	8,586	10,681	15,038	23,324	36,575	60,289	94,307	138,276	208,288	312,104	460,610	806,338		

* Calculated from the actual aggregates given in Table CXLIX. † From the Census Life Tables.

TABLE CLII.—Shewing the Numbers Dying per Month during each of the First Four Weeks of Life per 100,000 of the same Sex and Age, and during each of the First 12 Months of Life per 1,000,000 of the same Sex and Age. Australia, 1910-1915,

No.	Males—Cause.	Age at Death.				Age at Death.												Total
		Under 1 week.	1 week and under 2 weeks	2 weeks and under 3 weeks	3 weeks and under 1 mth.	Under 1 Mth.	1 mth. and under 2 mths.	2 mths. and under 3 mths.	3 mths. and under 4 mths.	4 mths. and under 5 mths.	5 mths. and under 6 mths.	6 mths. and under 7 mths.	7 mths. and under 8 mths.	8 mths. and under 9 mths.	9 mths. and under 10 mths.	10 mths. and under 11 mths.	11 mths. and under 12 mths.	Under 1 Year.
8	Whooping Cough	2	3	13	21	108	304	254	205	148	117	117	123	78	65	100	73	1,702
28	Pulmonary Tuberculosis ..	..	..	..	..	..	5	8	8	8	16	8	11	16	5	3	5	93
29	Acute Miliary ,, ..	..	..	..	..	..	0	3	5	0	5	3	8	0	3	5	0	32
30	Tubercular Meningitis ..	..	1	0	1	5	5	31	24	18	21	35	24	35	35	35	38	305
31	Abdominal Tuberculosis ..	..	1	0	0	3	10	10	32	21	18	16	11	11	11	13	32	188
34	Tuberculosis of Other Organs	..	..	..	..	..	5	3	0	3	3	0	0	8	3	0	3	26
35	Disseminated Tuberculosis	..	..	..	..	..	..	3	11	3	0	0	3	3	5	0	3	29
37	Syphilis	33	17	27	24	252	216	173	74	48	48	24	19	30	0	8	13	919
61	Meningitis	36	17	12	10	187	117	141	184	167	149	205	190	206	196	143	197	2,084
71	Convulsions	405	187	80	40	1,685	203	160	162	98	88	115	86	75	107	70	76	2,985
89	Acute Bronchitis	24	64	55	62	521	542	250	169	148	74	88	77	67	48	81	49	2,143
91	Broncho-Pneumonia ..	17	45	40	41	362	466	335	240	198	186	235	174	212	161	191	235	3,011
92	Pneumonia	43	37	29	31	347	276	165	162	172	117	139	130	145	172	124	97	2,058
104	Diarrhœa and Enteritis ..	64	151	186	172	1,446	1,826	2,324	2,505	2,427	2,331	2,125	1,843	1,663	1,423	1,183	1,157	22,303
109	Hernia, Intestinal Obstruction	37	17	8	8	169	36	37	63	74	101	99	86	62	62	24	22	839
150	Malformations	776	195	91	52	2,632	354	220	179	106	74	69	94	43	43	22	49	3,983
151	Congenital Debility, Icterus and Sclerema	7,193	990	649	355	21,681	2,046	1,122	724	511	430	357	227	185	124	89	130	28,410
152	Other Diseases peculiar to early infancy	1,713	184	79	24	4,690	99	52	0	0	0	0	0	0	0	0	0	5,003
153	Lack of Care	29	2	0	2	80	21	10	0	0	0	0	0	0	0	0	0	114
	Other Causes	480	232	192	109	2,432	750	464	424	384	322	363	241	354	358	337	340	6,854
		10,852	2,143	1,461	952	36,600	7,281	5,765	5,171	4,534	4,100	3,998	3,347	3,193	2,821	2,428	2,519	
	Population of males at the beginning of each period allowing for migration (on which the results are based).	399,823	389,936	388,016	386,716	385,413	382,661	380,528	378,654	377,052	375,642	374,239	373,226	372,236	371,409	370,753	370,087	
	Total Deaths (Males) on which results are based	9,889	1,924	1,306	1,311		2,796	2,200	1,963	1,713	1,543	1,499	1,251	1,190	1,049	901	933	31,288

No.	Females—Cause.	Age at Death.				Age at Death.												Total
		Under 1 week.	1 week and under 2 weeks	2 weeks and under 3 weeks	3 weeks and under 1 mth.	Under 1 Mth.	1 mth. and under 2 mths.	2 mths. and under 3 mths.	3 mths. and under 4 mths.	4 mths. and under 5 mths.	5 mths. and under 6 mths.	6 mths. and under 7 mths.	7 mths. and under 8 mths.	8 mths. and under 9 mths.	9 mths. and under 10 mths.	10 mths. and under 11 mths.	11 mths. and under 12 mths.	Under 1 Year.
8	Whooping Cough	1	2	20	24	132	398	310	202	162	85	108	122	152	122	106	106	2,008
28	Pulmonary Tuberculosis ..	1	0	2	0	8	3	0	3	8	14	8	14	8	11	0	8	85
29	Acute Miliary ,, ..	..	..	..	..	..	0	0	0	3	8	0	0	0	3	0	8	22
30	Tubercular Meningitis ..	..	..	..	..	..	0	14	22	22	27	19	22	25	33	42	39	264
31	Abdominal Tuberculosis ..	..	..	..	1	3	5	16	14	25	16	19	3	17	8	22	3	151
34	Tuberculosis of Other Organs	..	..	..	1	3	0	0	0	0	0	3	0	0	0	0	0	6
35	Disseminated Tuberculosis	..	..	..	..	..	0	0	5	0	0	3	0	0	0	-3	0	11
37	Syphilis	22	12	20	14	176	135	65	107	57	41	58	28	17	11	25	17	739
61	Meningitis	40	27	14	7	222	87	102	128	140	140	135	169	150	172	139	137	1,715
71	Convulsions	286	116	43	29	1,151	146	117	87	82	93	94	75	69	103	70	67	2,157
89	Acute Bronchitis	15	40	57	35	379	357	250	158	132	80	72	72	69	81	50	47	1,756
91	Broncho-Pneumonia ..	14	28	33	32	283	372	277	191	153	154	163	122	164	194	153	132	2,357
92	Pneumonia	25	22	26	17	228	173	106	137	118	146	149	136	103	133	136	95	1,657
104	Diarrhoea and Enteritis ..	37	96	106	95	835	1,432	1,876	2,144	2,197	1,937	1,666	1,544	1,328	1,368	1,159	992	18,456
109	Hernia, Intestinal Obstruction	18	13	0	9	99	32	11	27	57	77	77	53	44	25	19	11	530
150	Malformations	597	122	62	40	2,003	203	128	104	99	92	80	47	36	61	56	59	2,970
151	Congenital Debility, Icterus and Sclerema	5,579	809	518	289	17,375	1,418	878	710	419	352	265	243	155	142	128	114	22,22[illegible]
152	Other Diseases peculiar to early infancy	1,329	147	64	24	3,750	73	38	0	0	0	0	0	0	0	0	0	3,86[illegible]
153	Lack of Care	30	0	0	0	74	0	5	0	0	0	0	0	0	0	0	0	7[illegible]
	Other Causes	343	152	130	58	1,670	649	343	355	203	223	254	252	283	306	328	306	5,26[illegible]
		8,337	1,586	1,095	675	28,391	5,483	4,536	4,394	3,967	3,485	3,173	2,902	2,620	2,773	2,436	2,141	
	Population of females at the beginning of each period allowing for migration (on which the results are based).	380,827	373,570	372,209	371,276	370,516	368,534	366,933	365,413	364,077	362,943	361,949	361,078	360,333	359,557	358,927	358,426	
	Total Deaths (Females) on which results are based	7,259	1,365	939	770		2,026	1,668	1,609	1,447	1,267	1,150	1,049	945	998	875	768	24,135

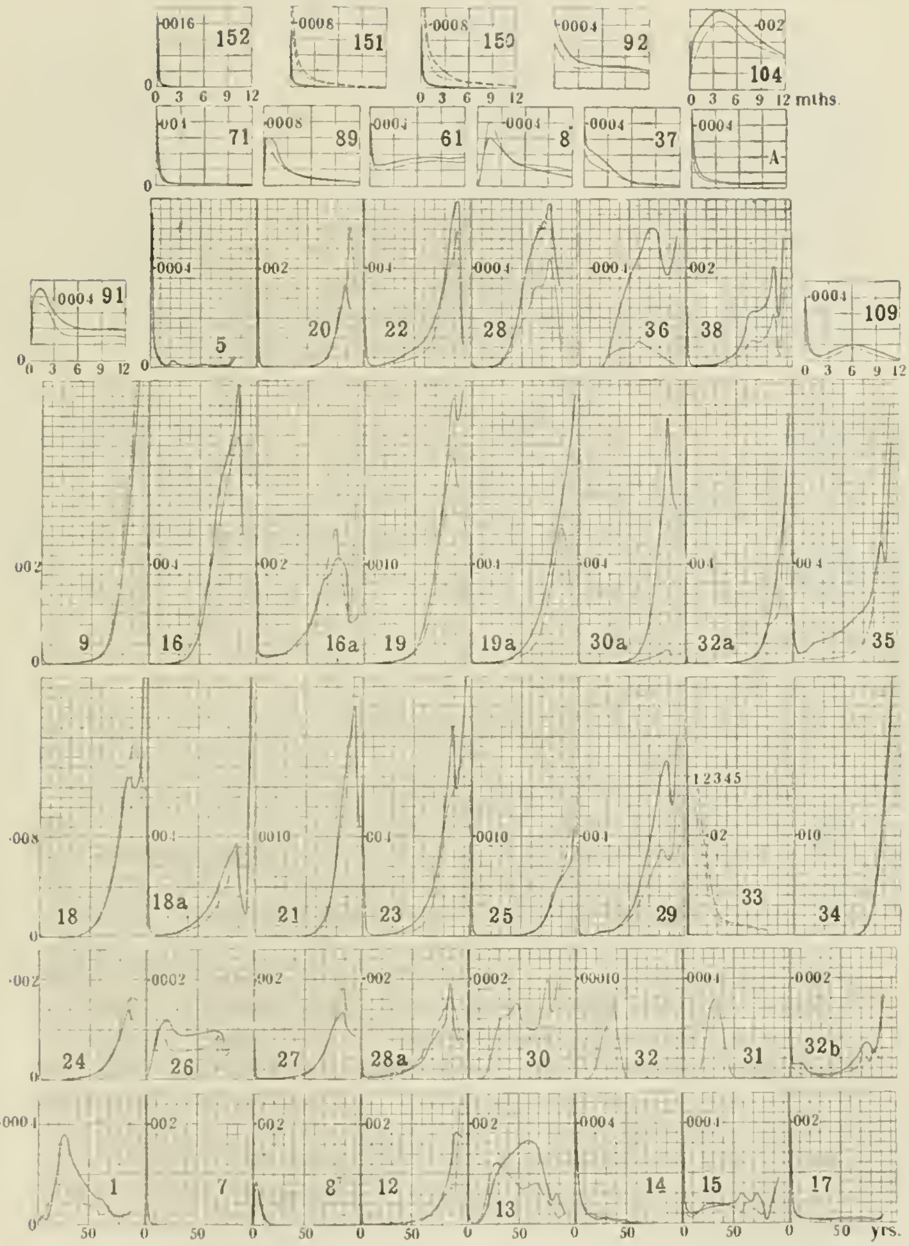

Fig. 104.

The 13 figures ruled into rectangles are death-rates for the first 12 months of life, the rates being shewn by the figures on one of the horizontal lines. The 38 figures ruled into smaller squares shew the death-rates for all ages of the diseases indicated by the numbers. For the index to the above curves see next page.

Index to Curves in Figure 104.

DEATH-RATES FOR ALL AGES.

†1. Typhoid Fever
5. Measles.
7. Whooping Cough.
8. Diphtheria and Croup.
9. Influenza.
12. Other Epidemic Diseases.
13. Tuberculosis of the Lungs.
14. Tuberculous Meningitis.
15. Other forms of Tuberculosis.
16. Cancer and other Malignant Tumours.
16*a*. Other General Diseases.
17. Simple Meningitis.
18. Cerebral Hæmorrhage and Softening.
18*a*. Other Diseases of the Nervous System.
19. Organic Diseases of the Heart.
19*a*. Other Diseases of the Circulatory System.
20. Acute Bronchitis.
21. Chronic Bronchitis.
22. Pneumonia.
23. Other Diseases of the Respiratory System.
24. Diseases of the Stomach.
25. Diarrhœa and Enteritis (all ages).
26. Appendicitis and Typhlitis.
27. Hernia, Intestinal Obstruction.
28. Cirrhosis of Liver.
28*a*. Other Diseases of the Digestive System.
29. Acute Nephritis and Bright's Disease.
30. Non-cancerous Tumours of Female Genital Organs.
30*a*. Other Diseases of the Genito-urinary System.
31. Puerperal Septicæmia.
32. Other Accidents of Pregnancy and Labour.
32*a*. Diseases of the Skin and Cellular Tissue.
32*b*. Diseases of the Organs of Locomotion.
33. Congenital Debility and Malformations.
34. Senile Debility.
35. Violent Death (Suicide excepted).
36. Suicide.
38. Unknown or Ill-defined Diseases.

DEATH-RATES FOR FIRST YEAR OF LIFE.

*8. Whooping Cough.
37. Syphilis.
61. Meningitis.
71. Convulsions.
89. Acute Bronchitis.
91. Broncho-Pneumonia.
92. Pneumonia.
104. Diarrhoea and Enteritis.
109. Hernia and Intestinal Obstruction.
150. Malformations.
151. Congenital Debility.
152. Other Diseases peculiar to Early Infancy.
A. Other Causes.

* These numbers, on Fig. 104, are identical with those of the "Detailed Nomenclatures of Diseases" of the International Commission, Session July 1909, at Paris.

† These numbers, on Fig. 104, are identical with those in Table CXLIX. to CLI., and where not marked "*a*" are those of the "Abridged Nomenclature" of diseases of 1909, where "*a*" or "*b*" added it denotes that the balance for the class in question is included.

The form of the mortality curves during the first year are given on the upper part of Fig. 104; see the Index thereto.

32. **Annual fluctuation of death-rates.**—The frequency of death from particular causes, and therefore generally, is affected by the season of the year, and though in the aggregate of deaths from all causes the seasonal effect is somewhat masked, it is not wholly obliterated. To ascertain rigorously the character of the annual periodicity, either generally or from a particular "cause," of death it is necessary to obtain the rates for small units of time, say equalised months; thus the rates δ_1, δ_2, ... δ_{12} must be obtained: these are sensibly independent of the fluctuations in the deaths and population during the month. Inasmuch, however, as deaths occur very rapidly in the first few days of life, any periodicity in birth-rate involves the death-rate; that is to say, the constitution of the population is not quite homogeneous, and a correction is—theoretically—necessary. The correction, however, is so small that it may be neglected. These last observations apply, *mutatis mutandis,* also to deaths from certain particular causes. The annual fluctuations of birth-rate, and the mode of solving have been indicated at length in Part XI., §§ 14-19, pp. 166-174. General factors for reducing the values given for calendar months to the values for equalised months must be so applied as to have regard to the average values at the beginning and end of the months.

Table CLIII. depends upon a total of 252,443 deaths of males [1], and 185,367 deaths of females occurring in an aggregate population of

[1] For example there were 3529 deaths from typhoid in the 9 years, of which 473 occurred in the month of January. These, when corrected, for the growth of population during the year, and altered so as to give the result for the exact twelfth of the mean length of the year, gave the basis for the calculation of the results in the table.

over 21,000,000 males and nearly 20,000,000 females. The numbers given in the table correspond to a population of 10 millions in each case.

In Table CLIV. the proportions of deaths occurring in months of equal length, when the population is constantly the same, are given.

Algebraically if b and e be the equalising *corrections* at the beginning and end of the month to D, the number of deaths, and P be the sum of the populations of the corresponding month for the whole period under review, the results in Tables CLIII. and CLIV. are respectively :—

$$(650)\ldots\ldots\delta = (D+b+e)/P\,; \quad (651)\ldots\ldots\rho = 12\delta\,/\,\Sigma\delta.$$

TABLE CLIII.—Shewing Average Number of Deaths due to Various Causes, per 10,000,000 Males, and per 10,000,000 Females respectively of all Ages during each Equalised Month of the Year. Based upon 9 Years' Experience (1907-1915) in Australia.

Cause of Death.	Sex	Jan.	Feb.	Mar.	April.	May.	June.	July.	Aug.	Sept.	Oct.	Nov.	Dec.	Year.
Typhoid Fever	M	221	231	247	223	167	108	63	54	50	51	92	164	1,671
	F	129	152	156	118	97	80	41	26	24	25	50	88	986
Whooping Cough	M	77	64	43	45	61	63	66	79	85	74	77	77	811
	F	100	83	71	63	64	63	92	94	102	102	94	89	1,017
Diphtheria and Croup ..	M	81	76	99	123	149	153	142	109	102	85	83	89	1,291
	F	95	88	86	147	151	165	145	128	116	90	107	83	1,401
Influenza	M	39	27	29	38	54	69	99	163	188	140	93	49	988
	F	48	27	28	36	45	56	94	160	191	146	97	59	987
Tuberculosis	M	613	581	585	580	646	641	713	701	692	669	610	590	7,651
	F	527	484	489	514	511	512	533	590	558	585	496	516	6,315
Cancer	M	637	659	604	638	603	594	571	595	619	643	622	658	7,443
	F	613	628	613	615	623	593	578	587	588	610	615	605	7,268
Diabetes	M	58	56	62	65	66	86	78	85	73	73	65	68	835
	F	74	74	68	80	82	94	102	98	109	99	90	93	1,063
Organic Diseases of the Heart	M	855	784	802	832	903	995	1,052	1,070	994	925	884	794	10,890
	F	725	613	650	667	697	834	950	891	780	744	629	670	8,850
Diseases of the Respiratory System	M	757	646	743	844	1,000	1,250	1,500	1,594	1,519	1,197	1,042	830	12,922
	F	519	472	471	581	723	895	1,083	1,217	1,088	895	726	609	9,279
Diarrhoea and Enteritis ..	M	1,021	941	866	764	503	265	203	166	185	338	782	1,069	7,103
	F	894	820	787	678	457	264	164	127	137	309	644	895	6,176
Infancy	M	662	697	695	703	686	719	734	656	683	614	666	680	8,196
	F	580	543	571	608	562	579	616	561	504	528	521	532	6,705
Old Age	M	692	664	629	671	754	857	905	873	836	733	722	732	9,068
	F	629	567	548	566	631	697	748	726	688	628	581	570	7,579
Total all Causes ..	M	10,406	9,681	9,469	9,633	9,604	9,881	10,411	10,309	10,215	9,570	9,984	10,146	109309
	F	8,152	7,667	7,391	7,724	7,702	7,897	8,279	8,411	7,895	7,697	7,802	7,967	94,584

TABLE CLIV.—Shewing for each Equalised Month the Average Relative Frequency of Death due to Various Causes, the Population being Constant throughout the Year. Based upon 9 Years' Experieuce (1907-1915). Australia.

Cause of Death.	Sex	Jan.	Feb.	Mar.	April.	May.	June.	July.	Aug.	Sept.	Oct.	Nov.	Dec.	Year.
Typhoid Fever	M	1.589	1.657	1.773	1.599	1.199	.779	.449	.391	.357	.367	.663	1.179	12.000
	F	1.567	1.844	1.905	1.436	1.178	.979	.494	.320	.288	.304	.612	1.071	12.000
Whooping Cough	M	1.139	.942	.630	.664	.910	.930	.975	1.176	1.258	1.095	1.138	1.144	12.000
	F	1.185	.978	.835	.742	.761	.744	1.082	1.107	1.205	1.206	1.105	1.052	12.000
Diphtheria and Croup ..	M	.752	.706	.921	1.141	1.381	1.426	1.316	1.015	.948	.790	.776	.829	12.000
	F	.810	.752	.739	1.258	1.295	1.414	1.241	1.098	.994	.774	.914	.711	12.000
Influenza	M	.468	.332	.349	.463	.657	.840	1.199	1.982	2.281	1.707	1.134	.593	12.000
	F	.580	.332	.345	.434	.550	.676	1.140	1.942	2 325	1.779	1.182	.714	12.000
Tuberculosis	M	.961	.912	.917	.910	1.012	1.006	1.118	1.100	1.086	1.049	1.003	.926	12.000
	F	1.001	.919	.929	.976	.971	.973	1.013	1.121	1.061	1.112	.913	.981	12.000
Cancer	M	1.027	1.062	.975	1.029	.972	.958	.921	.959	.998	1.036	1.164	1.061	12.000
	F	1.011	1.037	1.012	1.015	1.029	.979	.954	.969	.971	1.007	1.016	.998	12.000
Diabetes	M	.839	.811	.888	.927	.952	1.235	1 123	1.224	1.047	1.041	.938	.975	12.000
	F	.834	.837	.767	.901	.929	1.056	1.148	1.102	1.232	1.124	1.012	1.052	12.000
Organic Diseases of the Heart	M	.942	.864	.884	.917	995	1.096	1.159	1.179	1.095	1.020	.974	.875	12.000
	F	.983	.831	.882	.904	.945	1.130	1.288	1.208	1.058	1.008	.853	.909	12.000
Diseases of the Respiratory System	M	.703	.599	.690	.784	.929	1.161	1.392	1.480	1.411	1.112	.968	.771	12.000
	F	.671	.611	.610	.752	.935	1.157	1.400	1.573	1.407	1.157	.939	.788	12.000
Diarrhoea and Enteritis ..	M	1.725	1.590	1.463	1.290	.850	.448	.344	.281	.312	.570	1.321	1.805	12.000
	F	1.737	1.594	1.530	1.317	.889	.512	.319	.246	.266	.600	1.252	1.739	12.000
Infancy	M	.971	1.020	1.017	1.030	1.004	1.053	1.075	.961	.999	.899	.975	.996	12.000
	F	1.038	.971	1.002	1.089	1.005	1.037	1.102	1.003	.902	.915	.933	.951	12.000
Old Age	M	.916	.878	.833	.888	.997	1.134	1.197	1.155	1.106	.970	.955	.969	12.000
	F	.996	.897	.868	.896	.999	1.103	1.184	1.150	1.089	.995	.920	.903	12.000
Total all Causes	M	1.047	.974	.952	.969	.966	.994	1.047	1.037	1.027	.963	1.004	1.020	12.000
	F	1.034	.973	.938	.980	.977	1.002	1.050	1.067	1 002	.977	.990	1.011	12 000

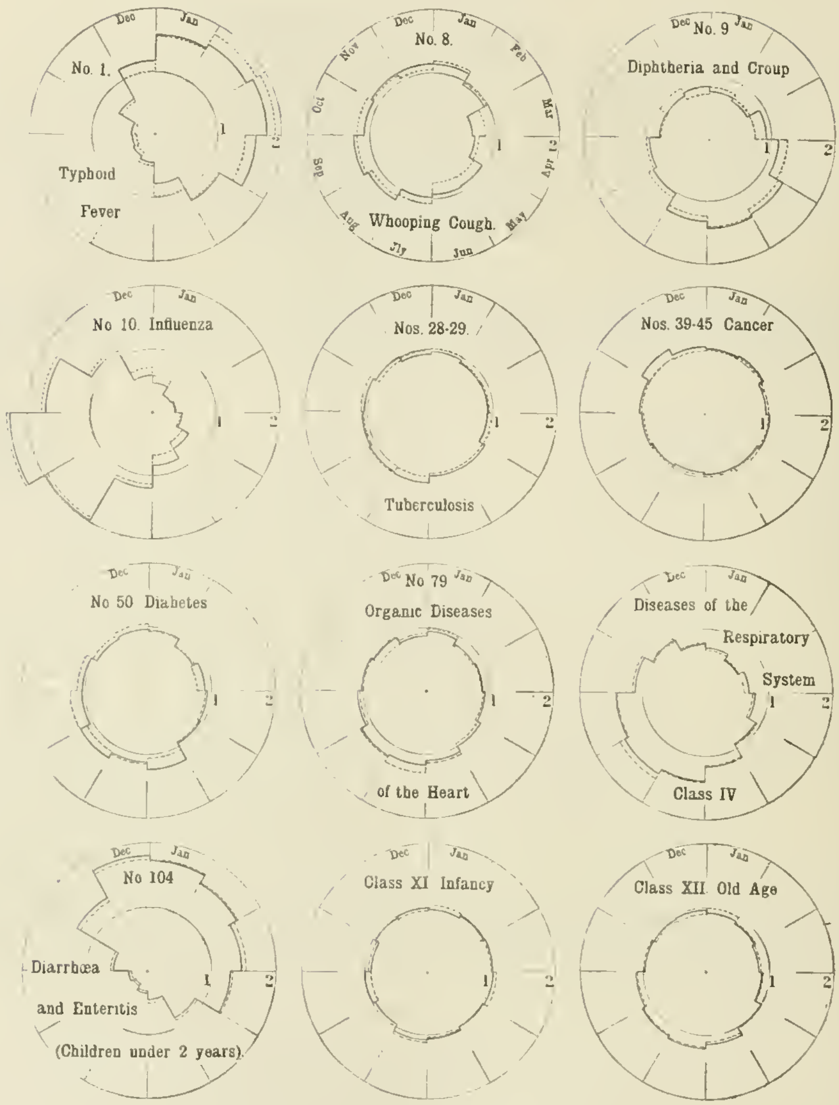

FIG. 105.

The distances from the centres of the circles shew the average ratios of the death-rate per month to the average rate for the entire year, the ratios for males being denoted by firm lines, and those for females by dotted lines, the succession of months being clockwise. In the case of absence of fluctuation the sector-boundaries would all be on the circle marked "1," *e.g.*, "Cancer." In the case of "Influenza" it will be seen that the September rate is more than double the average for the year.

33. **Studies of particular causes of death: voluntary death.**—Although the study of particular causes of death might appear not to belong to the general theory of population, it is really an essential. For example, if diseases, the incidence of which is characteristic of earlier life, be combatted, the consequence will be an increase in deaths from those which

characterise later years (*e.g.*, tuberculosis and cancer). Again statistics of voluntary death or *suicide*, are of special importance, inasmuch as they disclose the regularity of human conduct even in matters which might be thought to be peculiarly under individual control, and be imagined to lie outside regular law. But suicide follows well-defined laws, and even as regards the *mode* of death the regularity is remarkable, as the following table shews :—

TABLE CLV.—Mode of Voluntary Death. Australia 1907-15.

Mode of Death.	Number of Suicides.									Totals.	Ratio to Aggregate.	Range.			
	1907.	1908.	1909.	1910.	1911.	1912.	1913.	1914.	1915.			Lowest (*a*).	Mean.	Highest (*b*).	Mean of (*a*) & (*b*)*
Poison	57	88	70	79	93	128	127	121	105	868	.2079	57	96.4	128	92.5
Asphyxia	2	1	2	0	2	4	2	2	0	15	.0036	0	1.7	4	2.0
Hanging and Strangulation	71	68	67	72	69	79	79	72	84	661	.1583	67	73.4	84	75.5
Drowning	37	31	24	42	43	34	25	30	38	304	.0728	24	33.8	38	31.0
Firearms	129	146	138	134	133	168	163	201	196	1,408	.3373	129	156.4	201	165.0
Cutting Instruments ..	61	54	74	79	65	76	88	76	89	662	.1586	54	73.6	89	71.5
Precipitation from Height	6	4	7	3	2	0	6	4	4	36	.0086	0	4.0	7	3.5
Crushing	3	6	5	8	6	8	10	2	8	56	.0134	2	6.2	10	6.0
Other	19	15	11	15	33	17	16	26	13	165	.0395	11	18.3	33	22.0
Total, Males	385	413	398	432	446	514	516	534	537	4.175	1.0000	385	464	537	461
Poison	32	35	54	34	52	70	76	61	64	478	.5201	32	53.1	76	54.0
Asphyxia	0	0	0	0	1	0	1	0	1	3	.0033	0	0.3	1	0.5
Hanging and Strangulation	12	15	9	10	10	12	22	15	18	123	.1338	9	13.7	22	15.5
Drowning	19	14	19	19	13	11	14	17	21	147	.1600	11	16.3	21	16.0
Firearms	3	7	6	6	9	10	9	4	5	59	.0642	3	6.6	10	6.5
Cutting Instruments ..	5	6	5	13	9	8	4	3	6	59	.0642	3	6.6	13	8.0
Precipitation from Height	1	2	0	0	2	0	2	4	3	14	.0152	0	1.6	4	2.0
Crushing	2	2	1	0	0	1	1	2	2	11	.0120	0	1.2	2	2.0
Other	2	3	3	2	2	6	2	3	2	25	.0272	2	2.8	6	4.0
Total, Females ..	76	84	97	84	98	118	131	109	122	919	1.0000	76	102	131	103
Ratio of Females to Males	.197	.203	.244	.194	.220	.230	.254	.204	.227	.220		.194	.219	.254	2.37
Ratio of Males to Females	5.07	4.92	4.10	5.16	4.55	4.36	3.94	4.90	4.40	4.54		3.94	4.60	5.16	4.55

* It is worthy of note that the mean of the highest and lowest number of suicides in any year is sensibly equal to the arithmetic mean. The male population increased about 18.40 per cent. on the period covered, and the female 21.82 per cent.

The ratio of the total females of age 16 and above, to the total males of 16 and above, was about 1.10904, and of 21 and above was 1.12391. This would indicate a frequency of 4.097, or 4.042 to 1 for male, as compared with female suicides. But this relative frequency is very variable. On the whole it is rapidly increasing. The ratios of the death-rates of males and females according to age are as follow, viz. :—

Age.	10–14	15–19	20–24	25–29	30–34	35–39	40–44	45–49	50–54	55–59	60–64	65–69	70–74	75–79	80–84	85–89
Ratio of Death Rates	1.7	0.97	2.15	3.01	3.25	4.32	5.09	3.93	6.17	5.14	7.86	10.67	9.07	14.27	28.79	∞
Smoothed Ratio	.74	1.37	1.99	2.62	3.25	3.88	4.51	5.13	5.76	6.39	7.30	8.80	11.50	15.60	28.80	?

These results shew that the ratio of the rate of suicide by men to that of suicide by women increases about 0.125 per annum till about age 60, when it becomes more rapid. The general result is, that this rate ρ can be expressed between the ages 10 and 57.5 as :—

$$(652)\ldots\ldots\rho = 0.1256\ (x - 6.63)$$

after which the points lie upon the curve indicated by the numbers 6.39, 7.30, etc., in the preceding result as smoothed.

The annual fluctuation of suicide is fairly well-defined. By correcting the results so as to make them represent what would have been furnished by records of equal months, and a constant population[1] (as at the middle of the period), the following values are obtained, viz. :—

TABLE CLVI.—Number of Suicides per diem in a Population of 1,000,000 Persons. Australia, 1900 to 1915.

Period.	Jan.	Feb.	Mar.	April.	May.	June.	July.	Aug.	Sept.	Oct.	Nov.	Dec.
1900-1909*	.359	.371	.335	.336	.310	.284	.301	.326	.307	.353	.323	.345
1907-1915	.376	.356	.326	.335	.306	.262	.346	.295	.351	.358	.356	.381
Mean ..	.367	.364	.330	.335	.308	.273	.324	.310	.329	.356	.340	.363

* These results are given by $0.3291 + 0.0354 \sin(x + 72° 4') - 0.0117 \sin 2(x + 73 .22') + 0.0031 \sin 3(x + 12° 49') - 0.0142 \sin 4(x + 40° 52') - 0.0131 \sin 5(x + 0°.16') + 0.0104 \sin 6x$: Journ. Roy. Soc. N.S.W., xlv., p. 99.

The final mean results probably do not define the curve representing an indefinitely large number of cases. The results given are based upon only about 10,000 cases, and at least 10 times this number would be necessary to get satisfactory results. The distribution is more likely to be of the form.[2]

$$(653) \ldots y = A + B \sin x + C \cos x = A + b \sin(x + \beta) + c \cos(x + \gamma)$$

$$(654) \ldots A = (\Sigma_1^n y)/n\,;\ B = b \cos\beta - c \sin\gamma\,;\ C = b \sin\beta + c \cos\gamma.$$

[1] The population records give for the population at the middle of each month the following results, 00 omitted :—

	Jan.	Feb.	Mar.	April.	May.	June.	July.	Aug.	Sept.	Oct.	Nov.	Dec.
Males—	209,686	210,012	210,338	210,662	210,983	211,305	211,657	212,039	212,421	212,834	213,278	213,723
Females—	194,153	194,513	194,873	195,054	195,055	195,056	195,442	196,211	196,981	197,766	198,567	199,369

[2] See "Studies in Statistical Representation" (Statistical Applications of the Fourier series), by G. H. Knibbs, Journ. Roy. Soc., N.S.W., xlv., pp. 76-110, 1911; in particular pp. 97-110.

XVI.—MIGRATION.

1. **Migration.**—The effect of immigration, and indeed of migration generally, is to modify the age, sex, and race constitution of a community, and these facts are well illustrated in the statistics of any new country (*e.g.*, the Commonwealth of Australia). Concentrations of population due to seasonable or similar influences, or from other causes, may also become a factor of importance from particular points of view. For example, statistics of morbidity or of mortality, the object of which is to differentiate between urban and country hygienic conditions, may be materially affected even by temporary concentrations of populations in cities; for example, by the fact that serious impairments of health may lead to transfer to the cities for special treatment, with a consequent increase of the mortality and morbidity rates; and so on. Certain obvious economic consequences may, too, arise from such concentrations. For these reasons statistics for particular purposes are often limited as regards precision.

In countries where the migration of adults is a striking characteristic, the constitution of the population according to age ceases to be normal; but the aggregates obtained by inclusion of the group of countries between which the migration takes place, tend to restore the normality. In Australia financial arrangements between the component States have, among other things, led to records being kept (*a*) of oversea migration, (*b*) of interstate migration by sea, and to a partial record (*c*) of overland migration. All of these shew fluctuations of annual period.

Records of overland migration by road are not kept, but such migration is assumed to be in balance, that is to say, the immigration and emigration are supposed to be equal. It will be seen later that overland immigration by rail virtually balances the overland emigration.

2. **Proportion born in a country.**—The correlation of birth-place and age in any population is of sociologic importance.[1] In the following results, from the 1911 Australian Census, the "unspecified" cases (as to whether the birth-place was Australia or outside of Australia) have, for each age-group, been distributed in the proportion of the numbers given as born in and out of Australia, respectively. The results are as shewn in Table CLVII. hereunder and in Fig. 106. These disclose the fact that the initial preponderance of persons born in Australia diminishes very rapidly with age; this of course being due to the fact that the commencement of colonisation was at a point of time nearly identical with the birth of the present oldest inhabitants.

[1] An analysis of the Australian population will be found in the Census Report, Vol. I., pp. 120-125.

TABLE CLVII.—Shewing according to Age and Sex the Proportion of Persons Living in but not Born in Australia.[1]

Age last Birthday.	Proportion not Born in Australia. Males.	Proportion not Born in Australia. Females	Age last Birthday.	Proportion not Born in Australia. Males.	Proportion not Born in Australia. Females	Age last Birthday.	Proportion not Born in Australia. Males.	Proportion not Born in Australia. Females	Age last Birthday.	Proportion not Born in Australia. Males.	Proportion not Born in Australia. Females
0	.0036	.0036	15–19	.0403	.0243	50–54	.4536	.3244	85–89	.9792	.9756
1	.0106	.0103	20–24	.0699	.0513	55–59	.5875	.4915	90–94	.9781	.9886
2	.0160	.0166	25–29	.1866	.1100	60–64	.7014	.6485	95–99	.9569	.9449
3	.0202	.0193	30–34	.2290	.1531	65–69	.7572	.7181	00 and over	.9143	.8966
4	.0215	.0207	35–39	.2538	.1806	70–74	.8880	.8653			
5–9	.0249	.0242	40–44	.3007	.2083	75–79	.8952	.9318			
10–14	.0239	.0232	45–49	.3834	.2673	80–84	.9731	.9637			

The results in the table are graphed in Fig. 106, the Curves M, M′ and F, F′ denoting respectively the results for males and females. The irregular form of the curves is due to the age-peculiarities of the migration. As the population develops by natural increase the curves will tend to become similar to the dotted forms, the F and M curves to become identical, and both will approach more and more the base-line.

Proportions born in Australia.

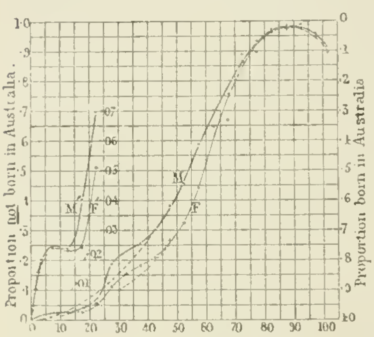

Fig. 106.

Curves M and F shew respectively the proportion of males and of females born in Australia. Curves M′ and F′ are plotted on ten times the vertical scale of Curves M and F.

It might be supposed that by comparing the Census results with the birth-registration results, reduced according to the mortality, so as to shew the number of survivors, the excess of immigrants over emigrants would appear. Such is not the case, however, notwithstanding the striking regularity of the results: see Census Report, Vol. I., pp. 93-94.

[1] See Census Report, Vol. II., pp. 130-1 for males, and pp. 132-3 for females. The unspecified according to age have been ignored. Let the total T of either sex be made up of I those born in Australia, O those born outside, and U the unspecified. Then the adjusted numbers I' and O' will be respectively, $I' = I \,.\, T/(T - U)$; $O' = O \,.\, T/(T - U)$.

Item.		N.S.W.	Vict.	Q'land.	S. Aus.	W. Aus.	Tas.	Total. (Cwlth.)
No. of Males	..	22,957	15,869	8,329	5,378	3,808	2,761	59,102
No. of Females		22,136	15,089	7,967	5,124	3,684	2,584	56,584
Ratio of excess of reduced registration Nos.	M	.0282	.0183	.0346	.0444	.0373	.0120	.02774
	F	.0321	.0323	.0336	.0357	.0231	.0240	.03174
	P	.0301	.0251	.0341	.0402	.0303	.0178	.02970

These excesses, ranging from about $1\frac{1}{4}$ to about $4\frac{1}{2}$ per cent., are only in part accounted for by the migration of infants (see p. 94 above referred to), in fact only one-tenth may be referred to migration. They disclose the necessity of fixing the age exactly by recording the *date of birth*. The practice of accepting loose statements as regards age is from every point of view most unsatisfactory for the purposes of accurate tabulation.

3. **Correlation, owing to migration, between age and length of residence.**—The length of residence of the proportion of persons *not* born in Australia, shown in Table CLVII. and Fig. 106, is furnished by data given in the Report of the Census, Vol. II., pp. 392-393. The middle of the age and length-of-residence groups may be regarded as a sufficiently accurate indication of the average value in both cases. It will then be seen that, for any given length of residence, there is an age at which the numbers are a maximum.

The maximum values are—for males—about as follows :—

MALES.

Average length of residence .. (yrs.)	0–1	1–2	2–3	3–4	4–5	5–10	10–15	15–20	20–25	25–30	30–35	35–40
Age giving maximum numbers .. (yrs.)	24.2	24.7	26.3	27.6	28.6	30.3	34.6	39.7	44.0 (28.6)	49.0 (31.6)	53.8 (37.0)	58.6 (42.5)
Approx. numbers ..	9,590	4,420	3,180	2,430	1,460	868†	1,148†	1,040†	3,000† (904)†	3,600† (1210)†	2,200† (740)†	1,120† (446)†
Total (and average) in age group	39 228	17,571	12,760	9.347	5,816	3,644	4,440	4,315	11,478	13,946	8,386	4,563

† These maximum numbers correspond to the maximum ages on the two lines above.

The above results shew that the curves are dimorphic and consequently that the relation between the frequency of migration and the age of the migrants is not simple. This is to be expected in a new country, where special tendencies in migration are likely to be in evidence from time to time. The complete record of migration requires that migration should be tabulated according to age, and for some purposes a Census is preferably tabulated under the headings "age at entry into the Commonwealth," and "duration of residence."[1] Only in this way can the relation between age and frequency be accurately and conveniently ascertained.

Before dealing in detail with the several classes of migration, the general theory may be indicated.

4. **The theory of migration.**—Migration, either into or out of any territory, varies according to age and sex. The character of these variations change greatly from time to time, when long periods are considered, but are ordinarily fairly constant for short periods. As between place and place, it is, no doubt, other things being equal, also a function of their distance apart.

[1] This is analogous to the tabulating with the headings "duration of marriage" with "age *at* marriage" instead of "age at the Census."

Let the ratio of the number migrating (T) in a unit of time (1 year) to the population (P) in which it occurs, be called the *migration ratio* (τ); then we shall have :—

$$(655)\ldots\ldots\tau = T/P = \Sigma_1^n T \,/\, \Sigma_1^n P$$

the last expression giving the weighted average over the n years included. That is, it is the ratio of the sums of the migrants to the aggregate of the mean populations of the same years.

Let the number of migrants consist of T_m males and T_f females, then the ratios of each of these to their sum may be called the *sex-ratios* a and γ respectively,[1] of the migration. Thus :—

$$(656)\ldots\ldots a = T_m/(T_m + T_f)\,;\; \gamma = T_f/(T_m + T_f).$$

Thus we shall have for the annual number of male and female migrants, respectively :—

$$(657)\ldots\ldots T_m = P\tau a\,;\; T_f = P\tau\gamma.$$

The male and female "migration ratios," however, are given by

$$(658)\ldots\ldots\tau_a = T_m/M\,;\; \tau_\gamma = T_f/F\,;$$

and if the number of males and females in the population be equal, we shall have :—

$$(659)\ldots\ldots\tau_a = 2a\,.\,\tau\,;\; \tau_\gamma = 2\gamma\,.\,\tau.$$

The components of the fluctuation of annual period are distinctly traceable—in many cases the causes can be assigned.

The "migration ratio" τ is not a population-ratio indicating the number of *different* persons migrating : it merely represents the relative quantity of migration independent of the individuals. It has a *fluctuation of annual period*, and minor periods within that, and these can be represented and dealt with as already indicated, see Part III., § 5, pp. 39, 40, in particular formulæ (90) to (101), and Part XI., §§ 16, 17, pp. 169-172.

[1] The first letters of ἀνήρ and γυνή.

The "migration-ratio" is a *function of age and of time*, and is probably in all cases polymorphic, that is :—

$$(660)\ldots\ldots\tau = \phi\,(x) = \Sigma(\,A'e^{\frac{1}{nx}}),\ \text{or} = \Sigma(\,Ax^m e^{-nx^r}\,)$$

in other words, it may be regarded as the sum of a series of curves of one or both of the types shown, see formulæ (23) to (39*a*), pp. 22 to 24, and formulæ (147) to (156), pp. 52 to 55. Like nearly all statistical curves it will probably not conform exactly to any simple expression. The variation with time will ordinarily be considerable in new countries.

The characteristics of the annual fluctuations are not quite identical for the sexes : hence each of the components (T_m and T_f) may be analysed separately, or the total ($T_m + T_f$) may be analysed, and the fluctuation of the sex-ratio, determined for individual months, may be analysed.

5. **Migration-ratios for Australia.**—The migration-ratios for Australia, determined as indicated by formulæ (655) to (659), are as follow :

TABLE CLVIII.—Shewing the Migration-ratios for Australia and the Sex-ratios of the Migration for Oversea and Interstate Sea Migration and for Migration by Railway.

To (I) or from (E)		Oversea Migration, 1909–1913.				Interstate Sea Migration, 1909–1913.				Interstate Migration by Railway, 1914–1916.			
		Males.	Females.	Persons.	Ratio Males to Total.	Males.	Females.	Persons.	Ratio Males to Total.	Males.	Females.	Persons.	Ratio Males to Total.
N.S.W.	I	.05237	.02644	.04003	.68549	.04557	.03251	.03935	.60666	.18966	.09635	.14426	.67506
,,	E	.03654	.01751	.02748	.69666	.04394	.03095	.03775	.60976	.19104	.09447	.14466	.68094
Vic.	I	.02195	.01336	.01763	.61875	.07226	.04807	.06009	.59749	.19580	.07766	.13582	.70974
,,	E	.01376	.00788	.01080	.63298	.07313	.04766	.06032	.60242	.19071	.08047	.13474	.69681
Qld.	I	.02284	.01472	.01912	.64953	.05516	.03251	.04482	.66902	.16801	.11287	.14238	.63129
,,	E	.00928	.00267	.00626	.80570	.05028	.03137	.04165	.65623	.16970	.11476	.14413	.62984
S. Aus.	I	.02199	.01048	.01632	.68336	.05873	.03534	.04720	.63097	.22646	.10576	.16490	.67288
,,	E	.01130	.00372	·00757	.75729	.05502	.03463	.04497	.62041	.23406	.10170	.16655	.68857
W. Aus.	I	.05561	.03502	.04676	.67788	.07460	.07288	.07386	.57566	..	..	..	..
,,	E	.02308	.00963	.01730	.76057	.08202	.07593	.07940	.58876	..	..	..	..
Tas.	I	.02129	.01104	.01630	.66986	.22592	.16258	.19505	.59379	..	..	..	..
,,	E	.00910	.00566	.00742	.62842	.24873	.18082	.21564	.59163	..	..	..	..

The table shows that as regards oversea migration, immigration is preponderant : in interstate sea migration it is also generally preponderant, the exceptions being—Victoria, "males" and "persons" ; Western Australia, "males," "females" and "persons." Interstate migration by railway shews an approximate equality between immigration and emigration, the balance on either side being variable.

That these results have very accordant values from year to year will appear from the following table :—

TABLE CLIX.—Interstate Immigration by Sea, 1909-1913.

Year	State.	Migration-ratios. Males.	Females.	Persons.	Ratio of Male Migrants to Total Migr'nts	State.	Migration-ratios. Males.	Females.	Persons.	Ratio of Male Migrants to Total Migr'nts	State.	Migration-ratios. Males.	Females.	Persons.	Ratio of Male Migrants to Total Migrn'ts
1909	N.S.W.*	.0401	.0278	.0343	.6134	Victoria.	.0651	.0407	.0528	.6099	Q'land.	.0575	.0323	.0460	.6801
1910		.0408	.0294	.0354	.6026		.0670	.0442	.0556	.5994		.0521	.0323	.0430	.6571
1911		.0487	.0347	.0420	6066		.0747	.0504	.0625	.5967		.0559	.0323	.0452	.6714
1912		.0488	.0359	.0427	.6008		.0763	.0526	.0644	.5915		.0522	.0311	.0425	.6641
1913		.0454	.0320	.0390	.6110		.0714	.0491	.0603	.5931		.0557	.0320	.0448	.6717
1909	S. Aust.	.0536	.0315	.0427	.6356	W. Aust.	.0693	.0691	.0692	·5705	Tas.	.1886	.1331	.1614	.5961
1910		.0555	.0349	.0453	.6209		.0790	.0757	.0776	.5808		.2017	.1396	.1713	.6007
1911		.0605	.0371	.0490	.6278		.0783	.0753	.0770	.5811		.2210	.1606	.1862	.5886
1912		.0610	.0362	.0488	.6352		.0709	.0707	.0708	.5694		.2465	.1792	.2139	.5935
1913		.0578	.0337	.0458	.6351		.0698	.0664	.0683	.5756		.2464	.1829	.2158	.5915

* Excluding Federal Territory.

TABLE CLX.—Shewing for the Years 1909 and 1913*, the Ratio of Male Migration to the Total Migration†, and the Proportion of Males, Females and Persons, under 12 Years of Age, to the Total Number of Emigrants. Australian Interstate Migration by Sea.‡

<table>
<tr><th>States from</th><th>N.S. Wales.</th><th>Victoria.</th><th>Queensland.</th><th>S. Australia.</th><th>W. Australia.</th><th>Tasmania.</th></tr>
<tr><td>To N.S. Wales.</td><td></td><td>Masc. M. .066
,, .059
.607 F. .095
.600 ,, .085
P. .078
,, .069</td><td>Masc. M. .061
,, .063
.665 F. .112
.667 ,, .118
P. .078
,, .082</td><td>Masc. M. .086
,, .070
.591 F. .109
.599 ,, .103
P. .095
,, .083</td><td>Masc. M. .161
,, .126
.573 F. .212
.598 ,, .163
P. .183
,, .141</td><td>Masc. M. .095
,, .087
.518 F. .079
.499 ,, .098
P. .087</td></tr>
<tr><td>To Victoria.</td><td>Masc. M. .061
,, .063
.612 F. .086
.578 ,, .082
P. .071
,, .071</td><td></td><td>Masc. M. .084
,, .085
.649 F. .119
.600 ,, .122
P. .096
,, .100</td><td>Masc. M. .065
,, .063
.647 F. .122
.677 ,, .099
P. .085
,, .075</td><td>Masc. M .182
,, .155
.539 F. .196
.550 , .182
P. .189
,, .167</td><td>Masc. M. .061
,, .059
.618 F. .088
.600 ,, .096
P. .071</td></tr>
<tr><td>To Queensland.</td><td>Masc. M. .059
,, .063
.687 F. .126
.678 ,, .116
P. .080
,, .080</td><td>Masc. M. .085
,, .081
.649 F. .136
.642 ,, .130
P. .103
,, .098</td><td></td><td>Masc. M. .033
,, .109
.831 F. .160
.567 ,, .155
P. .054
,, .129</td><td>Masc. M. .106
,, .270
.610 F. .067
.525 ,, .263
P. .091
,, .267</td><td>Nil.</td></tr>
<tr><td>To S. Australia.</td><td>Masc. M. .095
,, .078
.601 F. .054
.588 ,, .128
P. .111
,, .098</td><td>Masc. M. .063
,, .052
.659 F. .113
.674 ,, .109
P. .080
,, .070</td><td>Masc. M. .018
,, .156
.829 F. .089
.427 ,, .070
P. .030
,, .107</td><td></td><td>Masc. M. .112
,, .106
.629 F. .189
.635 ,, .161
P. .140
,, .126</td><td>Masc. M. .024
,, .000
.971 F. .053
.348 ,, .000
P. .029
,, .000</td></tr>
<tr><td>To W. Australia.</td><td>Masc. M. .162
,, .143
.556 F. .193
.566 ,, .174
P. .176
,, .156</td><td>Masc. M. .178
,, .148
.543 F. .184
,, .170
P. .181
,, .157</td><td>Masc. M. .058
,, .103
.340 F. .121
.439 ,, .181
P. .100
,, .091</td><td>Masc. M. .120
,, .112
.607 F. .191
.620 ,, .158
P. .148
,, .130</td><td></td><td>Masc. M. .097
,, .417
.633 F. .056
.343 ,, .087
P. .082
,, .200</td></tr>
<tr><td>To Tasmania.</td><td>Masc. M. .080
,, .081
.483 F. .072
.494 ,, .087
P. .076
,, .084</td><td>Masc. M. .054
,, .053
.620 F .082
.612 ,, .096
P. .065
,, 069</td><td>Masc. M. .100
,, .111
.409 F. .100
,, .154
P. .100
,, .136</td><td>Nil.</td><td>Nil.</td><td></td></tr>
</table>

* The upper figures are for the year 1909, the lower for the year 1913. † The masculinity of the migration in the table is the ratio of males to persons. ‡ Based upon the departures from and arrivals in the States indicated.

6. **Periodic fluctuations in migration.**—Periodic fluctuations of migration are exhibited alike by oversea migration, by interstate migration by sea, and by migration overland. The following tables give the variations for the first and second for Australia. Table CXLI. shews also the monthly variations of the sex-ratio (or masculinity) of the migration. To express these results by Fourier series, see Part III., § 5, pp. 38-40, and also Part XI., § 16, pp. 169-171.

TABLE CLXI.—Shewing Oversea Migration into and from Australia during the period 1909-1913, and its Fluctuations for "Persons" during the Year. (For equalised months and a constant population).

State.	1 or E	Totals for 1909-13.			Jan.	Feb.	Mar.	April.	May.	June.	July.	Aug.	Sept.	Oct.	Nov.	Dec.
		Persons.	Males.	Females												
N.S.W.	1	337,997	231,694	106,303	.809	1.110	1.180	1.096	.989	.992	.792 m	.876	.920	.985	1.091	1.070
	E	232,056	161,666	70,390	1.002	1.101	1.316	1.227	1.101	.884	.848	.806	.761 m	.896	.976	1.082
Masc.	I	.6855	..	..	.697	.665	.698	.679	.708	.708	.723	.689	.694	.679	.659 m	.689
	E	.6967	..	..	.692	.670	.642 m	.680	.712	.707	.721	.735	.709	.716	.712	.705
Vict.	I	116,603	72,148	44,455	.883	1.040	.990	.914	1.105	.856	.756	.832	.994	1.024	1 234	1.372
	E	71,425	45,211	26,214	1.189	1.288	1.529	1.386	.974	.801	.766	.627	.602 m	.720	.877	1.241
Masc.	1	.6187	..	..	.632	.612	.593	.637	.652	.659	.645	.636	.628	.613	.594	.565 m
	E	.6330	..	..	.663	.620	.580 m	.601	.631	.652	.658	.689	.664	.658	.649	.626
Qld.	1	58,507	38,002	20,505	.850	.717 m	1.114	.743	.963	1.165	.855	.989	1.172	1.264	.938	1 230
	E	19,161	15,438	3,723	1.125	1.244	1.434	1.213	1.022	.801	.697 m	.709	.777	.895	1.062	.991
Masc.	I	.6495	..	..	.650	.670	.595	.677	.672	.688	.637	.628	.711	.626	.670	.592 m
	E	.8057	..	..	.890	.854	.840	.702 m	.748	.777	.775	.812	.841	.848	.863	.865
S. Aust.	I	33,496	22,890	10,606	.902	1.004	1.037	.918	1.038	.854	.840	.731 m	.816	1.050	1.796	1.014
	E	15,529	11,760	3,769	.939	1.305	1.699	1.480	1.084	.808	.719	.723	.684 m	.726	.880	.953
Masc.	1	.6834	..	..	.733	.713	.714	.721	.654	.652	.698	.762	.747	.708	.568 m	.659
	E	.7573	..	..	.833	.769	.680 m	.691	.761	.811	.791	.826	.789	.802	.744	.737
W. Aust.	1	67,168	45,532	21,636	1.389	1.169	.702	.847	1.059	.860	1.095	.594 m	.842	.811	1.556	1.076
	E	24,846	18,897	5,949	.932	1.058	1.436	1.298	1.198	.895	.796	.797	.641 m	.679	.897	1,373
Masc.	1	.6779	..	..	.715	.703	.746	.724	.682	.662	.654	.708	.612	.653	.610 m	.707
	E	.7606	..	..	.793	.750	.683 m	.709	.718	.758	.772	.801	.806	.788	.781	.838
Tas.	I	15,633	10,472	5,161	.841	1.105	1.279	1.169	1.029	1.036	1.002	.840	1.040	.833	.821 m	1.005
	E	7,121	4,475	2,646	1.518	1.582	1.732	1.427	.755	.639	.499 m	.561	.590	.952	.625	1.120
Masc.	I	.6698	..	..	.622 m	.641	.686	.705	.704	.696	.689	.671	.675	.814	.624	.648
	E	.6284	..	..	.685	.580	.553 m	.583	.612	.652	.628	.637	.671	.722	.607	.690

The quantity underlined is the greatest, and that marked *m* the least during the year.

The two upper figures in each section are the relative average magnitudes of the migration for the month, the monthly average for the year being unity.

The two lower figures are the migration-ratios for the corresponding months, viz., the ratio of the migrants to the population of the State.

In Table CLXII. hereunder the fluctuations of interstate migration by sea are shewn, and the "migration-ratios" are also shewn.

TABLE CLXII.—Shewing the Fluctuations for "Persons" in the Interstate Migration by Sea in Australia for the Period 1909-1913.

(For equalised months and a constant population and the migration ratios × 1,000,000.)

	State.	Migrants.	Jan.	Feb.	Mar.	April.	May.	June.	July.	Aug.	Sept.	Oct.	Nov.	Dec.
		Fluctuation Ratio (Total = 12.000) and Migration-ratios for Persons.												
From N.S.W. P = 8438184.	To—Victoria	137,916	1.545	1.254	1.045	1.157	.891	.663	.595	.641	.797	1.164	.965	1.283
		16,344	2,105	1,707	1,423	1,575	1,214	903	*m* 810	873	1,085	1,586	1,314	1,748
	Q'land	109,542	1.195	1.011	1.059	1.225	1.318	1.256	.952	.883	.782	.706	.711	.902
		12,982	1,293	1,094	1,146	1,325	1,425	1,359	1,030	955	846	*m* 764	769	976
	S. Aust.	20,788	1.243	1.194	1.485	1.325	1.096	.776	.622	.592	.744	.860	.879	1.184
		2,464	255	245	305	272	225	159	128	*m* 122	153	176	180	243
	W. Aust.	16,218	1.068	1.253	1.270	1.366	1.333	.809	.663	.636	.763	.858	.871	1.160
		1,922	170	200	202	218	212	129	109	*m* 101	122	137	139	185
	Tas.	33,517	1.981	1.858	1.200	.907	.498	.437	.460	.448	.661	.730	.827	1.983
		3,972	659	615	397	300	165	*m* 145	152	148	219	242	274	656
	N. Terr.	825	.83	.60	.97	.217	.128	.108	.102	.87	.59	.93	.73	.93
		98	7	5	8	18	10	9	8	7	*m* 5	8	6	8
	Total ..	318,806	1.425	1.228	1.106	1.178	1.032	.859	.710	.701	.772	.925	.852	1.212
		37,781	4,489	3,866	3,482	3,708	3,251	2,704	2,237	*m* 2,207	2,429	2,911	2,682	3,816
From Victoria. $P = 6.613587$.	N.S.W.	145,326	1.354	1.274	1.343	1.297	.958	.738	.629	.610	.743	.709	1.088	1.257
		21,973	2,479	2,333	2,458	2,376	1,755	1,352	1,151	*m* 1,117	1,361	1,298	1,992	2,302
	Q'land.	25,828	.935	.909	.987	1.187	1.467	1.515	1.272	1.031	.760	.606	.652	.679
		3,905	304	296	321	387	477	493	414	336	247	*m* 197	212	221
	S. Aust.	28,006	1.266	1.231	1.233	1.212	1.083	.787	.701	.631	.779	.775	1.050	1.252
		4,235	447	435	435	428	382	278	247	*m* 223	275	274	371	442
	W. Aus.	46,031	1.082	1.339	1.438	1.307	1.220	.926	.716	.729	.710	.703	.864	.965
		6,960	627	777	835	758	708	537	415	422	412	*m* 407	501	560
	Tas.	153.568	1.614	1.253	1.041	1.009	.683	.594	.653	.602	.748	.786	1.112	1.905
		23,220	3,126	2,425	2,013	1,952	1,322	*m* 1,149	1,264	1,164	1,447	1,520	2,151	3,686
	N. Terr.	156	.92	.10	1.46	.39	2.46	1.15	.77	.92	1.38	.31	1.60	.54
		24	2	*m* 1	3	1	5	2	1	2	3	.6	3	1
	Total	398,915	1.390	1.246	1.207	1.174	.925	.758	.695	.649	.745	.736	1.041	1.434
		60,317	6,986	6,266	6,066	5,901	4,649	3,810	3,493	*m* 3,264	3,745	3,697	5,230	7,211

TABLE CLXII.—Shewing the Fluctuations for "Persons" in the Interstate Migration by Sea in Australia for the period 1909-13—*continued.*

	State.	Migrants.	Jan.	Feb.	Mar.	April	May.	June.	July.	Aug.	Sept.	Oct.	Nov.	Dec.
		FLUCTUATION RATIO (TOTAL = 12.000) AND MIGRATION-RATIOS FOR PERSONS.												
From Queensland. $P = 3,060,025$	N.S.W.	106,280	1.140	.957	.976	.890	.811	.781	.788	.905	1.036	1.057	1.037	1.622
		34,731	3,300	2,769	2,825	2,578	2,347	*m* 2,261	2,282	2,619	2,998	3,058	3,001	4,693
	Vict.	19,664	1.088	.914	.818	.959	.721	.773	.785	1.039	1.131	1.205	1.071	1.496
		6,426	582	490	438	514	*m* 386	414	420	556	606	645	573	801
	S. Aust.	593	1.23	.69	.93	.92	1.78	.63	1.11	.97	.69	1.03	1.05	.97
		194	20	10	20	14	29	*m* 10	17	16	11	16	17	16
	W. Aust.	325	1.11	1.22	.85	2.14	1.85	.85	.52	.59	.92	.89	.29	.77
		116	10	11	8	19	16	8	5	5	8	8	*m* 3	7
	Tas.	62	0.0	4.8	4.7	2.5	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
		20	0	8	8	*m* 4								
	N. Terr.	529	.78	.93	.95	1.47	.89	1.09	.77	1.16	1.11	1.04	1.13	.67
		173	11	13	14	21	13	16	11	17	16	15	16	*m* 9
	Total	127,453	1.130	.952	.953	.908	.804	.780	.788	.926	1.049	1.078	1.040	1.592
		41,651	3,924	3,302	3,307	3,151	2,791	*m* 2,708	2,736	3,213	3,640	3,743	3,610	5,526
From S. Australia. $P = 2052589$.	N.S.W.	22,115	1.285	1.422	1.494	1.231	.956	.694	.591	.560	.772	.870	.855	1.270
		10,774	1,153	1,277	1,341	1,105	858	623	531	*m* 503	693	781	768	740
	Vict.	25,828	1.222	1.445	1.188	1.069	.770	.793	.673	.626	.749	1.105	1.027	1.333
		12,583	1,281	1,515	1,245	1,121	807	831	706	*m* 657	786	1,160	1,077	1,397
	Q'land.	1,019	.77	1.21	.84	1.07	1.17	1.25	.98	.84	1 18	1.03	.66	1.00
		496	32	50	35	44	48	52	40	35	49	43	*m* 27	42
	W. Aust.	43,341	1.128	1.275	1.317	1.185	1.049	.869	.800	.783	.786	.911	.896	1.001
		21,115	1,984	2,243	2,318	2,086	1,845	1,529	1,408	*m* 1,377	1,383	1,603	1,577	1,760
	Total	92.303	1.188	1.357	1.318	1.163	.950	.810	.717	.686	.777	.856	.920	1.158
		44,969	7,941	8,348	7,501	6,681	5,568	4,617	4,331	*m* 4,455	5,614	6,862	6,954	10,528
From W. Australia. $P = 1436533$	N.S.W.	20,370	1.292	1.350	1.232	.967	.884	.804	.685	.578	.830	.930	1.062	1.385
		14,180	1,527	1,595	1 456	1,142	1,045	949	809	*m* 684	981	1,100	1,255	1,637
	Vict.	45,690	1.283	1.310	1.135	.978	.828	.609	.519	.608	.796	1.077	1.093	1.763
		31,806	3,400	3,476	3,007	2,592	2,195	1,615	*m* 1,375	1,612	2,110	2,852	2,898	4,674
	Q'land.	531	1.08	.78	1.22	.93	1.45	1.22	.68	.86	.77	1.02	.95	1 04
		370	33	24	38	28	45	38	*m* 21	26	24	31	29	32
	S. Aust.	47,205	1.088	1.182	1.096	1.050	.833	729	.775	.778	.913	1.032	1.012	1.514
		32,860	2,980	3,236	3,000	2,877	2,281	*m* 1,996	2,122	2,125	2,499	2,827	2,772	4,145
	N. Terr.	266	0.0	1.17	0 0	.267	.13	1.22	27	.45	0.0	3.39	.00	2.71
		185	0	18	..	41	*m* 2	26	4	7	..	52	..	42
	Total	114,062	1.200	1.262	1.134	1.010	.841	.698	.655	.673	.848	1.037	1.051	1.581
		79,401	7,941	8,348	7,501	6,681	5,568	4,617	*m* 4,331	4,455	5,614	6,862	6,954	10,528

TABLE CLXII.—Shewing the Fluctuations for "Persons" in the Interstate Migration by Sea in Australia for the Period 1909-13—*continued.*

	State.	Fluctuation Ratio (Total = 12.000) and Migration-ratios for Persons.												
		Mi-grants.	Jan.	Feb.	Mar.	April.	May.	June.	July.	Aug.	Sept.	Oct.	Nov.	Dec.
From Tasmania P = 959218.	N.S.W.	37,786	1.566	1.474	1.482	1.442	1 018	1.044	.825	.553	.735	.520	.613	.728
		3,939	5,140	4,837	4,865	4,735	3,343	3,427	2,708	1,815	2,411	*m* 1,709	2,013	2,388
	Vict.	168.563	1.606	1.619	1.432	1.223	.821	.686	.693	.680	.709	.876	784	.871
		17,572	23,512	23,700	20,970	17,911	12,014	10,053	10,149	*m* 9,952	10,389	12,831	11,485	12,756
	S. Aus.	339	.57	.67	4.01	3.33	.81	.78	0.0	.07	.28	.71	.60	.17
		35	17	20	118	98	24	23	..	*m* 2	8	21	18	5
	W. Aust.	158	0.0	.46	3.49	6.68	1.06	.31	..	..	..	..	..	..
		16	0	6	48	92	15	*m* 4	..	..	..	..	..	..
	Total	206,846	1.595	1.590	1.447	1.271	.857	.752	.715	.655	.713	.810	.752	.843
		21,564	28,669	28,563	26,001	22,839	15,396	13,506	12,857	*m* 11,768	12,809	14,560	13,516	15,150
From N. Territory. P = 10338.	N.S.W.	412	.87	.64	.61	.58	.75	1.11	.93	.41	1.72	1.81	1.49	1.08
		39,800	2,900	2,100	2,000	1,900	2,500	3,700	3,100	*m* 1,300	5,700	6,000	4,900	3,600
	Vict.	210	.57	1.66	1.37	.51	.40	.40	.57	.91	.80	1.49	1.26	2.06
		20,300	9,700	2,800	2,300	8,700	*m* 6,800	6,800	9,700	1,500	1,300	2,500	2,100	3,500
	Q'land.	387	.75	.53	.44	.84	1.02	.93	1.12	1.39	.65	1.77	1.24	1.33
		37,400	2,300	1,600	*m* 1,300	2,600	3,200	2,900	3,500	4,400	2,000	5,500	3,900	4,200
	W. Aust.	161	0.0	1.57	0.0	1.49	1.49	.75	1.12	.30	.37	2.01	.52	2.38
		15,600	0	2,000	0	1,900	1,900	9,700	1,400	*m* 3,900	4,800	2,600	6,800	3,100
	Total	1,170	.66	.91	.61	.78	.88	.87	.95	.81	1.02	1.76	1.23	1.52
		113,200	6,200	8,600	*m* 5,700	7,400	8,300	8,200	9,000	7,600	9,600	1,600	1,200	1,400

The upper figures are the relative average magnitudes of the migration for the month, the monthly average for the year being unity. Those underlined are the maximum-values and those marked "*m*" the minimum values during the year.

The small figures are the number of migrants ("persons") per 1,000,000 population in the State *from* which the migration takes place.

That *Land Migration* also shews marked periodicity is evident from Table CLXIII. It is worthy of notice that the total immigration for a year is sensibly equal to the total emigration for the same period, though the want of balance for individual months may be considerable.

TABLE CLXIII.—Shewing the Periodic Fluctuation of Overland Migration (by Railway) for equalised months and a Constant Population. Australia, 1914-1916. ("Persons.")

Month.	Arrivals or Immigration.				Departures or Emigration.			
	N.S.W.	Vic.	Q'land.	S. Aust.	N.S.W.	Vic.	Q'land.	S. Aust.
January ..	1.1517	1.0855	1.5297M	1.1928	1.2452	1.0973	1.2142	1.2352
February ..	.9824	.9812	1.0387	.9737	1.0044	.9437	.9738	1.0580
March ..	1.1099	.9427	1.2059	.9944	1.0339	1.0848	1.1270	1.0250
April ..	1.2085	1.1153	1.3487	1.1649	1.1811	1,1745M	1.2167	1.2827M
May ..	.9584	.9353	1.0189	.9761	.9498	.9928	.8907	1.0278
June ..	.8034*m*	.8360	.7474	.8007 *m*	.7888 *m*	.8533 *m*	.7072	.8643
July ..	.8389	.8269 *m*	.7472	.8528	.7999	.8921	.6995 *m*	.8904
August ..	.8734	.8985	.8454	.9445	.8938	.9179	.8231	.8488 *m*
September ..	.9394	1.0494	.7266	.9602	.9432	1.0177	.8349	.8892
October ..	.8700	1.1441	.7610	.9047	1.0000	.9115	.8703	.8970
November ..	.9091	.9884	.7117 *m*	.9350	.8809	.9970	.8219	.8725
December ..	1.3549M	1.1967 M	1.3188	1.3002 M	1.2790 M	1.1174	1.8207 M	1.1091
Mean No. for equalised mnth. Aggr. Popn. ..	67,102	48,188	24,278	18,063	67,007	47,804	24,516	18,244

M denotes the maximum and *m* minimum value.

7. **Migration and Age.**—If the ages of migrants of each sex are recorded at the moment of entry into or exit from any community, it is possible to know continuously the constitution of the population according to sex and age, once a population Census has been taken. Results forwarded to the compiling authority only at long intervals require corrections, of the type referred to in Part XI., §§ 7-9, pp. 152-160. The deduction of ages is best effectuated by referring all the results to the one point of time, say the end of the calendar year.

8. **Defects in migration records and the closure of results.**—Notwithstanding that elaborate care was taken as regards the record of emigration, it has been found in Australia that errors occur therein of considerable magnitude. From the 1901 Census and the intercensal records up to the Census of 1911, it appeared that, if the discrepancy were attributed wholly to this source of error, it would amount, in the case of males, to 0.1459 of the whole recorded male migrants outward (departures) and in the case of females to 0.0995 of the whole recorded female migrants outward. A still more extraordinary result was that apparently the island-continent of Australia was rapidly losing females.[1]

Suppose that a statistical element E_0 is accurately ascertained at any point of time (*e.g.*, as at a Census) and after n years is again accurately ascertained and found to be E_n; and further that the intervening changes are $e_1, e_2, \ldots e_n$. Then :—

$$(661)\ldots E_n = E_0 + k(e_1 + e_2 + \ldots + e_n)\,; \quad \text{or } k = (E_n - E_0)/(e_1 + e_2 + \ldots + e_n)$$

The quantity k may be called the *coefficient of proportional linear adjustment*, and E_1, E_2, etc., may be found by the successive additions, viz., of ke_1, ke_2, etc., instead of the unadjusted change. We may, however, correct the results as indicated in (662), that is :—

$$(662)\ldots E_n = E_0 + e_1 + e_2 + \ldots + e_n + \epsilon = E_0 + (e_1 + \kappa) + (e_3 + \kappa) + \ldots + (e_n + \kappa).$$

in which last expression $\kappa = \epsilon/n$, the total defect of closure, ϵ, being divided equally among the changes. Thus in this case E_1, E_2, etc., may be found by successive additions, viz., of $e_1 + \kappa$, $e_2 + \kappa$, etc. This may be called *simple linear adjustment*. The question as to whether one or the other or either is legitimate, must always be decided by the nature of the case, and obviously no general rule can apply.

[1] Upon a change being made on the system as between State and State, such that the aggregate of the State-increments of population gave the increment of population to the Commonwealth, this peculiarity vanished.

XVII.—MISCELLANEOUS.

1. **General.**—It is proposed in this part to refer to a number of miscellaneous matters, which have not been included in previous parts, and which either do not fall under any particular heading, or have been omitted from earlier consideration.

2 **Subdivision of population and other groups.**—The values of group-subdivisions, which are obtained by dividing groups by the middle ordinate, are given earlier, see Part VI., § 4, pp. 80-81. These formulæ are not always applicable. Two questions often arise, viz. (i.) the value of the subdivisions or (ii.) of their ratios to each other.

Considering first the subdivision of a group g into two parts, let it be supposed that the function, representing a series of groups, viz., $g_{-4}, \ldots . g_{-1}, g, g_1 \ldots . g_4$, is $a+bx+cx^2+$ etc., then we shall have[1] :—

(663)......

$$g_{-\frac{1}{2}} = \tfrac{1}{2}g - \tfrac{1}{256}[61(g_1-g_{-1}) - 44(g_2-g_{-2}) + 19(g_3-g_{-3}) - 3\tfrac{1}{2}(g_4-g_{-4}) + \text{etc.}]$$

$g_{-\frac{1}{2}}$ denoting the portion of the group g on the negative side of the middle ordinate of that group. This formula is in general suitable about maxima and minima values, but *may*, of course, be inappropriate if $g_k - g_{-k}$ increase more rapidly than the coefficients diminish. It may often be employed, however, when *pairs* of terms in the square brackets are sharply convergent.

Another process of arriving at values for the subdivision of groups into halves is the following :—Let the values of the successive groups be C, B, A, M, A', B' and C', and M, the group to be divided. Then the portion next to A is[2] :—

$$(664)..\ M' = \tfrac{1}{2}M - \tfrac{1}{2048}[201(A'-A) - 44(B'-B) + 5(C'-C) - ..]$$

which in many cases gives substantially the same result as (663), though it is not an identical formula, and apparently might be regarded as not in agreement therewith.

---

[1] This is deduced by finding, in terms of the groups themselves, the values of the constants a, b, etc., of the curve : and then integrating between the limits which give the first half of the group to be subdivided.

[2] This is easily derived from the usual formula for interpolation into the middle, viz., $F(\tfrac{1}{2}) = F + \tfrac{1}{2}a' - \tfrac{1}{8}b_0 + \tfrac{3}{128}d_0 - \tfrac{5}{1024}f_0 +$ etc., by regarding the aggregates $C, C+B, C+B+A, C+B+A+M$, etc., as successive totals represented by ordinates represented by $\alpha + \beta x + \gamma x^2 +$ etc. ; finding the value to the middle ordinate of group M and subtracting $C+B+A$.

In the case of groups rapidly increasing or rapidly diminishing in amount—as for example the numbers dying at the beginning of life in 0–1, 1 to 2, etc., days, weeks, months or years, the following method of subdivision may be followed :—

Let it be required to divide each of a series of larger groups A, B, C, etc , for equal limits of a variable into s smaller groups, viz., $a_1, a_2, \ldots a_s$; $b_1, \ldots b_s$: $c_1, \ldots c_s$; etc., and suppose that $B=mA$; $C=m'B=mm'A$: etc. Then if $m'=m$, etc., it is obvious that the successive values of the smaller groups will be :—

$$(665)..(a_1 + na_1 + n^2a_1 + \ldots) + (b_1 + nb_1 + n^2b_1 + \ldots) + \text{etc.} = (a_1 + \ldots + n^{s-1}a_1) + (n^s a_1 + \ldots + n^{2s-1}a_1) + (n^{2s}a_1 + \ldots n^{3s-1}a_1)$$

in which n is the sth root of m and m'. The brackets shew the groups, the sum of which give the original values A, B, C, etc. Since from each of any three adjoining groups an equal quantity Q may be cut off or added, so that the altered values A', B', C' will be A', $m_0 A'$, $m_0^2 A'$, we can constitute the group-divisions by adding a common value Q/s to each of a series of quantities of the type of (665) above, n_0 in this case being the sth root of m_0. Hence we have :—

$$(666)\ldots\ldots a\,(1 + n + n^2 + \ldots\, n^{s-1}) = A' = A - Q\,;$$

from which, since n is known, a can consequently be determined, and the series a, na, n^2a, etc., to which, if a common quantity $q=Q/s$ is added we obtain a_1, a_2, etc. Thus :—

$$(667)\ldots\ldots a_1 = A'/(1 + n + n^2 + \ldots + n^{s-1}) + q = a + q\,;$$

$$a_2 = na + q\,; \quad a_3 = n^2a + q\,;\ \text{etc.}$$

In applying this method practically, any group may be subdivided by treating it as B, and dividing it according to the indications of the groups on either side A and C.

3. **The measure of precision in statistical results.**—Statistical results, expressed without regard to their possible or probable error, often suggest the attainment of a precision far beyond that which the data can furnish. For example, if the ratio of the survivors after one year be given (as in life-tables) to 5 decimals, the results *imply* for Australian data an average precision of *age* for the first year of 1.1 hour, or at its terminal of 0.4 hour. For other countries it will be much the same. Again, in the case of the instantaneous rate of mortality at the end of the first year, the expression to 5 places of decimals *implies* a precision, in the *time* or epoch to which it may be deemed to apply, of 8 days. In both cases the apparent precision is illusory,[1] forasmuch as the recording of the facts and their actual

[1] See Census Report, Vol. III., p. 1215, and also p. 1212.

variableness does not conform to this order of precision. For example, births and deaths are not recorded as regards age to 0.4 hour per annum even on the average: nor can the point of time to which they may appropriately be referred be deemed to be ascertained to 8 days or its equivalent in a decade. Actuarial tables are often carried to 7 places of decimals. A unit in the last place is (on the average) for ages 1 to 2 about equivalent to an age-difference of 2 minutes, and, owing to the diminution of death-rate with the lapse of time, also to about the same as to the point of time to which the result is presumed to apply.

Let u and y denote respectively functions of time (t) and of age (x), then if :—

$$(668)\ldots\Delta u = Idt\,;\ \Delta y = J\Delta x\,;\ \text{or } I = du/dt\,;\ J = dy/dx$$

in the limit, I and J are the *ratios of relative importance*—as compared with the units of u and y—of precision in the units respectively of t and x. These ratios serve as guides in fixing the relative accuracy required in the data giving the two co-ordinates. If in graphing results, the units on the axis of abscissæ are, respectively, I and J times the units on the axis of ordinates, then the curve will make an angle of 45° with either axis, and this, in so far as it is practicable to follow it, is the best scale-relation between ordinate and abscissa for any graph intended to be used for analysis.

The life-tables published in connection with the Australian Census of 3rd April, 1911, follow the usual tradition as regards the number of figures to which the results are expressed. It is not, however, implied that the precision indicated is realised, they merely are followed for the sake of consistency in the results.

By suitable combinations of arithmetical and graphical methods results can be obtained to any required degree of practical precision.[1]

4. **Indirect relations.**—It is often necessary to establish statistical relations which really depend upon some intermediary statistical relation. For example, the average number of children born to an individual, or "average issue" may be related to age of "mothers," and such a relation would, of course, be a direct one. For certain purposes, however, (*e.g.*, social insurance) the average issue may be required as related to the age of fathers. The later relation, though physically indirect, is a regular and important one. Nevertheless, it is one which may be deduced by means of certain data from direct relations; at the same time it is not preferable to obtain it in this way. The relations according to "wives" and "husbands" are both given immediately by the Census, and the relation so given is, in general, to be preferred to the deduced relation: see Fig. 107.

[1] If the value of I or J is not between the limits $\frac{1}{4}$ to 4, the natural scale for both co-ordinates is not ordinarily satisfactory in graphing a function; however, the mode of variation of the greater co-ordinates will assist in the determination of a truly smoothed curve.

Fertility Curves.

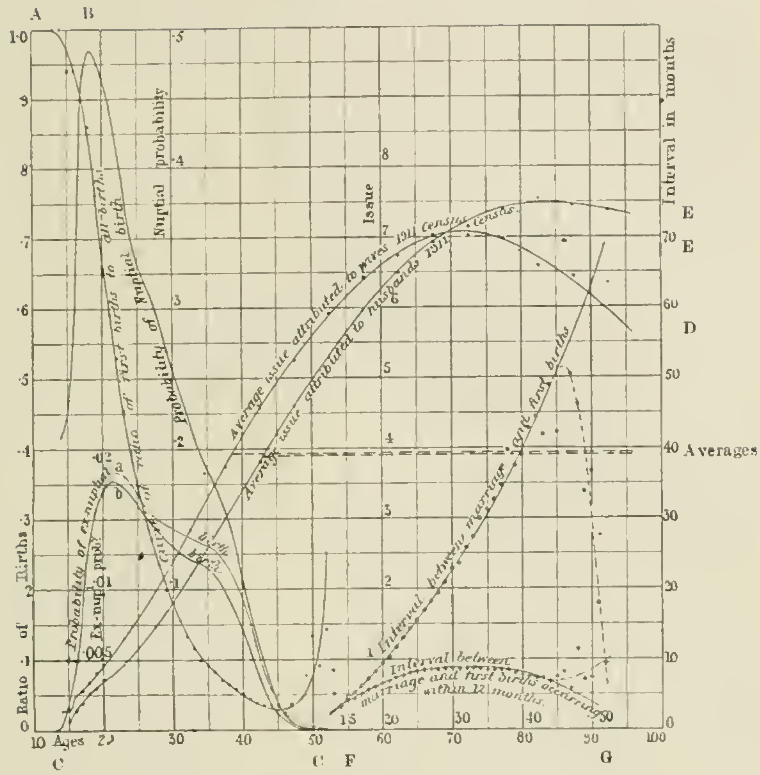

Fig. 107.

Curve A shews the ratio, according to age, of first births to all births.
Curve B shews the probability, according to age, of a nuptial birth ; see also p. 242 and p. 243.
Curve Ca shews the probability, according to age, of an ex-nuptial birth on the assumption (i.) that they are attributable wholly to the never-married.
Curve Cb shews the probability, according to age, of an ex-nuptial birth on the assumption (ii.) that they are attributable equally to the never-married, widowed, and divorced.
Curve D shews the average issue, according to age, of wives at the Census of 1911.
Curve E shews the average issue, according to age, as related to husbands at the Census of 1911.
Curve FE′ shews the average interval, according to age, between marriage and first-births.
Curve FG shews the average interval, according to age, between marriage and first-births, occurring within 1 year of marriage.

5. **Limits of uncertainty.**—The limits of an uncertainty in any deduced quantity may be due to possible errors in the numbers upon which it is founded, or upon an uncertainty as to the particular quantity which should be employed. The first cause of uncertainty is sufficiently illustrated by the ratio of, say, first births to all births : for prediction purposes the smoothed numbers in Table CLXIV. are really more probable than the crude numbers : see Fig. 107.

The second cause of uncertainty is illustrated in the following example :—If the "never married," the "widowed" and the "divorced" are regarded as a homogeneous class, the probability of a case of ex-nuptial maternity during one year is found by dividing the number of births in one year by the sum of the average numbers in the three

classes. If, however, they are not homogeneous as regards this probability, a more accurate result might be obtained by dividing by the never married. The general probability must lie between the two results : see the curves marked Ca and Cb on Fig 107, and the results in columns marked I. and II. respectively in Table CLXIV.

It may be noted that the characteristics of a variation may be wholly changed by restriction within limits. This is seen by taking the interval according to age between marriage and a first birth, when the consideration is restricted to the lapse of 12 months, or is indefinite : see the curves FG and FE′ respectively.

TABLE CLXIV.—Shewing Rates of First to All Births, and Probabilities of Ex-nuptial Maternity. Australia, 1907-14.

Age.		Ratio of First to all Births.		Probability Ex-nuptial Maternity based on assumption		Age.		Ratio of First to all Births.		Probability Ex-nuptial Maternity based on assumption	
		Crude.	Smoothed.	I	II.			Crude.	Smoothed.	I.	II.
12	..	..	1.0000	.000015	.000015						
13	..	1.0000	.9970	.000062	.000062	34	..	.0994	.1040	.01325	.0118
14	..	1.0000	.9930	.00037	.00037	35	..	.0923	.0920	.0130	.0115
15	..	.9404	.9715	.0016	.0015	36	..	.0817	.0825	.0127	.0110
16	..	.9407	.9430	.0042	.0042	37	..	.0703	.0730	.0122	.0103
17	..	.9130	.9035	.0085	.0085	38	..	.0640	.0640	.0115	.0094
18	..	.8602	.8450	.0131	.0131	39	..	.0583	.0560	.01045	.0084
19	..	.7627	.7627	.0157	.0162	40	..	.0524	.0485	.0095	.0072
20	..	.6594	.66[illegible]4	.0172	.0173	41	..	.0437	.0425	.0076	.0058
21	..	.5912	.5912	.0180	.0179	42	..	.0381	.0370	.0059	.0044
22	..	.5285	.5170	.01835	.0181	43	..	.0352	.0338	.0043	.0030
23	..	.4534	.4485	.0181	.0179	44	..	.0351	.0310	.0030	.0018
24	..	.3[illegible]60	.3960	.0176	.0174	45	..	.0349	.0285	.0020	.00123
25	..	.3482	.3482	.0169	.0161	46	..	.0244	.0295	.0012	.00080
26	..	.3098	.3080	.0160	.0154	47	..	.0360	.0338	.00085	.00040
27	..	.2722	2710	.0154	.0147	48	..	.0255	.0428	.00055	.00022
28	..	.2352	.2370	.0149	.0141	49	..	.0769	.0555	.00035	.00013
29	..	.1[illegible]87	.2065	.0146	.0135	50	..	.1333	.0790	.00022	.00009
30	..	.1800	.1795	.0143	.0130	51	..	.0909	.1230	.00012	.00006
31	..	.1555	.1560	.0140	.0126	52	..	.1429	.2500	.00004	.00004
32	..	.1324	.1355	.0137	.0123	53	..	..	..	.00002	.00002
33	..	.1160	.1182	.0135	.0120	54	..	..	..	.00001	..
34	..	.0994	.1040	.01325	.0118	55	..	..	..	..	..

I. denotes the ratio of births to the never-married ; II., the ratio of births to the aggregate of the never-married, widowed and divorced.

6. **The theory of "happenings" or "occurrence frequencies."**—In order to establish a rational theory of, and to completely interpret, the frequency curves met with in the various elements of the statistics of population, a theory of the frequency of occurrences of various kinds is a first requisite, and the type-forms of distribution established by Prof. K. Pearson and his co-workers are a contribution thereto, based upon the application of the theory of probability, plus certain empirical assumptions by means of which assymetrical forms of various kinds are deduced. Recently a foundation has been laid of a perfectly general theory of the frequency of occurrences, by Prof. Sir Ronald Ross. This latter seems to have had its birth in an attempt made in 1866 by Dr. Farr to develop a definite theory of an epidemic (cattle plague)[1]. In 1873-5 Dr. G. H.

[1] Dr. William Farr, "On the Cattle Plague," Journ. Soc. Sci., 20th Mar., 1866.

Evans endeavoured to extend Farr's theory to other epidemics.[1] The subject was again reopened by Dr. J. Brownlee[2] in a series of very significant contributions, and later, by Ross. Quite recently the last-named has put forward a definite theory, the fundamental elements of which are outlined in this section.[3] Although the main object was initially the determination of a basis for a theory of epidemics, the results are entirely general, and may be called the theory of "occurrences" or "happenings."

The differential equation of independent occurrences, reduced to its simplest expression, may be deduced as follows :—

Suppose a population P to consist of two parts. viz., A a part which is unaffected, and Q a part which[4] is affected[4], by any "happening,"[5] so that $P = A + Q$. Suppose also that some portion, viz., hdt, of the unaffected part becomes affected in the time dt, and also that a portion rdt of the affected part reverts to the unaffected part in the same element of time, so that the element of increase of the affected part is $(h - r)\,dt$; and finally let bdt, mdt, idt and edt denote in the unaffected part, the rates of *birth*, *death* (or mortality), *immigration* and *emigration* respectively; and Bdt, Mdt, Idt and Edt denote the similar rates in the affected part. Obviously therefore :—

$$(669)\ldots dP = A\,(b - m + i - e)dt + Q(B - M + I - E)dt = (Av + Q\,V)dt\,;$$

$$(670)\ldots dA = A(b - m + i - e - h)dt + Q(B + r)dt = \{A(v - h) + Q(B + r)\}\,dt\,;$$

$$(671)\ldots dQ = Ahdt + Q(-M + I - E - r)dt = \{Ah + Q\,(V - B - r)\}\,dt\,;$$

[1] Dr. G. H. Evans, "Some arithmetical considerations on the progress of epidemics," Trans. Epidemiol. Soc. London, Vol. 3, Pt. III., p. 551, 1873-5.

[2] Dr. J. Brownlee (i.) Theory of an Epidemic," Proc. Roy. Soc. Edin., Vol. 26, Pt. IV., p. 484, 1906; (ii.) "Certain considerations on the causation and course of epidemics," Proc. Roy. Soc. Med., Lond., June 1909; (iii.) "The mathematical theory of random migration and epidemic distribution," Proc. Roy. Soc., Edin., Vol. 31, Pt. II., p. 261, 1910; (iv.) "Periodicity in infectious disease," Proc. Roy. Phil. Soc., Glasgow, 1914; (v.) "Investigations into the theory of infectious diseases, etc., Public Health, Lond., Vol. 28, No. 6, 1915; (vi.) "On the curve of the epidemic," Brit. Med. Journ., May 8, 1915.

[3] Lieut.-Col. Prof. Sir Ronald Ross. (i.) "The logical basis of the sanitary policy of mosquito reduction." Cong. Arts and Sci., St. Louis, U.S.A., Vol. 6, p. 89, 1904, and Brit. Med. Journ., May 13, 1905; (ii.) The prevention of malaria in Mauritius," Waterlow and Sons, Lond., 1908, p. 29-40; (iii.) The prevention of malaria, J. Murray, Lond., 1910; 2nd Edit., 1911; Addendum on "the theory of happenings," 1911; (iv.) Some quantitative studies in epidemiology, Nature, Lond., Oct. 5, 1911; (v.) "Some *a priori* pathometric equations," Brit. Med. Journ., Mar. 27, 1915; (vi.) "An application of the theory of probabilities to the study of *a priori* pathometry"; Proc. Roy. Soc., Lond., Vol. 92, ser. A., July 14, 1915, pp. 204-230. See also H. Waite, "Mosquitoes and Malaria," Biometrika, Lond., Oct. 1910, Vol. 7, No. 4, p. 421.

[4] The affection may be of any nature, such as a disease, etc., and the supposition is quite general.

[5] The "happening" is the becoming affected, and is equally general with the preceding supposition.

and writing v and V for the algebraic sum of the quantities in the brackets in (669), the final forms of the preceding equations are given as is necessary of course, $dP = dA + dQ$. It may be noted that only dA and dQ contain terms representing the *happening* (h) *and reverting elements* (r), and that $QBdt$ appears in (670) but not in (671), because, in general at least, the progeny of the affected part are not affected at the instant of birth. Although the *variation elements* b, m, i, e and B, M, I, E will, if long periods are considered, generally be functions of time, they *may be regarded as constant* when short periods only are under review. Consequently for elementary cases mean values may be taken without sensible error,[1] similarly in regard to the *reverting element*.[2]

The most important element is the *happening element*, h, which it is to be clearly understood ordinarily falls on both groups (A and Q) alike. Should, however, it fall upon individuals already affected, it merely reaffects them and does not cause them to pass from one group to the other. Hence, though the total number of "happenings" is $P.hdt = (A+Q)\,hdt$, the number $Qhdt$ are already affected and must not be taken account of. The actual number of *new cases* Gdt, say, is thus only $Ahdt$. Thus :—

$$(672)\ldots\ldots Gdt/Phdt = \frac{A}{P}\ ;\ \text{or } G = hA = h\,(P - Q)$$

"Happenings" may be divided into two classes, viz. :—(a) those in which the frequency of the happening is *independent* of—and (b) those on which is *dependent* upon—the number of individuals already affected.[3] In independent happenings h and G are constants, in dependent happenings they are functions of Q.

---

[1] If, as is often the case, the "happenings" have no effect on the birth, death and migration rates, then we may have $b = B$, $m = M$, $i = I$, $e = E$, and consequently $v = V$, which may also occur fortuitously though the several terms differ. In general b is less than B in marriages, m than M in accidents, while in certain alarming epidemics (*e.g.*, cholera, plague, malaria) i is greater than I, and e less than E, in which case v is greater than V. In fatal accidents $M = 1$, and B, I and E are all 0, which value may also be assigned when considering happening among the *same* individuals. If a surrounding population be not affected $I = 0$; if affected individuals cannot move $E = 0$.

[2] In the case of "independent happenings"—see later, rdt denotes merely the proportion of affected individuals who may become reaffected in the time dt. (*e.g.*, by divorce in marriage). In "dependent happenings" it implies loss of capacity for affecting others (*e.g.*, in infectious disease it implies both immunity and loss of infectivity). In some diseases r may be zero (*e.g.*, leprosy and organic diseases, fatal accidents); it may be of small value (*e.g.*, many zymotic diseases); it may be of high value (*e.g.*, snake-bite, heat-stroke, etc.), and it may be unity (*e.g.*, slight accidents).

[3] To the former belong cases which are attributable to what may be called external causes (*e.g.*, accidents, non-infectious diseases, etc.); to the latter belong all cases attributable to propagation from individual to individual (*e.g.*, infectious diseases, etc.).

In *independent happenings*, therefore, the happening falls upon the same proportion (hdt) of the population in every element of time. Put $x = Q/P$ and $P - Q$ for A, then equations (669) and (671) give :—

$$(673)\ldots\ldots dP/dt = vP - (v - V)\,xP$$

$$(674)\ldots\ldots d\,(xP)/dt = hP\,(1 - x) + (V - B - r)\,xP$$

and by differentiation :—

$$(675)\ldots\ldots d\,(xP)/dt = xdP/dt + Pdx/dt.$$

From these three last equations, we have after dividing by P, and eliminating $d\,(xP)/dt$ and dP/dt :—

$$(676)\ldots\ldots dx/dt = h - (h + v - V + B + r)\,x + (v - V)\,x^2$$

which gives one form of integral if $v - V = 0$, and a different one if v and V are unequal.

When the sum of the variation elements of the affected group is constant the case may be called the *equivariant case*, the total population is unaltered.[1] Putting :—

$$(677)\ldots\ldots K = h + B + r\,;\ L = h/K\,;\ y = L - x\,;\quad \text{hence}$$

$$(678)\ldots dx/dt = -\,dy/dt = K\,(L - x) = Ky\,;\ dy/y = -\,Kdt\,;$$

which gives on integrating :—

$$(679)\ldots\ldots \log y = -\,Kt + C, \text{ or } y = y_0\,e^{-Kt},$$

y_0 being the value of y at the beginning of the "happening." Consequently, since $y_0 = L - x_0$:—

$$(680)\ldots\ldots x = L - (L - x_0)\,e^{-Kt}$$

viz., the proportion of the total population affected at the time t, the proportion being x_0 when $t = 0$.[2]

When v is not equal to V, we have the *general case of independent happenings* which involves the integration of (676). This may be written in the form :—

$$(681)\ldots dx/dt = K\,(L - x)\,L' - x) \equiv K\,(\alpha - \beta - x)\,(\alpha + \beta - x)$$

[1] An example would be the occurrence of slight accidents in which case $r = 1$, or the attainment of a certain standard of wealth tending to diminish simultaneously the birth, death, and migration rates of the affected by an equal decrement. If the progeny of the affected are also affected B should be omitted from (670), and inserted in (671), and will disappear in (674) and (676).

[2] Obviously in (673) if $v - V = 0$, a differential equation of the same form as (678) is obtained, hence $P = P_0 e^{vt}$, formula (2), p. 10 herein.

in which $a = (h + B + r + K)/2K$ and $\beta = \sqrt{(a^2 - h/K)}$, the roots $L = a - \beta$ and $L' = a + \beta$, being always real and positive when $v > V$. This gives :—

$$(682)\ldots x = L - (L' - L)\ (L - x_0) / \{(L' - x_0)\ e^{2K\beta t} - (L - x_0)\}$$

which simplifies slightly if $x_0 = 0$. The relative number of the affected depends upon whether K, that is whether $v - V$ is positive or negative. the former being usually the case in injurious happenings and the latter the case in beneficial ones. This gives :—

$$(683)\ldots P = P_0 e^{vt}.\ Le^{-KLt}/(L' - L)\ ;\ \text{or}\ P_0 e^{vt}.\ -Le^{-KL't}/(L - L)$$

the former expression being appropriate when K is positive, the latter when it is negative.

Among *dependent happenings* the case of *proportional happenings* is important as a first approximation to the study of the infection of a community. In this instance h is a function of Q and consequently of t. If each affected individual affects c others in a unit of time the total happenings in the time dt will be $cQdt$. The number of new cases per element of time may be taken as probably :—

$$(684)\ldots\ldots Gdt/cQdt = A/P\ ;\ \text{or}\ G = cQ\ (1 - x)\ ;\ h = cx.$$

This gives :—

$$(685)\ldots dx/dt = Kx(L - x),\ \text{in which}\ K = c - v + V;\ L = 1 - (B + r)/K,$$

from which may be obtained :—

$$(686)\ldots\ldots x = L / \{1 + (L/x_0 - 1)\ e^{-KLt}\}.$$

This gives regular bell-shaped curves : x_0 and Q_0 can never be zero.

Sufficient has been indicated to shew the value and reach of Prof. Ross's analysis of the question, and to render evident the fact that it is the foundation of a rational theory of "occurrences" of any kind, which can be numerically defined, in a population.

7. **Actual statistical curves do not coincide with elementary type forms.**—The importance of a rational theory of "happenings" does not consist in the fact that the curves deduced from elementary suppositions, meticulously correspond to actual statistical frequencies, but in the fact that deduced types give the general configuration. Since in actual cases what may be called the *frequency of initiation* is variable, the deduced forms of frequency at any given moment are only partially applicable to actual cases. Moreover any assymetrical and polymorphic curve, and indeed even any regular curve, can be built up in an infinite number of ways. The dissection of a curve into additive components is therefore,

in general, purely empirical. Although this is so, when extra-mathematical reasons exist for the acceptance of an hypothesis of constituent elements, whose origins, and general characters, are known, it may be possible to effect an analysis into components which yields a real and not merely a formal interpretation.[1]

In general, type-curves, the interpretation of which is impossible and is ignored outside certain selected points (*e.g.*, the points where they meet the axis of abscissæ) are logically unsatisfactory.

The function of a "theory of happenings" and of the "theory of probability," is therefore one of guidance in interpretation, and of deciding as to the applicability or otherwise of particular types of mathematical expression for the representation of the change of frequency with change of the variable. Mere arithmetical tests of the "goodness of fit" of particular mathematical expressions are significant or otherwise according as they conform to what is known *a priori*, or is deducible from *a priori* considerations, and these must certainly be taken in conjunction with the observations over the whole range of experience.[2]

8. **International norm-graphs and type-curves.** — The function served by the creation of norms has been indicated in Part VIII., § 6, p. 102. When norms for every important population-character have been computed, it is desirable that they should be *graphed and used internationally.* This could be done by printing squared graphed paper, with the norm shewn thereon, say in pale colour (or by a very fine line). The graphing of the same character on such paper for any particular population, would then immediately disclose the nature of its deviation from the normal. In this way the population phenomena could be graphically studied in their comparative relationships.

An extension of the system would be for each country to shew by pale tint not only the international norm, but also its own norm for (say) the previous decade.

Type-curves for international use would also greatly assist in the work of a better technical reduction of statistical results. The forms desirable or necessary would doubtless be more readily recognised when the international norms had been obtained.

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[1] For example in the harmonic analysis of tides, the forms and periods of the components are determined by celestial positions (*i.e.*, of the sun, moon, "anti-sun," "anti-moon," etc.), and the elements to be ascertained are merely the epoch of each component and its amplitude.

[2] For example, to systematically vary the representation of facts in order to agree with some adopted mathematical expression to which it is thought they *ought* to conform, is only to delude oneself. The character of terminal conditions is often known *a priori*, and the mathematical expression representing the facts should not be merely one in substantial arithmetical agreement with the frequency, but one which expresses as accurately as may be the law of its change. Similarly, the adoption of an expression which disturbs the observed critical values of the frequency, vitiates the results.

9. **Derivative elements from population-theory.**—It is beyond the purpose of this monograph to discuss the various derivative branches of the theory of population; such, for example, as the estimation from probate-records of the aggregate of private wealth; of the economic value of an average man or woman; of the economic value of different classes of persons; the cost of, and economic value of, education, etc. The present increasing length of life tends to give a higher average economic value—other things being equal—to an individual: the average wealth possessed per individual is probably also increasing.

Although all that relates to population may, in a comprehensive view, be regarded as belonging to its theory, it is quite appropriate that purely economic questions should be separated out. Therefore, while results obtained by means of the development of the population-theory are essential and are of the first order of importance, in any attempt, for example, to reach decisions as to the economic aspects of population, the questions that arise are so extensive that they must be treated independently. Nevertheless, the value of a suitably developed theory of population is not seen until it is viewed in the light of all its applications among which the economic is but one.

Similar observations apply to the anthropometric elements of the population. These are probably correlated with elements treated hereinbefore: nevertheless, it is preferable to deal with them independently.

10. **Tables for facilitating statistical computations.**—Mathematical tables of various kinds have been prepared for facilitating statistical computations, among which may be specially mentioned "Tables for Statisticians and Biometricians," by Prof. Karl Pearson, F.R.S., etc. In this monograph the following tables are solely for facilitating the computation or illustrating the mode of deducing quantities which enable required quantities to be found by inspection:—

Tables	I.	VI.	XVII. & XVIII.	XXXVI. & XXXVII.	XL.	LXV.	LXVI.
Pages	20	77	123	159	163	217	219-220

Tables	LXVII.	LXXV.	LXXXI.	CXLIV.
Pages	221-222	247	266	398

11. **Statistical integrations and general formulae.**—Reference has already been made in Part VI., §§ 6-8, pp. 82-84, to statistical integrations, and references were given to various tables, see p. 82. The integration of functions of a single variable is the subject of one of the Cambridge Tracts in Mathematics and Mathematical Physics, No. 2. This and the works previously mentioned will enable most integrations occurring in practical cases to be effected. For convenience the following are given:—

Table of Integrals and Limits.

$$\int dx = x = 2\arctan\sqrt{\left(\frac{1-\cos x}{1+\cos x}\right)} : \quad \int(ax \pm b)\,dx = \frac{1}{2}(ax^2 \pm 2bx) = \frac{1}{2a}(ax \pm b)^2 ;\ C = \frac{b^2}{2a} :$$

$$\int\frac{dx}{x} = \log x = \log\left(\frac{x}{a} + \frac{x}{b}\right) : \int\frac{dx}{ax \pm b} = \frac{1}{a}\log(ax \pm b) : \int\frac{mx+n}{ax+b}dx = \frac{m}{a}x + \frac{an-bm}{a^2}\log(ax+b) :$$

$$\int\frac{dx}{ax^2+bx} = \frac{1}{b}\log\frac{x}{ax+b} : \quad \int\frac{dx}{(x+a)(x+b)} = \frac{1}{b-a}\log\frac{x+a}{x+b} : \quad \int\frac{dx}{ax^2+bx+c} = \int\frac{dx}{\phi},\ \text{put } \beta = b^2-4ac,$$

$$\text{then } \int\frac{dx}{\phi} = \frac{1}{\sqrt{\beta}}\log\frac{2ax+b-\sqrt{\beta}}{2ax+b+\sqrt{\beta}} = \frac{2}{\sqrt{\beta}}\log\frac{2ax+b-\sqrt{\beta}}{\sqrt{\phi}} = \frac{2}{\sqrt{-\beta}}\arctan\frac{2ax+b}{\sqrt{-\beta}} = \frac{1}{\sqrt{-\beta}}\arcsin\frac{(2ax+b)\sqrt{-\beta}}{2a\phi}$$

$$= \frac{2}{\sqrt{-\beta}}\arcsin\frac{2ax+b}{2\sqrt{(a\phi)}},\ = -\frac{2}{2ax+b} \text{ when } \beta = 0 : \qquad \int\frac{x\,dx}{ax^2+bx+c} = \int\frac{x\,dx}{\phi} = \frac{1}{2a}\log\phi - \frac{b}{2a}\int\frac{dx}{\phi} =$$

$$= \frac{1}{2a}\log\phi - \frac{b}{2a\sqrt{\beta}}\log\frac{2ax+b-\sqrt{\beta}}{2ax+b+\sqrt{\beta}} = \frac{1}{2a}\log\phi - \frac{b}{a\sqrt{-\beta}}\arctan\frac{2ax+b}{\sqrt{-\beta}} = \frac{b}{a(2ax+b)} + \frac{1}{a}\log(2ax+b),$$

(when $\beta = 0$;

$$\int\frac{mx^2+nx+p}{ax+b}dx = \frac{mx^2}{2a} + \frac{an-bm}{a^2} + \frac{a^2p-abn+b^2m}{a^3}\log(ax+b) : \quad \int x^2dx = \tfrac{1}{3}x^3 = \tfrac{1}{3}(x^{\frac{3}{2}}+a^{\frac{3}{2}})(x^{\frac{3}{2}}-a^{\frac{3}{2}}).$$

$$\int x^3dx = \tfrac{1}{4}x^4 = \tfrac{1}{4}\left(x^2+ax+\frac{a^2}{2}\right)\left(x^2-ax+\frac{a^2}{2}\right), \text{ etc.} : \qquad \int\frac{dx}{x(ax^2+bx+c)} = \int\frac{dx}{x\phi};\ \text{put } \beta = b^2-4ac$$

$$\int\frac{dx}{x\phi} = \frac{1}{2c}\log\frac{x^2}{\phi} - \frac{b}{2c}\int\frac{dx}{\phi},\ \text{see above} : \quad \int\frac{dx}{x^2\phi} = \frac{1}{cx} - \frac{b}{2c^2}\log\frac{x^2}{\phi} + \frac{b^2-2ac}{2c^2}\int\frac{dx}{\phi},\ \text{ see above} :$$

$$\int x^m dx = \frac{x^{m+1}}{m+1} = \frac{ax^{m+1}+b}{a(m+1)} : \quad \int\frac{dx}{x^m} = -\frac{1}{(m-1)x^{m-1}} = -\frac{a+bx^{m-1}}{(m-1)ax^{m-1}} :$$

$$\int x^{-(1+\epsilon)}dx = \frac{x^{-\epsilon}-1}{-\epsilon} = \frac{e^{-\epsilon\log x}-1}{-\epsilon} = \log x - \frac{1}{2!}\epsilon(\log x)^2 + \frac{1}{3!}\epsilon^2(\log x)^3 - \frac{1}{4!}\epsilon^3(\log x)^4 + \ldots = \log x \text{ for } \epsilon = 0$$

$$\int\sin(ax+b)\,dx = -\frac{1}{a}\cos(ax+b) : \quad \int\cos(ax+b)\,dx = \frac{1}{a}\sin(ax+b) : \int\tan x\,dx = -\log\cos x :$$

$$\int\tan(ax+b)\,dx = -\frac{1}{a}\log\cos(ax+b) : \int\cot(ax+b)\,dx = \frac{1}{a}\log\sin(ax+b) :$$

$$\int\frac{dx}{\sin(ax+b)} = \frac{1}{a}\log\tan\tfrac{1}{2}(ax+b) : \quad \int\sin ax\sin bx\,dx = \frac{1}{2}\left\{\frac{\sin(a-b)x}{a-b} - \frac{\sin(a+b)x}{a+b}\right\} :$$

<u>General Formulæ.</u> $\int f(x)\,dx = xf(0) + \frac{x^2}{2!}f'(0) + \frac{x^3}{3!}f''(0) + \quad : \quad U_1 = \int U\,dx : U_2 = \int U_1\,dx :$

$$U' = \frac{d}{dx}U : U'' = \frac{d}{dx}U' \qquad \int U\,dx = xU - \frac{x^2}{2!}U' + \frac{x^3}{3!} - \ldots = xU - \int xU'dx : \int U'V\,dx = UV - \int UV'dx :$$

$$\int UV\,dx = UV_1 - U'V_2 + U''V_3 - \quad : \int x^mV\,dx = x^mV_1 - mx^{m-1}V_2 + m(m-1)x^{m-2}V_3 - \quad . \int U^mU'dx = \frac{U^{m+1}}{m+1}$$

$$\int(U'V+UV')\,dx = UV : \quad \int(aU+b)^mU'dx = \frac{(aU+b)^{m+1}}{a(m+1)} : \int\frac{U}{(U+V)^m}dx = \int\frac{dx}{(U+V)^{m-1}} - \int\frac{V}{(U+V)^m}dx :$$

$$\int\frac{U'}{U}dx = \log U = -\frac{1}{m}\log\frac{1}{U^m} : \int\frac{U'}{U^m}dx = -\frac{1}{(m-1)U^{m-1}} : \int\frac{VU'}{U^2}dx = -\frac{V}{U} + \int\frac{V'}{U}dx \ldots$$

$$\int\frac{U'V-UV'}{V^2}dx = \frac{U}{V} : \int\frac{U'V-UV'}{UV}dx = \log\frac{U}{V} : \int\frac{U'V-UV'}{U^2+V^2}dx = \arctan\frac{U}{V} : \int\frac{U'V^2+V'U^2}{UV(U+V)}dx = \log\frac{UV}{U+V} :$$

$$\int U\log x\,dx = U_1\log x - \int\frac{U_1}{x}dx : \quad \int V\log U\,dx = V_1\log U - \int\frac{U'V_1}{U}dx : \int U(\log x)^n dx =$$

$$= U_1(\log x)^n - n\int\frac{U_1(\log x)^{n-1}}{x}dx = U_1(\log x)^n - nU_2(\log x)^{n-1} + n(n-1)U_3(\log x)^{n-2} - \quad :$$

$$\int Ue^{ax}dx = \frac{1}{a}Ue^{ax} - \frac{1}{a}\int U'e^{ax}dx = \frac{e^{ax}}{a}\left\{U - \frac{U'}{a} + \frac{U''}{a^2} - \ldots \frac{(-1)^n}{a^n}U^{(n)}\right\} : \quad \int\frac{e^x}{x}dx = \int\frac{dz}{\log z},\ e^x = z ;$$

$$\int Ua^xdx = \frac{Ua^x}{\log a} - \frac{1}{\log a}\int U'a^xdx \quad U_1a^x - \log a\int U_1a^xdx = \frac{Ua^x}{\log a} - \frac{U'a^x}{(\log a)^2} + \frac{U''a^x}{(\log a)^3} - \quad = U_1a^x - U_2a^x\log a +$$

$$\int a^{mx+n}dx = \frac{a^{mx+n}}{m\log a} : \text{ thus } \int e^{mx}dx = \frac{1}{m}e^{mx} : \text{ and } \int e^{mx+n}dx = \frac{1}{m}e^{mx+n} :$$

$$\int x^ma^xdx = \frac{a^x}{\log a}\left\{x^m - \frac{mx^{m-1}}{\log a} + \frac{m(m-1)x^{m-2}}{(\log a)^2} - \quad \pm \frac{m(m-1)\ \cdot 2.1}{\log a^m}\right\} : \int\frac{e^{ax}}{x}dx = \log x + \frac{ax}{1!} + \frac{1}{2}\frac{(ax)^2}{2!} +$$

$$+ \frac{1}{3}\frac{(ax)^3}{3!} + \ldots : \int x^ma^{nx}dx = \int\left\{1 + \frac{nx\log x}{1} + \ldots + \frac{n^kx^k(\log x)^k}{k!} + \quad\right\}x^mdx :$$

Table of Integrals and Limits.

$\int \frac{a^x}{x^m}dx = -\frac{a^x}{(m-1)x^{m-1}} - \frac{a^x \log a}{(m-1)(m-2)x^{m-2}} - \frac{a^x(\log a)^2}{(m-1)(m-2)(m-3)x^{m-3}} - \frac{a^x(\log a)^{m-2}}{(m-1)\cdots 2.1\,x} + \frac{(\log a)^{m-1}}{(m-1)\cdots 2.1}\int\frac{a^x}{x}dx$:

$\int x^{nx}dx = x\{1-\frac{nx}{2^2}+\frac{n^2x^2}{3^3}-\frac{n^3x^3}{4^4}+\cdots\} + \frac{nx^2\log x}{1!}\{\frac12-\frac{nx}{3^2}+\frac{n^2x^2}{4^3}-\cdots\}+\frac{n^2x^3(\log x)^2}{2!}\{\frac13-\frac{nx}{4^2}+\frac{n^2x^2}{5^3}-\cdots\}+\cdots$

$\int x^{m+nx}dx = \int x^m\{1+nx\log x+\frac{1}{2!}(nx\log x)^2+\frac{1}{3!}(nx\log x)^3+\cdots\}dx$:

$\int\frac{a^x}{x}dx = \log x + x\log a + \frac12\frac{(x\log a)^2}{2!}+\frac13\frac{(x\log a)^3}{3!}+\cdots$: $\int\frac{a^x}{x^2}dx = -\frac{a^x}{x}+\log a\int\frac{a^x}{x}dx$:

$\int e^{ax}\frac{\cos^n}{\sin^n}bx\,dx = \frac{e^{ax}\frac{\cos^{n-1}}{\sin^{n-1}}bx\,(a\frac{\cos}{\sin}bx \pm nb\frac{\sin}{\cos}bx)}{a^2+n^2b^2} + \frac{n(n-1)b^2}{a^2+n^2b^2}\int e^{ax}\frac{\cos^{n-2}}{\sin^{n-2}}bx\,dx$:

$\int x^m e^{ax}\frac{\sin}{\cos}bx\,dx = x^m e^{ax}\frac{a\frac{\sin}{\cos}bx \mp b\frac{\cos}{\sin}bx}{a^2+b^2} - \frac{m}{a^2+b^2}\int x^{m-1}e^{ax}(a\frac{\sin}{\cos}bx\mp b\frac{\cos}{\sin}bx)\,dx =$

$= e^{ax}\left\{\frac1\theta x^m\frac{\sin}{\cos}(bx-\zeta)-\frac{m}{\theta^2}x^{m-1}\frac{\sin}{\cos}(bx-2\zeta)+\cdots\pm\frac{m(m-1)\cdots1}{\theta^{m+1}}\sin[bx-(m+1)\zeta]\right\}$, in which

$a+b\sqrt{-1}=\theta(\cos\zeta+\sqrt{-1}\sin\zeta)$: $\int x^m\log x\,dx=\frac{x^{m+1}}{m+1}(\log x-\frac{1}{m+1})$, true also if $m=0$.

$\int(a+bx)^m\log x\,dx = \frac{(a+bx)^{m+1}}{(m+1)b}\log x - \frac{1}{(m+1)b}\int\frac{(a+bx)^{m+1}}{x}dx$: $\int(a+bx)\log x\,dx =$

$= \frac{\log x}{2b}\{(a+bx)^2-a^2\} - x(a+\frac14 bx)$: $\int(a+bx)^2\log x\,dx = \frac{\log x}{3b}\{(a+bx)^3-a^3\}-x(a^2+\frac12abx+\frac19b^2x^2)$

$\int x^m\log(a+bx)\,dx = \frac{x^{m+1}}{m+1}\log(a+bx)-\frac{b}{m+1}\int\frac{x^{m+1}}{a+bx}dx$: $\int\frac{\log x}{(a+bx)^m}dx = -\frac{\log x}{(m-1)b(a+bx)^{m-1}}+$

$+\frac{1}{(m-1)b}\int\frac{dx}{x(a+bx)^{m-1}} = -\frac{\log x}{(m-1)b\rho^{m-1}}+\frac{1}{(m-1)ab}\{\frac{1}{(m-2)\rho^{m-2}}+\frac{1}{(m-3)a\rho^{m-3}}+\frac{1}{(m-4)a^2\rho^{m-4}}+\cdots+\frac{1}{2a^{m-4}\rho^2}+$

$+\frac{1}{a^{m-3}\rho}\} + \frac{1}{(m-1)a^{m-1}b}\log\frac{x}{\rho}$: $\int x^m(\log x)^n dx = \int z^n e^{(m+1)z}dz$, $\log x = z$, $= \frac{x^{m+1}\log x^n}{m+1} -$

$-\frac{n}{m+1}\int x^m(\log x)^{n-1}dx = \frac{x^{m+1}}{m+1}\{(\log x)^n-\frac{n}{m+1}(\log x)^{n-1}+\frac{n(n-1)}{(m+1)^2}(\log x)^{n-2}\cdots\cdots\pm\frac{n(n-1)\cdots2.1}{(m+1)^n}\}$:

Limits $\underset{x=1}{L}\frac{x^m-1}{x-1}=m$: $\underset{x=a}{L}\frac{x^m-a^m}{x^n-a^n}=\frac mn a^{m-n}$: $\underset{x=\infty}{L}\frac{\log(x+1)}{\log x}=1$: Denote $\log(\log x)$ by $\log^2 x$;

then $\underset{x=\infty}{L}\frac{\log^k(x+1)}{\log^k x}=1$: $\underset{x=\infty}{L}\frac{\log_a^k x}{\log_e^k x}=\frac{1}{\log_e a}$: $\underset{x=\infty}{L}(1+\frac1x)^x=\underset{x=\infty}{L}(1-\frac1x)^{-x}=e=2\cdot718281828459045235$

$\underset{x=0}{L}(1+x)^{\frac1x}=e$: $\underset{x=\infty}{L}(1+x)^{\frac1x}=1$: $\underset{x=\infty}{L}(1+\frac yx)^x=\underset{x=0}{L}(1+xy)^{\frac1x}=e^y$: $\underset{x=0}{L}\frac{a^x-1}{x}=\log a$ ∴ $\underset{x=0}{L}\frac{e^x-1}{x}=1$

$\underset{x=\infty}{L}(\log^k(x+1)-\log^k x)=0$: $\underset{x=\infty}{L}(1+\frac12+\frac13+\cdots+\frac1x-\log x)=\gamma=0\cdot5772156649\cdots$ = Euler's const.

$\underset{x=\infty}{L}\{(1+\frac12+\frac13+\cdots\frac1x)/\log x\}=1$: $\underset{\substack{x=\infty\\a>1}}{L}a^x/x=\infty$: $\underset{x=\infty}{L}a^x x^{-m}=\infty$ if m be $+$: $\underset{x=0}{L}e^{\frac{a^x-b^x}{x}}=\frac ab$:

$\underset{x=+0}{L}x\log_e x=0$: $\underset{x=\infty}{L}\log_e x/x^k=\infty$: $\underset{x=0}{L}x^n\log_e x=-\infty$: $\underset{m=\infty}{L}\frac{x^m}{m!}=0$: $\underset{\substack{m>-1\\m<-1}}{L}\frac{m(m-1)\cdots\cdots(m-n+1)}{n!}=\frac{0}{\infty}$

$\underset{x=+0}{L}x^x=1$ ∴ $0^0=1$ (not invariably) : $\underset{x=+0}{L}x^{x^m}=1$: $\underset{x=+0}{L}x^{x^{-m}}=0$: $\underset{x=0}{L}x^{\frac{1}{\log(e^x-1)}}=e$: $Lu^v=e^{Lv(u-1)}$, if the exp is determ.

$\underset{x=1}{L}(1+\log x)^{\frac{1}{x-1}}=e$: $L(1^r+2^r+\cdots m^r)/m^{r+1}=\frac{1}{r+1}$: $\underset{m=\infty}{L}\{(a+1)^r+(a+2)^r+\cdots+(a+m)^r\}/m^{r+1}=\frac{1}{r+1}$

Minimum Values. $\Gamma(x=1\cdot4616321\cdots)=0\cdot8856032$: x^x for $x=0\cdot3678794=0\cdot6922007$.

$\underset{x=a}{L}(1+\phi x)^{\psi x}=\underset{x=a}{L}[(1+\phi x)^{\frac{1}{\phi x}}]^{\phi x\cdot\psi x}$; if $\phi a=0$, $\psi a=\infty$, and $\phi a\cdot\psi a=m$, then $\underset{x=a}{L}\Phi(x)=e^m$.

$\underset{x=0}{L}e^{\frac{a^x-b^x}{x}}=\frac ab$: $\underset{x=a}{L}\frac{e^{mx}-e^{ma}}{x-a}=me^{ma}$: $\underset{x=0}{L}$ [illegible] ; $n!=\sqrt{(2\pi n)}(n/e)^n e^{\frac{\theta}{12n}}$, $0<\theta<1$.

In the above Tables the sign "=" merely denotes that a following quantity is also a value of the integral, not that it is necessarily equal, as will be obvious in the first few examples. The general formulæ facilitate the integration of many types of expression occurring in statistics.

XVIII. CONCLUSION.

1. **The larger aim of population statistic.**—Inasmuch as population is the foundation-element of all branches of social statistics, its complete study is both of practical value and general interest. For this reason an adequate "theory of population" has become a necessity. Moreover, international relationships have made it evident that the proper co-ordination of the whole world's statistical method and effort has become an imperious need of civilisation.[1] This emphasises the importance of the formation of a basis for international comparisons, and is a desideratum yearly becoming more urgent. Only by a sufficiently wide survey of human facts can the required *norms* of all sorts be established, norms which represent the characters of the great unit constituted by the aggregation of all the nations. It is only in the comparatively slow secular changes of these norms, that the drift of mankind in the gross can be unequivocally revealed ; when that drift is ascertained, the quicker and more marked variations of individual nations and populations can then be forced to disclose the real significance of their differentiating tendencies. The limits of human expansion are much nearer than popular opinion imagines; the difficulty of future food supplies will soon be of the gravest character ; the exhaustion of sources of energy necessary for any notable increase of population or advance in the standards of living, or both combined, is perilously near. Within periods of time, insignificant compared with geologic ages, the multiplying force of living things, man included, must receive a tremendous check. The present rate of increase in the world's population cannot continue for four centuries,[2] and the extraordinary increase in the standard of living which has characterised the last few decades must quickly be brought to a standstill, or be determined by the destructive forces of human extravagance. Very soon world-politic will have to face the question whether it is better that there should be larger numbers and more modest living, or fewer numbers and lavish living ; whether world-morality should aim at the enjoyment of life by a great multitude, or aim at the restriction of life experience to a few, that they may live in relative opulence. The statistician of the

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[1] This is the raison d'etre of the " Institute International de Statistique," and the ideal aim which its activities have in view.

[2] If we take the present population as about 1,700,000,000, and the annual rate of increase as 0.01159 the increase doubles the population in 60.15 years, and gives a population 3.16 times as great in 100 years ; thus in 200 years the population will have increased 10-fold, and in 400 years 100-fold We thus get, at the end of successive centuries, the following populations in round numbers :—100 years, 5,380,000,000 ; 200 years, 17,040,000,000 ; 300 years, 53,930,000,000 ; 400 years, 170,710,000,000.

future will utilize all discovery of the mysterious play, and no less cryptic, limitation of life-force to make prediction sure. Given co-ordinated international effort, there would be no difficulty in so directing future statistical technique that all countries and all analysts could add their quota in a form suitable for the wider study of the drift of mankind in the more important relations of civic, national, and international life.

In earlier days monarchs utilized statistic as a basis for judging the probability of success in operations of war and plunder. That use has not disappeared, but the plexus of relations, which, through the fructifying power of science, the modern world has seen established, particularly in the realms of industry and commerce, has shewn a growing measure of economic solidarity in the affairs of mankind. The modern world responds to everything that profoundly touches any one nation. By the conditions of modern life mankind tends to be welded into a unit. By the magic of invention, humanity has been quickened ; distance—if not annihilated—has been immensely shortened ; life has been enriched in the potentialities of material and psychical enjoyment, and be it said also in the plane of its possible intellectual and moral effort. The destiny of mankind will therefore be the supreme problem of those statisticians of the future, who have an adequate outlook on that science and art with which it is their privilege to concern themselves. For the craftsman with acute and microscopic vision there are a multitude of analyses to be made ; for one with the capacity for reaching wide generalisations there is no end of larger work, while for him who is happily able to see both the trees *and* the forest of the statistical landscape, there is the most far-reaching task of all, the creation of a statistical world-picture, which shall reveal the secrets of man's place in the many-sided world of social-economics, using that word in its fullest and most ideal sense.

2. **The impossibility of any long-continued increase of population at the present rate.**—An increase of population at the rate of 1 per cent. per annum is often regarded as unduly slow ; the increase for the United States between 1790 and 1860 was nearly 3 per cent. per annum, a rate which has recently also been attained in Australia. That this rate cannot possibly last even five centuries is a fact, however, that, though immensely important, is not realised.

It has been contended in reply to Malthus that experience has shewn that food-production will advance even more rapidly than the growth of population. It can do so for only a very limited time. The false inference has been drawn from this fact that therefore almost any population can be provided for. The point demands attention, for the argument is a plausible one. Notwithstanding this it is invalid, as can easily be shewn.[1]

[1] See L. Hirsch, La théorie de la population de Th. R. Malthus, Bibliothèque Universelle, Dec. 1916, No. 252, pp. 553-567, and Jan. 1917, No. 253, pp. 141-154.

If the earth's present population be taken as low even as 1,500,000,000 persons (which is, of course, an underestimate), and its land area, excluding the Arctic and Antarctic continents, be assumed to be, say, 33,000,000,000 acres ; and if further it is supposed that by some means it is possible to make the whole of this land-area yield an average of as much as 22.8 bushels of food-corn per acre, per annum, the total yield would be only 752,400,000,000 bushels.

In Australia, and in fact generally, the food-corn consumption is on the average, about equivalent to 5.7 bushels per annum, viz., one-fourth of the amount above assumed, which means that the total population which could be fed with 5.7 bushels of food-corn per annum together with other foods in like proportion, would be only 132,000,000,000. At a rate of increase of population of 0.01 per annum,[1] somewhat less than the rate for all countries which have accurate statistics, it would require only 450 years to exhaust the food requirement mentioned (more exactly 449.96 years). That no possible increase of the earth's reproductiveness can materially affect the question can also be readily shewn. For—to postulate the impossible—let it be supposed that every acre of area on the earth's entire surface could produce as much as 228 bushels, that is, ten times the above amount, with other foods in like proportion : this being done, it would take less than 700 years (681.37 years) for the population to exhaust the food supply. The fundamental element in Malthus' contention is thus seen to be completely established. Even a low rate of increase must soon exhaust the possibilities of food-supply, and as we have seen already the material of the earth is inadequate to provide bodies for any long-continued increase quite apart from the food question. It is quite clear therefore that statistical analyses of the world's progress in various ways will soon become of the highest order of importance.

3. **Need for analysis of existing statistical material.**—At present there exists a large and accumulating mass of unanalysed material. Numerical data have in many instances already become a burden, and in other cases threaten to become one. But when their significance has been penetrated they seem no longer tedious ; they have been transformed into illuminating and interesting facts.

Here, however, we need a word of warning. The problem of all so-called knowledge is to subsume what we know—or think we know—under suitable elementary conceptions, conceptions, in fact, that are within our intellectual grasp, and that we can mentally handle. As in physics the Boyle-Charles gaseous laws, the molecular law of equal numbers in equal volumes at equal pressures and temperatures, and the conception of mass

[1] The number of years in which a population is doubled is given by the following quantities divided by the increase. When the increase or

divisor is	.000	.010	.020	.030	.040	.050
the numerator is	.6931	.6966	.7001	.7035	.7069	.7103

(See also footnote p. 31.)

as independent of velocity,[1] are but crude statements of the actual facts, so crude that their elementary simplicity entirely disappears when necessary qualifications are made, so likewise does a deeper knowledge of statistic reveal that relations subsisting among crude data are subject to corrections that, not infrequently, are very elaborate. The more simple and obvious of these relations constitute a kind of rough frame-work about which more subtle and accurate conceptions may cluster,[2] or, to change the figure, they are a skeletal foundation on which the body of justly conceived statistic is to be built up.

4. **The trend of destiny.**—To the extent man is ignorant, he is both the puppet of Fortune, and the victim of Desire. Anyone who has seriously reflected upon the facts of the last ten decades must realise that, within the next ten, tremendous problems will arise for solution and these will touch fundamentally the following matters, viz. :—

(i.) The multiplying power of the human race ;

(ii.) The organic constitution of Nature and the means at human disposal for avoiding the incidence of its unfavourable aspects ; *i.e.*, eugenics in its wider sense ;

(iii.) The enhancing of the productivity of Nature, and the limits of its exploitation ;

(iv.) The mechanism of the social organism, and the scheme of its control ;

(v.) Internationalism and the solidarity of humanity.

For the adequate study of these matters, not only will the mere technique of the collection and analysis of statistic require to be much advanced, but the popular opinion as to the value of the effort will also have to progress. Given, however, an intelligent public opinion, as to the utility of statistical inquiries, there would be some ground for hope that the great questions, the analysis of which would throw light upon human destiny, could be properly attacked. It is for educational departments, worthy of the name, to create such opinion by the mechanism of their systems, in order that each human being should be sufficiently interested to cordially co-operate, by accurately furnishing the necessary data in the taking of a census of population or wealth. Census-taking is a costly operation, but it is the foundation of all branches of statistic that have a direct human interest. Its value and the facility of using it would be immensely increased if it were meticulously accurate. The importance of technique and of precision, matters apparently of little moment, can be rightly estimated only when the ultimate aim of all statistical inquiry is realised to be "the study of man's destiny" as the denizen of a world of limitations.

[1] According to modern views "mass" in matter becomes infinite when its velocity equals that of light.

[2] Thus, for example, the crude mortality of one population may agree with that of another, but when corrected may seriously differ, shewing either a better or worse state of things regarding the conditions of human life.

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APPENDIX A.

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